Time Scales Analysis
Lecture 1
Introduction to Time Scales: Motivation, examples

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In mathematics, time scale calculus is a unification of integral and differential calculus with the calculus of finite differences. Time scale calculus was introduced in 1988 by the German mathematician Stefan Hilger. However, similar ideas have been used before and go back at least to the introduction of the Riemann–Stieltjes integral, which unifies sums and integrals. This connection is valuable because many real-world systems exhibit both continuous and discrete behaviors, making the time scales framework and the Riemann-Stieltjes integral applicable to diverse areas such as economic maximization, traffic dynamics, and biological models. The time scales calculus has applications in any field that requires simultaneous modelling of discrete and continuous data. It gives a new definition of a derivative such that if one differentiates a function defined on the real numbers then the definition is equivalent to standard differentiation, but if one uses a function defined on the integers then it is equivalent to the forward difference operator. By setting the time scale to be the real numbers ($\mathbb{R}$), a dynamic equation becomes a differential equation, and by choosing the integers ($\mathbb{Z}$), it becomes a difference equation.

The theory has found applications in various fields, such as:

1. **Biology**: Modeling population dynamics, like insect populations that have continuous growth during a season and then a discrete period of dormancy.
2. **Engineering**: Analyzing complex systems with both continuous and discrete components.
3. **Economics**: Studying models of economic growth and household consumption.
4. **Pharmacokinetics**: The time-scale calculus has been used to model and solve pharmacokinetic problems involving multiple drug doses, which inherently blend continuous absorption processes with discrete drug intake events.

The theory has gained significant attention and is an active area of research, with numerous papers published on its theory and applications. As a result of the active development of the theory, various branches of it have been established.

Dynamic Calculus on Time Scales

Dynamic calculus on time scales is the study of the definition, properties, and applications of the Hilger derivative or delta derivative of a function. The process of finding the Hilger derivative is called Hilger differentiation or delta differentiation. Given a function and a point in the domain, the Hilger derivative at that point is a way of encoding the small-scale behavior of the function near that point. By finding the Hilger derivative of a function at every point in its domain, it is possible to produce a new function, called the Hilger or delta derivative function of the original function.

List of Dynamic Calculus Topics

- Sum rule in Hilger differentiation.
- Constant factor rule in Hilger differentiation.
- Linearity of Hilger differentiation.
- Power rule.
- Chain rule.
- Product rule.
- Quotient rule.
- Inverse functions and Hilger differentiation.
- Stationary point.
- Maxima and minima.
- First Hilger derivative test.

List of Dynamic Calculus Topics

- Second Hilger derivative test.
- Extreme value theorem.
- Hilger differentiation operator.
- Taylor's theorem.
- L'Hôpital's rule.
- General Leibniz rule.
- Mean value theorem.
- Elementary Hilger functions.
- Rolle's theorem.

Hilger Integral Calculus

Hilger integral calculus is the study of the definitions, properties, and applications of two related concepts, the indefinite Hilger integral and the definite Hilger integral. The process of finding the value of a Hilger integral is called Hilger integration. The indefinite Hilger integral, also known as the Hilger antiderivative, is the inverse operation to the Hilger derivative. F is an indefinite Hilger integral of f when f is a Hilger derivative of F. The definite Hilger integral inputs a function and outputs a number, which gives the algebraic sum of areas between the graph of the input and the x-axis. The technical definition of the definite Hilger integral involves the limit of a sum of areas of rectangles, called a Hilger Riemann sum.

List of Hilger Integral Calculus Topics

- Sum rule in Hilger integration.
- Constant factor rule in Hilger integration.
- Linearity of Hilger integration.
- Arbitrary constant of Hilger integration.
- Fundamental theorem of dynamic calculus.
- Hilger integration by parts.
- Hilger integration by substitution
- Hilger differentiation under the Hilger integral sign.

Dynamic Calculus of Functions of Several Variables

Dynamic calculus of functions of several variables extends single-variable dynamic calculus to analyze functions with multiple inputs, focusing on partial dynamic derivatives, the dynamic gradient, the Hilger Jacobian, and the total dynamic differential to understand rates of change, dynamic tangent planes, and the sensitivity of the function to each input variable. Key concepts include partial dynamic derivatives, which measure change with respect to one variable while holding others constant; the total dynamic differential, which approximates a small change in the function's output; the dynamic gradient vector, which points in the direction of the greatest rate of increase; and the Hilger Jacobian matrix, a generalization of the dynamic gradient for vector-valued functions.

