Time Scales Analysis
Lecture 2

Forward Graininess Function, Backward Jump Operator, Classification of Points, Topology in Time Scales

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Definition

Define the forward graininess function $\mu : \mathbb{T} \to \mathbb{R}$ as follows

$$\mu(t) = \sigma(t) - t, \quad t \in \mathbb{T}.$$

Example

Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then, using the computations in Example 98, we find

$$\begin{aligned} \mu(t) &= \sigma(t) - t \\ &= t + h - t \\ &= h, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then, using the computations in Example 99, we get

$$\mu(t) = \sigma(t) - t = 3t - t = 2t, \quad t \in \mathbb{T}.$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$. Then, using the computations in Example 100, we find

$$\begin{aligned} \mu(t) &= \sigma(t) - t = \left(\sqrt[k]{t} + 1\right)^k - t \\ &= t + \binom{k}{1}\sqrt[k]{t^{k-1}} + \binom{k}{2}\sqrt[k]{t^{k-2}} + \cdots + \binom{k}{k-1}\sqrt[k]{t} - t \\ &= \binom{k}{1}\sqrt[k]{t^{k-1}} + \binom{k}{2}\sqrt[k]{t^{k-2}} + \cdots + \binom{k}{k-1}\sqrt[k]{t}, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$ are the harmonic numbers. Then, using the computations in Example 101, we find

$$\mu(H_n) = \sigma(H_n) - H_n = H_{n+1} - H_n = \frac{1}{n+1}, \quad n \in \mathbb{N}.$$

Example

Let $\mathbb{T} = P_{1,3}$. Then, using the computations in Example 102, we find

$$\begin{aligned} \mu(t) &= \sigma(t) - t \\ &= \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty} [4k, 4k+1) \\ 3 & \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{4k+1\}. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. Then, using the computations in Example 106, we find

$$\begin{aligned} \mu(t) &= \sigma(t) - t \\ &= \begin{cases} \frac{1}{3^{m+1}} \quad \text{if} \quad t \in C_1, \quad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ 0 \quad \text{if} \quad t \in \mathbb{T} \backslash C_1. \end{cases} \end{aligned}$$

Example

Let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n > 0$, $nn \in \mathbb{N}$,

$$t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbb{N}, \quad \text{and} \quad \mathbb{T} = \{t_n : n \in \mathbb{N}\}.$$

Then, using the computations in Example 109, we find

$$\begin{aligned} \mu(t_n) &= \sigma(t_n) - t_n \\ &= t_{n+1} - t_n \\ &= \alpha_n, \quad n \in \mathbb{N}. \end{aligned}$$

Example

Let $\mathbb{T} = \{t_n = -\frac{1}{n} : n \in \mathbb{N}\} \cup \mathbb{N}_0$. Then, using the computations in Example 110, we find

$$\begin{aligned} \mu(t) &= \sigma(t) - t \\ &= \begin{cases} -\frac{t^2}{t-1} \quad \text{if} \quad t \in \{t_n = -\frac{1}{n} : n \in \mathbb{N}\}, \quad t = t_n, \\ 1 \quad \text{if} \quad t \in \mathbb{N}_0. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \bigcup \{0, 1\}$. Then, using the computations in Example 112, we find

$$
\begin{aligned}
\mu(t) &= \sigma(t) - t \\
&= \begin{cases} \sqrt{t} - t & \text{if} \quad t \in \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ \frac{1}{2} & \text{if} \quad t = \frac{1}{2} \\ 0 & \text{if} \quad t \in \{0, 1\}. \end{cases}
\end{aligned}
$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

Then, using the computations in Example 114, we find

$$\begin{aligned} \mu(t) &= \sigma(t) - t \\ &= \begin{cases} 0 & \text{if} \quad t = 0 \\ \frac{1}{4} & \text{if} \quad t = \frac{1}{2} \\ 0 & \text{if} \quad t = 1 \\ \frac{1}{4} & \text{if} \quad t = \frac{3}{2} \\ 0 & \text{if} \quad t = 2 \end{cases} \end{aligned}$$

Example

$$\mu(t) = \begin{cases} 0 \quad \text{if} \quad t = \frac{5}{2} \\ t \quad \text{if} \quad t \in U \backslash \left\{ \frac{1}{2} \right\} \\ \frac{t-1}{2} \quad \text{if} \quad t \in (1-U) \backslash \left\{ \frac{1}{2} \right\} \\ t-1 \quad \text{if} \quad t \in (1+U) \backslash \left\{ \frac{3}{2} \right\} \\ \frac{t-2}{2} \quad \text{if} \quad t \in (2-U) \backslash \left\{ \frac{3}{2} \right\} \\ t-2 \quad \text{if} \quad t \in (2+U) \backslash \left\{ \frac{5}{2} \right\}. \end{cases}$$

Definition

The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as follows

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition we put $\sup \emptyset = \inf \mathbb{T}$. Then, $\rho(t) = t$ if t is a minimum of $\mathbb{T}$.