List of Dynamic Calculus of Functions of Several Variables Topics

- Sum rule in Hilger partial differentiation.
- Constant factor rule in Hilger partial differentiation.
- Linearity of Hilger partial differentiation.
- Power rule.
- Chain rule.
- Product rule.
- Quotient rule.
- Inverse functions and Hilger partial differentiation.
- Stationary point.
- Maxima and minima.
- First Hilger partial derivative test.

Hilger Integral Calculus of Functions of Several Variables

Hilger integral calculus of functions of several variables is known as multivariable Hilger calculus or multivariate dynamic calculus, and it extends the concepts of single-variable calculus to multiple dimensions through the use of multiple Hilger integrals. Hilger multiple integrals, such as double and triple integrals, allow for the calculation of quantities like areas, volumes, and total amounts within regions in 2D or 3D space by Hilger integrating a function over several variables.

List of Hilger Integral Calculus Topics

- Sum rule in Hilger integration.
- Constant factor rule in Hilger integration.
- Linearity of Hilger integration.
- Fundamental theorem of dynamic calculus.
- Hilger integration by parts.
- Hilger integration by substitution
- Hilger differentiation under the Hilger integral sign.

A dynamic equation (DE) is an equation relating a function to its Hilger or delta derivatives. If the function is of only one variable, we call the equation an ordinary dynamic equation (ODE). Equations relating the partial delta derivatives of a function of several variables are called partial dynamic equations (PDEs).

List of Dynamic Equations Topics

- Existence of solutions.
- Uniqueness of solutions.
- Multiplicity of solutions.
- Dependence on initial data.
- Dependence on parameters.

List of Dynamic Equations Topics

- Initial value problems.
- Boundary value problems.
- Initial boundary value problems.
- Stability of solutions.
- Oscillations of solutions.
- Qualitative analysis of solutions.

Subclasses of Dynamic Equations

- Functional Dynamic Equations.
- Fuzzy Dynamic Equations.
- Fuzzy Dynamic Inclusions.
- Impulsive Dynamic Equations.
- Fuzzy Impulsive Dynamic Equations.
- Impulsive Functional Dynamic Equations.
- Fuzzy Impulsive Functional Dynamic Equations.

Integral equations on time scales are equations in which an unknown function appears under a Hilger or delta integral sign. Various classification methods for Hilger integral equations exist. A few standard classifications include distinctions between linear and nonlinear; homogeneous and inhomogeneous; Fredholm and Volterra; first order, second order, and third order; and singular and regular integral equations.[These distinctions usually rest on some fundamental property such as the consideration of the linearity of the equation or the homogeneity of the equation.

List of Integral Equations Topics

- Existence of solutions.
- Uniqueness of solutions.
- Multiplicity of solutions.
- Dependence on initial data.
- Dependence on parameters.

List of Integral Equations Topics

- Initial value problems.
- Boundary value problems.
- Initial boundary value problems.
- Stability of solutions.
- Oscillations of solutions.
- Qualitative analysis of solutions.

Subclasses of Integral Equations on Time Scales

- Integro-Dynamic Equations on Time Scales.
- Impulsive Integral Equations on Time Scales.
- Functional Integral Equations on Time Scales.
- Fuzzy Integral Equations.
- Fuzzy Integro-Dynamic Equations on Time Scales.
- Fuzzy Impulsive Integral Equations on Time Scales.

Numerical analysis on time scales is the study of algorithms that use numerical approximation (as opposed to symbolic manipulations) for the problems of time scales analysis. It is the study of numerical methods that attempt to find approximate solutions of problems rather than the exact ones. Numerical analysis on time scales finds application in all fields of engineering and the physical sciences, the life and social sciences like economics, medicine, business and even the arts. Examples of numerical analysis on time scales include: dynamic equations as found in celestial mechanics (predicting the motions of planets, stars and galaxies), numerical linear algebra in data analysis, simulations in biology and theory of populations.

Key Concepts in Numerical Analysis on Time Scales

- Direct methods.
- Iterative methods.
- Generation of errors.
- Propagation of errors.

Key Concepts in Numerical Analysis on Time Scales

- Round-off.
- Truncation error.
- Numerical stability.
- Well-posed problems.
- Computing values of functions.
- Interpolation.
- Extrapolation.
- Regression.