Note that $\rho(t) \leq t$ for any $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{Z}$ such that $t = hn$. Hence, applying the definition for backward jump operator, we find

$$\begin{aligned} \rho(t) &= \sup\{s \in \mathbb{T} : s = hm, \quad m \in \mathbb{Z}, \quad s < hn\} \\ &= h(n-1) \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Assume that $t = 1$. Then

$$\rho(1) = 1.$$

2. Assume that $t > 1$. Then $t = 3^l$ for some $l \in \mathbb{N}$. Hence, applying the definition for backward jump operator, we find

$$\begin{aligned} \rho(t) &= \sup\{s \in \mathbb{T} : s = 3^k, \quad k \in \mathbb{N}_0, \quad s < 3^l\} \\ &= 3^{l-1} \\ &= \frac{3^l}{3} \\ &= \frac{t}{3}. \end{aligned}$$

Example

Consequently

$$\rho(t) = \begin{cases} 1 & \text{if} \quad t = 1 \\ \frac{t}{3} & \text{if} \quad t > 1. \end{cases}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, $k \in \mathbb{N}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Assume that $t = 0$. Then
$$\rho(0) = 0.$$
2. Let $t > 0$. Then there is an $n \in \mathbb{N}$ such that $t = n^k$. Hence, $n = \sqrt[k]{t}$. Now, applying the definition for backward jump operator, we find
$$\begin{aligned} \rho(t) &= \sup\{s \in \mathbb{T} : s = l^k, \quad l \in \mathbb{N}_0, \quad s < n^k\} \\ &= (n-1)^k \\ &= (\sqrt[k]{t} - 1)^k. \end{aligned}$$

Example

Consequently

$$\rho(t) = \begin{cases} 0 \quad \text{if} \quad t = 0 \\ \\ (\sqrt[k]{t} - 1)^k \quad \text{if} \quad t > 0. \end{cases}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$ arbitrarily. We have the following cases.

1. Assume that $n = 0$. Then

$$\rho(H_0) = H_0.$$

2. Assume that $n \geq$. Then, applying the definition for backward jump operator, we find

$$\begin{aligned} \rho(H_n) &= \sup\{s \in \mathbb{T} : s = H_l, \quad l \in \mathbb{N}_0, \quad s < H_n\} \\ &= H_{n-1}. \end{aligned}$$

Consequently

$$\rho(t) = \begin{cases} H_0 & \text{if} \quad n = 0 \\ H_{n-1} & \text{if} \quad n \geq 1. \end{cases}$$

Example

Let $\mathbb{T} = P_{1,3}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Assume that $t = 0$. Then

$$\rho(0) = 0.$$

2. Assume that $t > 0$. Then we have the following subcases.
   1. $t \in (0,1] \bigcup \bigcup_{k=1}^{\infty} (4k, 4k+1]$. Then

$$\rho(t) = t.$$

   2. $t = 4k$, $k > 0$. Then $k = \frac{t}{4}$ and

$$\begin{aligned} \rho(t) &= 4(k-1)+1 \\ &= 4k-4+1 \\ &= 4k-3 \end{aligned}$$

Example

Consequently

$$\rho(t) = \begin{cases} t \quad \text{if} \quad t \in [0,1] \bigcup \bigcup_{k=1}^{\infty} (4k, 4k+1] \\ \\ t-3 \quad \text{if} \quad t \in \bigcup_{k=1}^{\infty} \{4k\}. \end{cases}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. Let also, C_2 be the set of all right-hand end points of the intervals that are removed from the interval $[0, 1]$, i.e.,

$$C_2 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\}, \quad 1 \leq k \leq m \right\}.$$

Take $t \in \mathbb{T}$ arbitrarily. Then we have the following cases.
Assume that $t \in C_2$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}$$

and

$$\rho(t) = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}$$

Example

If $t \in \mathbb{T} \backslash C_2$, then $\rho(t) = t$.
Consequently

$$\rho(t) = \begin{cases} t - \frac{1}{3^{m+1}} \quad \text{if} \quad t \in C_2, \quad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\ \\ t \quad \text{if} \quad t \in \mathbb{T} \backslash C_2. \end{cases}$$

Example

Let $\mathbb{T} = \left\{ t_n = \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N} \right\}$, where $\alpha_n > 0$, $n \in \mathbb{N}_0$. Take $t_n \in \mathbb{T}$ arbitrarily for some $n \in \mathbb{N}$. We have the following cases. Assume that $n = 1$. Then

$$\rho(t_1) = t_1.$$

Assume that $n > 1$. Then

$$\begin{aligned} \rho(t_n) &= \sup \left\{ t_m = \sum_{k=0}^{m-1}, \quad m \in \mathbb{N}, : t_m < t_n \right\} \\ &= \sum_{k=0}^{n-2} \alpha_k \\ &= \sum_{k=0}^{n-1} \alpha_k - \alpha_{n-1} = t_n - \alpha_{n-1}. \end{aligned}$$

Example

Consequently

$$\rho(t_n) = \begin{cases} t_1 & \text{if} \quad n = 1 \\ \\ t_n - \alpha_{n-1} & \text{if} \quad n \geq 2. \end{cases}$$

Example

Let $\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases. $t = t_n$ for some $n \in \mathbb{N}$. If $n = 1$, then $\rho(t_1) = t_1$. If $n \geq 2$, then $n = -\frac{1}{t_n}$ and

$$\rho(t_n) = -\frac{1}{n-1} = -\frac{1}{-\frac{1}{t_n} - 1} = \frac{t_n}{t_n + 1}.$$

Let $t \in \mathbb{N}_0$. If $t = 0$, then $\rho(0) = 0$. If $t \geq 1$, then $\rho(t) = t - 1$.

Example

Consequently

$$\rho(t) = \begin{cases} t \quad \text{if} \quad t \in \{-1, 0\} \\ \frac{t}{1+t} \quad \text{if} \quad t \in \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 2 \right\} \\ t - 1 \quad \text{if} \quad t \in \mathbb{N}. \end{cases}$$

Example

Let $\mathbb{T} = \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases. If $t = \left(\frac{1}{2}\right)^{2^n}$ for some $n \in \mathbb{N}_0$, then

$$\begin{aligned}\rho(t) &= \left(\frac{1}{2}\right)^{2^{n+1}} \\ &= \left(\frac{1}{2}\right)^{2^n \cdot 2} = \left(\left(\frac{1}{2}\right)^{2^n}\right)^2 = t^2.\end{aligned}$$

Let $t = 0$. Then $\rho(0) = 0$. Let $t = 1$. Then $\rho(1) = \frac{1}{2}$.

Example

Consequently

$$\rho(t) = \begin{cases} t^2 & \text{if} \quad t \in \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \\ 0 & \text{if} \quad t = 0 \\ \frac{1}{2} & \text{if} \quad t = 1. \end{cases}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

We will find $\rho(t)$, $t \in \mathbb{T}$. Taket $t \in \mathbb{T}$ arbitrarily. We have the following cases. If $t = \frac{1}{2^n}$ for some $n \in \mathbb{N}$, then

$$\rho(t) = \frac{1}{2^{n+1}} = \frac{1}{2} \cdot \frac{1}{2^n} = \frac{t}{2}.$$

Let $t \in (1 - U)$. If $t = \frac{1}{2}$, then

$$\rho\left(\frac{1}{2}\right) = \frac{1}{4}.$$

If $t = 1 - \frac{1}{2^n}, \quad n \geq 2$, then $\frac{1}{2^n} = 1 - t, \quad n \geq 2$, and

$$\rho(t) = 1 - \frac{1}{2^{n-1}} = 1 - \frac{2}{2^n} = 1 - 2(1 - t) = 2t - 1.$$

Example

Let $t \in (1+U)$ and $t = 1 + \frac{1}{2^n}$ for some $n \in \mathbb{N}$. Then $\frac{1}{2^n} = t - 1$ and

$$\rho(t) = 1 + \frac{1}{2^{n+1}} = 1 + \frac{1}{2} \cdot \frac{1}{2^n} = 1 + \frac{t-1}{2} = \frac{t+1}{2}.$$

Let $t \in (2-U)$. If $t = \frac{3}{2}$, then $\rho\left(\frac{3}{2}\right) = \frac{5}{4}$. If $t = 2 - \frac{1}{2^n}$ for some $n \in \mathbb{N}$, $n \geq 2$, then $\frac{1}{2^n} = 2 - t$ and

$$\rho(t) = 2 - \frac{1}{2^{n-1}} = 2 - \frac{2}{2^n} = 2 - 2(2-t) = 2t - 2.$$

Example

Let $t \in (2+U)$ and $t = 2 + \frac{1}{2^n}$, $n \in \mathbb{N}$. Then $\frac{1}{2^n} = t - 2$ and

$$\rho(t) = 2 + \frac{1}{2^{n+1}} = 2 + \frac{1}{2} \cdot \frac{1}{2^n} = 2 + \frac{t-2}{2} = \frac{t+2}{2}.$$

If $t = 0$, then $\rho(0) = 0$. If $t = 1$, then $\rho(1) = 1$. If $t = 2$, then $\rho(2) = 2$.