Key Concepts in Numerical Analysis on Time Scales

- Solving equations.
- Solving systems.
- Solving eigenvalue problems.
- Solving singular problems.
- Optimization.
- Evaluating Hilger integrals.
- Dynamic equations.
- Partial dynamic equations.
- Integral equations.

Dynamic geometry on time scales unifies the classical differential geometry and discrete geometry. It uses the techniques of single variable time scales calculus, vector time scales calculus, linear algebra and multilinear algebra.

Concepts of Dynamic Geometry on Time Scales

- Dual spaces.
- Tangent spaces.
- Curves.
- Surfaces.
- Manifolds.

Concepts of Dynamic Geometry on Time Scales

- Dynamic differential forms.
- Hilger integration of forms.
- Volume forms.
- Tensor calculus.
- Vector calculus.

Convex Analysis on Time Scales

Convex analysis is the branch of time scales analysis devoted to the study of properties of convex functions and convex sets, often with applications in convex minimization, a subdomain of optimization theory.

Programme of the Lecture Course

The main aim of this cycle of lectures on time scales analysis is to introduce the participants to the basic terms and facts of time scales analysis and how to operate with them.

The programme of the course includes the following topics.

- Time Scales
  - Definition. Examples
  - Forward Jump Operators, Backward Jump Operators and Graininess Functions
  - A Classification of Points
  - The Topology of Time Scales
  - Functions and Jump Operators
  - The Induction Principle

Time Scales Differentiation

- Definition for Delta Derivative. Examples
- Basic Rules for Delta Differentiation
- Higher Order Delta Differentiation
- Nabla Derivatives
- Delta Mean Value Theorems
- Delta Increasing and Delta Decreasing Functions
- Delta Convex and Delta Concave Functions
- Extreme Values
- Completely Delta Differentiable Functions
- One-Sided Delta Derivatives
- Delta Chain Rules
- Delta L'Hopital's Rule

- Regulated, Rd-Continuous and Delta Pre-Differentiable Functions
- Delta Indefinite Integral
- The Darboux Delta Integral
- The Riemann Delta Integral
- Other Definition for the Riemann Delta Integral
- Properties of the Riemann Integral
- Improper Delta Integrals of the First Kind
- Improper Integrals of the Second Kind
- Delta Monomials
- The Taylor Formula
- Survey on Nabla Integrals

Time Scales Elementary Functions

- Hilger's Complex Plane
- Delta Regressive Functions
- The Delta Exponential Functions
- Delta Hyperbolic Functions
- Delta Trigonometric Functions

Definition

A time scale is an arbitrary nonempty closed subset of the real numbers.

We will denote a time scale by the symbol $\mathbb{T}$. We suppose that a time scale $\mathbb{T}$ has the topology that inherits from the real numbers with the standard topology.

Example

The sets $[-1, 4]$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$,

$$\left\{-2, \quad -1, \quad -\frac{1}{2}, \quad 0, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad 2, \quad 3, \quad 6\right\},$$

and

$$\{1\} \bigcup \left\{\frac{1}{n} + 1\right\}_{n\in\mathbb{N}} \bigcup \{3\} \bigcup \left\{\frac{4}{n^2} + 3\right\} \bigcup \{9\} \bigcup \left\{\frac{7}{n^4} + 9\right\}_{n\in\mathbb{N}}$$

are time scales.

Example

The sets $(-3, 7)$, $[0, 5)$, $(1, 7]$ and $\left\{\frac{5}{n} + 3\right\}_{n\in\mathbb{N}}$ are not time scales.

Example

Let $a, b > 0$. The sets

$$P_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$$

are time scales.

Example

The set of harmonic numbers

$$
\begin{aligned}
H_0 &= 0, \\
H_n &= \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N},
\end{aligned}
$$

is a time scale.

Example

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers with $\alpha_n > 0$, $n \in \mathbb{N}_0$. Define

$$
t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbb{N}.
$$

Then the set $\mathbb{T} = \{t_n : n \in \mathbb{N}\}$ is a time scale.