Example

Consequently

$$\rho(t)=\begin{cases}\frac{1}{2}t \quad \text{if} \quad t\in U\\ \frac{1}{4} \quad \text{if} \quad t=\frac{1}{2}\\ 2t-1 \quad \text{if} \quad t\in(1-U)\setminus\left\{\frac{1}{2}\right\}\\ \frac{t+1}{2} \quad \text{if} \quad t\in(1+U)\\ \frac{5}{4} \quad \text{if} \quad t=\frac{3}{2}\end{cases}$$

Example

$$\rho(t) = \begin{cases} 2(t-1) & \text{if} \quad t \in (2-U) \setminus \left\{\frac{3}{2}\right\} \\ \frac{t+2}{2} & \text{if} \quad t \in (2+U) \\ 0 & \text{if} \quad t = 0 \\ 1 & \text{if} \quad t = 1 \\ 2 & \text{if} \quad t = 2. \end{cases}$$

Definition

The backward graininess function $\nu : \mathbb{T} \to \mathbb{R}$ is defined as follows

$$\nu(t) = t - \rho(t), \quad t \in \mathbb{T}.$$

Note that $\nu(t) \geq 0$, $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then, using Example 14, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= t - (t - h) \\ &= h, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then, using Example 15, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t = 1 \\ \frac{2t}{3} & \text{if} \quad t > 1. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, $k \in \mathbb{N}$. Then, using Example 15, we find

$$\begin{aligned}
\nu(t) &= t - \rho(t) \\
&= \begin{cases} 0 & \text{if} \quad t = 0 \\ t - (\sqrt[k]{t} - 1)^k & \text{if} \quad t > 0. \end{cases}
\end{aligned}$$

Observe that

$$\begin{aligned}
t - (\sqrt[k]{t} - 1)^k &= t - \left(t - \binom{k}{1} \sqrt[k]{t^{k-1}} + \binom{k}{2} \sqrt[k]{t^{k-2}} - \cdots - 1 \right) \\
&= \binom{k}{1} \sqrt[k]{t^{k-1}} - \binom{k}{2} \sqrt[k]{t^{k-2}} + \cdots + 1.
\end{aligned}$$

Example

Consequently

$$\nu(t) = \begin{cases} 0 \quad \text{if} \quad t = 0 \\ \\ \binom{k}{1}\sqrt[k]{t^{k-1}} - \binom{k}{2}\sqrt[k]{t^{k-2}} + \cdots + 1 \quad \text{if} \quad t > 0. \end{cases}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Using Example 19, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 \quad \text{if} \quad n = 0 \\ H_n - H_{n-1} \quad \text{if} \quad n \geq 1. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Using Example 20, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t \in [0,1] \bigcup \bigcup_{k=1}^{\infty} (4k, 4k+1] \\ 3 & \text{if} \quad t \in \bigcup_{k=1}^{\infty} \{4k\}. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. Using Example 22, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} \frac{1}{3^{m+1}} \quad \text{if} \quad t \in C_2, \quad t = \sum_{k=1}^{k} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\ 0 \quad \text{if} \quad t \in \mathbb{T} \backslash C_2. \end{cases} \end{aligned}$$

Example

Let

$$\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N} \right\}.$$

Using Example 24, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad n = 1 \\ \alpha_{n-1} & \text{if} \quad n \geq 2. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$. Using Example 26, we find

$$
\begin{aligned}
\nu(t) &= t - \rho(t) \\
&= \begin{cases} 0 & \text{if} \quad t \in \{-1, 0\} \\ \frac{t^2}{t+1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 2\right\} \\ 1 & \text{if} \quad t \in \mathbb{N}. \end{cases}
\end{aligned}
$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$. Using Example 28, we find

$$\begin{aligned} \nu(t) &= t = \rho(t) \\ &= \begin{cases} t - gt^2 & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \\ 0 & \text{if} \quad t \in \{0, 1\}. \end{cases} \end{aligned}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

Then, using Example 30, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} \frac{t}{2} & \text{if} \quad t \in U \\ 0 & \text{if} \quad t = 0 \\ 1 - t & \text{if} \quad t \in (1 - U) \backslash \left\{ \frac{1}{2} \right\} \\ \frac{t-1}{2} & \text{if} \quad t \in (1 + U) \end{cases} \end{aligned}$$

Example

$$\nu(t) = \begin{cases} \frac{1}{4} & \text{if} \quad t = \frac{3}{2} \\ 2 - t & \text{if} \quad t \in (2 - U) \setminus \left\{\frac{3}{2}\right\} \\ \frac{t-2}{2} & \text{if} \quad t \in (2 + U) \\ 0 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t = 2. \end{cases}$$

Example

We will show that σ is in general no continuous. Consider

$$\mathbb{T} = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0.$$

We have

$$\sigma(0) = 1, \quad \sigma\left(-\frac{1}{n}\right) = -\frac{1}{n+1}, \quad n \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \lim_{n\to\infty} \sigma\left(-\frac{1}{n}\right) &= -\lim_{n\to\infty} \frac{1}{n+1} = 0 \\ &\neq 1 \\ &= \sigma(0) = \sigma\left(\lim_{n\to\infty}\left(-\frac{1}{n}\right)\right). \end{aligned}$$

Example

We will show that ρ is in general not continuous. Consider

$$\mathbb{T} = [-2, -1] \cup \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}} \cup \mathbb{N}_0.$$

Then

$$\rho(0) = -1,$$

$$\rho\left(\frac{1}{n}\right) = \frac{1}{n+1}, \quad n \in \mathbb{N}.$$

Hence,

$$\lim_{n\to\infty} \rho\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\neq -1 = \rho(0) = \rho\left(\lim_{n\to\infty} \frac{1}{n}\right).$$

Definition

The backward graininess function $\nu : \mathbb{T} \to \mathbb{R}$ is defined as follows

$$\nu(t) = t - \rho(t), \quad t \in \mathbb{T}.$$

Note that $\nu(t) \geq 0$, $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then, using Example 14, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= t - (t - h) \\ &= h, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then, using Example 15, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t = 1 \\ \frac{2t}{3} & \text{if} \quad t > 1. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, $k \in \mathbb{N}$. Then, using Example 15, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 \quad \text{if} \quad t = 0 \\ t - (\sqrt[k]{t} - 1)^k \quad \text{if} \quad t > 0. \end{cases} \end{aligned}$$

Observe that

$$\begin{aligned} t - (\sqrt[k]{t} - 1)^k &= t - \left(t - \binom{k}{1} \sqrt[k]{t^{k-1}} + \binom{k}{2} \sqrt[k]{t^{k-2}} - \cdots - 1 \right) \\ &= \binom{k}{1} \sqrt[k]{t^{k-1}} - \binom{k}{2} \sqrt[k]{t^{k-2}} + \cdots + 1. \end{aligned}$$

Example

Consequently

$$\nu(t) = \begin{cases} 0 \quad \text{if} \quad t = 0 \\ \\ \binom{k}{1}\sqrt[k]{t^{k-1}} - \binom{k}{2}\sqrt[k]{t^{k-2}} + \cdots + 1 \quad \text{if} \quad t > 0. \end{cases}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Using Example 19, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad n = 0 \\ H_n - H_{n-1} & \text{if} \quad n \geq 1. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Using Example 20, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t \in [0,1] \bigcup \bigcup_{k=1}^{\infty} (4k, 4k+1] \\ 3 & \text{if} \quad t \in \bigcup_{k=1}^{\infty} \{4k\}. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. Using Example 22, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} \frac{1}{3^{m+1}} \quad \text{if} \quad t \in C_2, \quad t = \sum_{k=1}^{k} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\ \\ 0 \quad \text{if} \quad t \in \mathbb{T} \backslash C_2. \end{cases} \end{aligned}$$

Example

Let

$$\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N} \right\}.$$

Using Example 24, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad n = 1 \\ \alpha_{n-1} & \text{if} \quad n \geq 2. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$. Using Example 26, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} 0 & \text{if} \quad t \in \{-1, 0\} \\ \frac{t^2}{t+1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 2\right\} \\ 1 & \text{if} \quad t \in \mathbb{N}. \end{cases} \end{aligned}$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$. Using Example 28, we find

$$
\begin{aligned}
\nu(t) &= t - \rho(t) \\
&= \begin{cases} t - t^2 & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \\ 0 & \text{if} \quad t \in \{0, 1\}. \end{cases}
\end{aligned}
$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

Then, using Example 30, we find

$$\begin{aligned} \nu(t) &= t - \rho(t) \\ &= \begin{cases} \frac{t}{2} & \text{if} \quad t \in U \\ 0 & \text{if} \quad t = 0 \\ 1 - t & \text{if} \quad t \in (1 - U) \setminus \left\{ \frac{1}{2} \right\} \\ \frac{t-1}{2} & \text{if} \quad t \in (1 + U) \end{cases} \end{aligned}$$

Example

$$\nu(t) = \begin{cases} \frac{1}{4} \quad \text{if} \quad t = \frac{3}{2} \\ 2 - t \quad \text{if} \quad t \in (2 - U) \setminus \left\{ \frac{3}{2} \right\} \\ \frac{t-2}{2} \quad \text{if} \quad t \in (2 + U) \\ 0 \quad \text{if} \quad t = 0 \\ 0 \quad \text{if} \quad t = 1 \\ 0 \quad \text{if} \quad t = 2. \end{cases}$$

For any element of any time scale the following classification holds.