Example

(Cantor Set) Consider the interval $K_0 = [0, 1]$. We obtain a subset K_1 of K_0 by removing the open "middle third" of K_0, i.e., the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from K_0. The set K_2 is obtained by removing the two open middle thirds of K_1, i.e., the two open inetrvals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from K_1. Proceeding in this manner, we obtain a sequence $\{K_n\}_{n\in\mathbb{N}_0}$ of sub sets of the interval $[0, 1]$. In the figure below are shown the sets K_0, K_1, K_2, K_3 and so forth

Example

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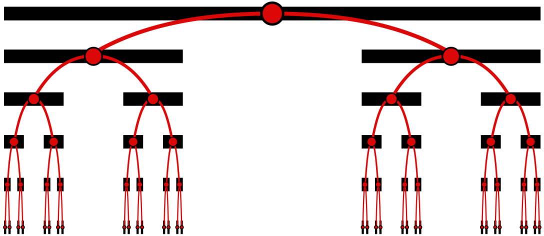
Figure: Expansion of a Cantor Set.



Example

The Cantor set C is now defined as follows

$$C = \bigcap_{n=0}^{\infty} K_n.$$

The Cantor set is a time scale. Any its element x can be represented in its ternary expansion as follows

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \quad \text{where} \quad a_j \in \{1, 2, 3\}, \quad j \in \mathbb{N}.$$

This expansion is unique unless x is of the form $p3^{-k}$ for some integers p and k. In this case, x has two expansions

1. $a_j = 0$ for $j > k$.
2. $a_j = 2$ for $j > k$.

Example

Assume that p is not divisible by 3. One of these expansions will have $a_k = 1$ and the other wiill $a_k = 0$ or $a_k = 2$. We have that

$$a_1 = 1 \quad \text{if and only if} \quad \frac{1}{3} < x < \frac{2}{3}$$

and

$$a_1 \neq 1 \quad \text{and} \quad a_2 = 1 \quad \text{if and only if} \quad \frac{1}{9} < x < \frac{2}{9} \quad \text{or} \quad \frac{7}{9} < x < \frac{8}{9}$$

and so forth.

Example

If

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j} \quad \text{and} \quad y = \sum_{j=1}^{\infty} \frac{b_j}{3^j},$$

then $x < y$ if and only if there exists an $n \in \mathbb{N}$ such that $a_n < b_n$ and $a_j = b_j$ for $j < n$. Thus, the Cantor set C is the set of all $0 \le x \le 1$ that have a base-3 expansion

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j} \quad \text{with} \quad a_j \neq 1 \quad \text{for any} \quad j.$$

Example

The set

$$\mathbb{T} = \{t_n = -\frac{1}{n} : n \in \mathbb{N}\} \cup \mathbb{N}_0$$

is a time scale.

Example

The set

$$[0,1] \cup \left\{1 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup (2,3] \cup \left\{3 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$$

is a time scale.

Example

Let

$$U = \left\{ \frac{1}{2^n} : n \in \mathbb{N}_0 \right\}.$$

Then the set

$$\{0\} \cup U \cup (1 - u) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup (3 - U) \cup (3 + U) \cup \{1, 2, 3, 4\}$$

is a time scale.

Exercise

Check if the following sets are time scales

1. $2^{\mathbb{N}_0}$.
2. $(-1, 1] \cup [2, 3] \cup [4, 8]$.
3. $\left\{-\frac{1}{2n} : n \in \mathbb{N}\right\} \cup 2\mathbb{N}_0$.
4. $U \cup (2 - U) \cup (2 + U)$, $U = \left\{\frac{1}{4^n} : n \in \mathbb{N}_0\right\}$.
5. $[0, 2] \cup \left\{2 + \frac{1}{n}\right\}_{n \in \mathbb{N}} \cup (3, 5] \cup 7^{\mathbb{N}_0}$.

We start by defining the forward jump operator.

Definition

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ in the following manner

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$. Then, $t = \sigma(t)$ if t is a maximum of $\mathbb{T}$.

Note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{Z}$ such that $t = hn$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned} \sigma(t) &= \inf\{s = hp, p \in \mathbb{Z} : hp > hn\} \\ &= h(n+1) \\ &= hn + h \\ &= t + h. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = 3^n$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned} \sigma(t) &= \inf\{3^s, s \in \mathbb{N}_0 : 3^s > 3^n\} \\ &= 3^{n+1} \\ &= 3 \cdot 3^n \\ &= 3t. \end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$ is fixed. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = n^k$. Hence, $n = \sqrt[k]{t}$. Now, applying the definition for forward jump operators, we arrive at