Definition

For $t \in \mathbb{T}$ we have the following cases.

1. If $\sigma(t) > t$, then we say that t is right-scattered.
2. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then we say that t is right-dense.
3. If $\rho(t) < t$, then we say that t is left-scattered.
4. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that t is left-dense.
5. If t is left-scattered and right-scattered at the same time, then we say that t is isolated.
6. If t is left-dense and right-dense at the same time, then we say that t is dense.

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. By Example 98, we have that

$$\sigma(t) \quad = \quad t + h > t, \quad t \in \mathbb{T}.$$

Thus, any point of $\mathbb{T}$ is right-scattered. Now, using Example 14, we get

$$\rho(t) \quad = \quad t - h < t, \quad t \in \mathbb{T}.$$

Therefore any point of $\mathbb{T}$ is left-scattered. Hence, we conclude that any point of $\mathbb{T}$ is isolated.

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Assume that $t = 1$. By Example 99, we have

$$\sigma(1) = 3 > 1,$$

i.e., $t = 1$ is right-scattered. By Example 15, we have

$$\rho(1) = 1.$$

Since $1 = \inf \mathbb{T}$, we conclude that $t = 1$ is not left-dense.

2. Let $t > 1$. By Example 99, we have $\sigma(t) = 3t > t$. Thus, t is right-scattered. By Example 15, we get

$$\rho(t) = \frac{t}{3} < t,$$

i.e., t is left scattered. Hence, we conclude that t is isolated.

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, $k \in \mathbb{N}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Let $t = 0$. By Example 100, we have

$$\sigma(0) \;=\; 1 > 0,$$

   i.e., $t = 0$ is right-scattered. By Example 17, we obtain $\rho(0) = 0$. Since $0 = \inf \mathbb{T}$, we conclude that $t = 0$ is not left-dense.

2. Let $t > 0$. By Example 100, we get

$$\sigma(t) \;=\; (\sqrt[k]{t} + 1)^k > t,$$

   i.e., t is right-scattered. B y Example 17, we find

$$\rho(t) \;=\; (\sqrt[k]{t} - 1)^k < t,$$

   i.e., t is left-scattered. Therefore t is isolated.

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$. We have the following cases.

1. Let $n = 0$. Then, by Example 101, we get $\sigma(H_0) = H_1$, i.e., H_0 is right-scattered. By Example 19, we have $\rho(H_0) = H_0$. Since $H_0 = \inf \mathbb{T}$, we conclude that H_0 is not left-dense.
2. Let $n > 0$. By Example 101, we get

$$\sigma(H_n) \;=\; H_{n+1} > H_n.$$

Then H_n is right-scattered. By Example 19, we get

$$\rho(H_n) \;=\; H_{n-1} < H_n,$$

i.e., H_n is left-scattered. Then H_n is isolated.

Example

Let $\mathbb{T} = P_{1,3}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases.

1. Let $t \in \bigcup_{k=0}^{\infty} (4k, 4k+1)$. By Example 102, we get $\sigma(t) = t$, i.e., t is right-dense. By Example 20, we find $\rho(t) = t$, i.e., t is left-dense. Thus, t is dense.
2. Let $t = 0$. By Example 102, we obtain $\sigma(0) = 0$, i.e, $t = 0$ is right-dense. By Example 20, we find $\rho(0) = 0$. Since $0 = \inf \mathbb{T}$, we conclude that 0 is not left-dense.
3. Let $t \in \bigcup_{k=1}^{\infty} \{4k\}$. By Example 102, we get $\sigma(t) = t$, i.e., t is right-dense. By Example 20, we find $\rho(t) = t - 3 < t$, i.e., t is left-scattered.
4. Let $t \in \bigcup_{k=0}^{\infty} \{4k+1\}$. By Example 102, we find $\sigma(t) = t + 3 > t$, i.e., t is right-scattered. By Example 20, we find $\rho(t) = t$, i.e., t is left-dense.

Example

Let $\mathbb{T} = C$, where C is the Cantor set. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases. Let $t \in C_1$. By Example 106, we have $\sigma(t) = t + \frac{1}{3^{m+1}} > t$, i.e., t is right-scattered. By Example 22, we find $\rho(t) = t$. If $t \neq 0$, then it is left-dense. If $t = 0$, then it is not left-dense because $0 = \inf \mathbb{T}$. Let $t \in C_2$. By Example 106, we get $\sigma(t) = t$, i.e., t is right-dense. By Example 22, we find

$$\rho(t) \;\; = \;\; t - \frac{1}{3^{m+1}} < t,$$

i.e., t is left-scattered. Let $t \in T \backslash C_1$. We have the following subcases. Let $t \in C_2$. By Example 106, we find $\sigma(t) = t$, i.e., t is right-dense. By Example 22, we obtain $\rho(t) = t - \frac{1}{3^{m+1}} < t$, i.e., t is left-scattered. Let $t \in T \backslash C_2$. By Example 106, we arrive at $\sigma(t) = t$, i.e., t is right-dense. By Example 22, we have $\rho(t) = t$. If $t \neq 0$, then it is left-dense. If $t = 0$, then it is not left-dense.

Example

Let $t \in \mathbb{T} \backslash C_2$. We have the following subcases. Let $t \in C_1$. By Example 106, we find

$$\sigma(t) = t + \frac{1}{3^{m+1}} > t,$$

i.e., t is right-scattered. By Example 22, we find $\rho(t) = t$. If $t \neq 0$, then it is left-dense. If $t = 0$, then is left-dense. Let $t \in T \backslash C_1$. By Example 106, we have $\sigma(t) = t$, i.e., t is right-dense. By Example 22, we have $\rho(t) = t$. If $t \neq 0$, then it is left-dense and hence dense. If $t = 0$, it is not left-dense.

Example

Let $\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : \alpha_k > 0, \quad k \in \mathbb{N}_0, \quad n \in \mathbb{N} \right\}$. Take $t \in \mathbb{T}$ arbitrarily.

Then there is a $n \in \mathbb{N}$ such that $t = \sum_{k=0}^{n-1} \alpha_k$. We have the following cases.

Let $n = 1$. By Example 109, we have

$$\sigma(t) = \sum_{k=0}^{n} \alpha_k > t,$$

i.e., t is right-scattered. By Example 24, we arrive at $\rho(t) = t$. Since $t = \inf \mathbb{T}$, we conclude that t is not left-dense.

Example

Let $n > 1$. By Example 109, we find

$$\sigma(t) \;=\; \sum_{k=0}^{n} \alpha_k > t,$$

i.e., t is right-scattered. By Example 24, we find

$$\rho(t) \;=\; t - \alpha_{n-1} < t,$$

i.e., t is left-scattered. Therefore t is isolated.

Example

Let $\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases. Let $t = -1$. By Example 110, we have $\sigma(-1) = -\frac{1}{2} > -1$, i.e., $t = -1$ is right-scattered. By Example 26, we find $\rho(-1) = -1$. Since $-1 = \inf \mathbb{T}$, we conclude that $t = -1$ is not left-dense. Let $t = 0$. By Example 110, we have $\sigma(0) = 1 > 0$, i.e., $t = 0$ is right-scattered. By Example 26, we obtain $\rho(0) = 0$. Since $0 > \inf \mathbb{T}$, we conclude that $t = 0$ is left-dense. Let $t \in \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \backslash \{-1\}$. By Example 110, we have $\sigma(t) = -\frac{t}{t-1} > t$, i.e., t is right-scattered. By Example 26, we get $\rho(t) = \frac{t}{t+1} < t$, i.e., t is left-scattered. Thus, t is isolated. Let $t \in \mathbb{N}$. By Example 110, we have $\sigma(t) = t + 1 > t$, i.e., t is right-scattered. By Example 26, we obtain $\rho(t) = t - 1 < t$, i.e., t is left-scattered. Thus, t is isolated.

Example

Let $\mathbb{T} = \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \bigcup \{0, 1\}$. Take $t \in \mathbb{T}$ arbitrarily. We have the following cases. Let $t = 0$. By Example 112, we have $\sigma(0) = 0$, i.e., $t = 0$ is right-dense. By Example 28, we get $\rho(0) = 0$. Since $0 = \inf \mathbb{T}$, we conclude that $t = 0$ is not left-dense. Let $t = \frac{1}{2}$. By Example 112, we have $\sigma\left(\frac{1}{2}\right) = 1$, i.e., $t = \frac{1}{2}$ is right-scattered. By Example 28, we get $\rho\left(\frac{1}{2}\right) = \frac{1}{4} < \frac{1}{2}$, i.e., $t = \frac{1}{2}$ is left-scattered. Thus, $t = \frac{1}{2}$ is isolated. Let $t = 1$. By Example 112, we have $\sigma(1) = 1$. Since $1 = \sup \mathbb{T}$, we conclude that $t = 1$ is not right-dense. By Example 28, we obtain $\rho(1) = \frac{1}{2} < 1$, i.e., $t = 1$ is left-scattered. Let $t \in \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\}$. Then there is a $n \in \mathbb{N}$ such that $t = \left(\frac{1}{2}\right)^{2^n}$. By Example 112, we have $\sigma(t) = \sqrt{t} > t$, i.e., t is right-scattered. By Example 28, we obtain $\rho(t) = t^2 < t$, i.e., t is left-scattered. Thus, t is isolated.