$$\begin{aligned} \sigma(t) &= \inf\{s^k, s \in \mathbb{N}_0 : s^k > n^k\} \\ &= (n+1)^k \\ &= \left(\sqrt[k]{t} + 1\right)^k. \end{aligned}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$ arbitrarily. Then, applying the definition for forward jump operators, we find

$$\begin{aligned}
\sigma(H_n) &= \inf\{H_s, s \in \mathbb{N}_0 : H_s > H_n\} \\
&= \inf\left\{H_s, s \in \mathbb{N}_0 : \sum_{k=1}^{s} \frac{1}{k} > \sum_{k=1}^{n} \frac{1}{k}\right\} \\
&= \sum_{k=1}^{n+1} \frac{1}{k} \\
&= H_{n+1}.
\end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$\begin{aligned} \mathbb{T} &= \bigcup_{k=0}^{\infty} [4k, 4k+1] \\ &= [0,1] \cup [4,5] \cup [8,9] \cup [12,13] \cup \ldots . \end{aligned}$$

If $t \in [0,1)$, then, applying the definition for forward jump operators, we find

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t. \end{aligned}$$

Example

If $t = 1$, then

$$\begin{aligned} \sigma(1) &= \inf\{s \in \mathbb{T} : s > 1\} \\ &= 4. \end{aligned}$$

Let now, $k \in \mathbb{N}$ be arbitrarily chosen. If $t \in [4k, 4k+1)$, then we have

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t. \end{aligned}$$

Example

If $t = 4k + 1$, then

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > 4k+1\}\\ &= 4(k+1)\\ &= 4k+4\\ &= 4k+1+3\\ &= t+3.\end{aligned}$$

Example

Therefore

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty} [4k, 4k+1) \\ \\ t+3 & \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{4k+1\}. \end{cases}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. We will find $\sigma(t)$ for $t \in \mathbb{T}$. For this aim, let C_1 denote the set of all left-hand end points of the open intervals that are removed. Then

$$C_1 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \le k \le m \right\}.$$

With C_2 we will denote the set of all right-hand end points of the open intervals that are removed. We have

$$C_2 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \le k \le m \right\}.$$

Take $t \in C$ arbitrarily. We have the following cases.
Let $t \in C_1$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}.$$

Example

Hence, we obtain

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\}\\ &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}\\ &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} + \frac{1}{3^{m+1}}\\ &= t + \frac{1}{3^{m+1}}.\end{aligned}$$

Let $t \in C_2$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}.$$

Hence,

Example

Let $t \in \mathbb{T}\backslash(C_1 \cup C_2)$. Then

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t. \end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}} & \text{if} \quad t \in C_1, t = \sum\limits_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ t & \text{if} \quad t \in \mathbb{T}\backslash C_1. \end{cases}$$

Example

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers with $\alpha_n > 0$, and

$$t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbb{N},$$

and

$$\mathbb{T} = \{t_n : n \in \mathbb{N}\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned} \sigma(t_n) &= \inf\left\{x \in \mathbb{T} : s = \sum_{k=0}^{n-1} \alpha_k, \quad s > t_n\right\} \\ &= \sum_{k=0}^{n} \alpha_k = \sum_{k=0}^{n-1} + \alpha_n = t_n + \alpha_n. \end{aligned}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$n = -\frac{1}{t_n}$$

and

$$\begin{aligned} \sigma(t_n) &= \inf \left\{ s \in \mathbb{T} : s = -\frac{1}{m}, m \in \mathbb{N}, s > t_0 \right\} \\ &= -\frac{1}{n+1} = -\frac{1}{-\frac{1}{t_n} + 1} = -\frac{t_n}{t_n - 1}. \end{aligned}$$

Example

Next, if $t \in \mathbb{N}_0$, then

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

$$= t + 1.$$

Consequently

$$\sigma(t) = \begin{cases} -\frac{t}{t-1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\}, \quad t = t_n \\ t + 1 & \text{if} \quad t \in \mathbb{N}_0. \end{cases}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned} \sigma(t_n) &= \inf\{s \in \mathbb{T} : s > t_n\} \\ &= \left(\frac{1}{2}\right)^{2^{n-1}} = \left(\frac{1}{2}\right)^{2^n \cdot \frac{1}{2}} \\ &= \left(\left(\frac{1}{2}\right)^{2^n}\right)^{\frac{1}{2}} = \sqrt{t_n}. \end{aligned}$$

Example

Next,

$$t_0 = \frac{1}{2} \quad \text{and} \quad \sigma(t_0) = 1$$

and

$$\sigma(0) \;=\; 0, \quad \sigma(1) = 1.$$

Consequently

$$\sigma(t) = \begin{cases} \sqrt{t} \quad \text{if} \quad t \in \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ 1 \quad \text{if} \quad t = \frac{1}{2} \\ 0 \quad \text{if} \quad t = 0 \\ 1 \quad \text{if} \quad t = 1. \end{cases}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{0, 1, 2\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. We have the following cases.
Let $t = 0$. Then

$$\sigma(0) = 0.$$

Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}.$$

Let $t = 1$. Then

$$\sigma(1) = 1.$$

Example

Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}.$$

Let $t = 2$. Then

$$\sigma(2) = 2.$$

Let $t = \frac{5}{2}$. Then

$$\sigma\left(\frac{5}{2}\right) = \frac{5}{2}.$$

Example

Let $t \in U \backslash \left\{\frac{1}{2}\right\}$. Then

$$t = \frac{1}{2^n}$$

and

$$\sigma(t) = \frac{1}{2^{n-1}} = \frac{2}{2^n} = 2t.$$

Let $t \in (1-U) \backslash \left\{\frac{1}{2}\right\}$. Then $t = 1 - \frac{1}{2^n}$ and $\frac{1}{2^n} = 1 - t$. Hence,

$$\sigma(t) = 1 - \frac{1}{2^{n+1}} = 1 - \frac{1}{2} \cdot \frac{1}{2^n} = 1 - \frac{1-t}{2} = \frac{1+t}{2}.$$

Example

Let $t \in (1+U) \setminus \left\{\frac{3}{2}\right\}$. Then

$$t = 1 + \frac{1}{2^n}.$$

Hence,

$$\frac{1}{2^n} = t - 1$$

and

$$\sigma(t) = 1 + \frac{1}{2^{n-1}} = 1 + \frac{2}{2^n} = 1 + 2(t-1) = 2t - 1.$$

Example

Let $t \in (2-U)\setminus\left\{\frac{3}{2}\right\}$. Then

$$t = 2 - \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = 2 - t.$$

Hence,

$$\sigma(t) = 2 - \frac{1}{2^{n+1}} = 2 - \frac{1}{2}\cdot\frac{1}{2^n} = 2 - \frac{2-t}{2} = \frac{t+2}{2}.$$

Example

Let $t \in (2+U) \setminus \left\{\frac{5}{2}\right\}$. Then

$$t = 2 + \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = t - 2.$$

Hence,

$$\sigma(t) = 2 + \frac{1}{2^{n-1}} = 2 + \frac{2}{2^n} = 2 + 2(t-2) = 2(t-1).$$

Example

Consequently

$$\sigma(t) = \begin{cases} 0 \quad \text{if} \quad t = 0 \\ \frac{3}{4} \quad \text{if} \quad t = \frac{1}{2} \\ 1 \quad \text{if} \quad t = 1 \\ \frac{7}{4} \quad \text{if} \quad t = \frac{3}{2} \\ 2 \quad \text{if} \quad t = 2 \\ \frac{5}{2} \quad \text{if} \quad t = \frac{5}{2} \\ 2t \quad \text{if} \quad t \in U \setminus \left\{ \frac{1}{2} \right\} \\ \frac{1+t}{2} \quad \text{if} \quad t \in (1-U) \setminus \left\{ \frac{1}{2} \right\} \end{cases}$$

Example

$$\sigma(t) = \begin{cases} 2t - 1 & \text{if} \quad t \in (1+U) \setminus \left\{ \frac{3}{2} \right\} \\ \frac{t+2}{2} & \text{if} \quad t \in (2-U) \setminus \left\{ \frac{3}{2} \right\} \\ 2(t-1) & \text{if} \quad t \in (2+U) \setminus \left\{ \frac{5}{2} \right\}. \end{cases}$$

Exercise

Find $\sigma(t)$, $t \in \mathbb{T}$, where

1. $\mathbb{T} = h\mathbb{Z} + k$, $h > 0$, $k \in \mathbb{R}$.
2. $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}$.
3. $\mathbb{T} = P_{3,7} \cup [4, 6]$.
4. $\mathbb{T} = 11^{\mathbb{N}_0} \cup \{0\}$.
5. $\mathbb{T} = [1, 2] \cup [3, 4] \cup [7, 8] \cup 9^{\mathbb{N}}$.