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and $\mathbb{T} = \{0\} \cup U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{1, 2\}$. Take $t \in \mathbb{T}$ arbitrarily. Then, we have the following cases. Let $t = 0$. By Example 114, we have $\sigma(0) = 0$, i.e., $t = 0$ is right-dense, By Example 30, we get $\rho(0) = 0$. Since $0 = \inf \mathbb{T}$, we conclude that $t = 0$ is not left-dense. Let $t = \frac{1}{2}$. By Example 114, we have $\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$, i.e., $t = \frac{1}{2}$ is right-scattered. By Example 30, we get $\rho\left(\frac{1}{2}\right) = \frac{1}{4}$, i.e., $t = \frac{1}{2}$ is right-scattered. Thus, $t = \frac{1}{2}$ is isolated. Let $t = 1$. By Example 114, we have $\sigma(1) = 1$, i.e., $t = 1$ is right-dense. By Example 30, we find $\rho(1) = 1$, i.e., $t = 1$ is left-dense. Thus, $t = 1$ is dense. Let $t = \frac{3}{2}$. By Example 114, we have $\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$, i.e., $t = \frac{3}{2}$ is right-scattered. By Example 30, we find $\rho\left(\frac{3}{2}\right) = \frac{5}{4}$, i.e., $t = \frac{3}{2}$ is left-scattered. Thus, $t = \frac{3}{2}$ is isolated. Let $t = 2$. By Example 114, we have $\sigma(2) = 2$, i.e., $t = 2$ is right-dense. By Example 30, we get $\rho(2) = 2$, i.e., $t = 2$ is left-dense. Thus, $t = 2$ is dense. Let $t = \frac{5}{2}$. Then, by Example 114, we get $\sigma\left(\frac{5}{2}\right) = \frac{5}{2}$. Since $\frac{5}{2} = \sup \mathbb{T}$, we conclude that $t = \frac{5}{2}$ is not right-dense. By Example 30, we find $\rho\left(\frac{5}{2}\right) = \frac{9}{8}$. Thus, $t = \frac{5}{2}$ is left-scattered.

Example

Let $t \in U \backslash \left\{\frac{1}{2}\right\}$. By Example 114, we have $\sigma(t) = 2t > t$, i.e., t is right-scattered. By Example 30, we get $\rho(t) = \frac{t}{2} < t$, i.e., t is left-scattered. Thus, t is isolated. Let $t \in (1 - U) \backslash \left\{\frac{1}{2}\right\}$. By Example 114, we have $\sigma(t) = \frac{1+t}{2} > t$, i.e., t is right-scattered. By Example 30, we find $\rho(t) = 2t - 1 < t$, i.e., t is left-scattered. Thus, t is isolated. Let $t \in (1 + U) \backslash \left\{\frac{3}{2}\right\}$. By Example 114, we have $\sigma(t) = 2t - 1 > t$, i.e., t is right-scattered. By Example 30, we find $\rho(t) = \frac{t+1}{2} < t$, i.e., t is left-scattered. Thus, t is isolated. Let $t \in (2 - U) \backslash \left\{\frac{3}{2}\right\}$. By Example 114, we have $\sigma(t) = \frac{2+t}{2} > t$, i.e., t is right-scattered. By Example 30, we find $\rho(t) = 2t - 2 < t$, i.e., t is left-scattered. Thus, t is isolated. Let $t \in (2 + U) \backslash \left\{\frac{5}{2}\right\}$. By Example 114, we have $\sigma(t) = 2(t - 1) > t$, i.e., t is right-scattered. By Example 30, we find $\rho(t) = \frac{t+2}{2} < t$, i.e., t is left-scattered. Thus, t is isolated.

Definition

Let $a, b \in \mathbb{T}$, $a \leq b$. Define closed, half open and open time scales intervals as follows

$$\begin{aligned}
[a, b]_{\mathbb{T}} &= \{x \in \mathbb{T} : a \leq x \leq b\}, \\
[a, b)_{\mathbb{T}} &= \{x \in \mathbb{T} : a \leq x < b\}, \\
(a, b]_{\mathbb{T}} &= \{x \in \mathbb{T} : a < x \leq b\}, \\
(a, b)_{\mathbb{T}} &= \{x \in \mathbb{T} : a < x < b\},
\end{aligned}$$

respectively.

Example

Let $\mathbb{T} = 3\mathbb{Z}$. Then

$$\begin{aligned} [-3, 12]_{\mathbb{T}} &= \{-3, 0, 3, 6, 9, 12\}, \\ [-3, 12)_{\mathbb{T}} &= \{-3, 0, 3, 6, 9\}, \\ (-3, 12]_{\mathbb{T}} &= \{0, 3, 6, 9, 12\}, \\ (-3, 12)_{\mathbb{T}} &= \{0, 3, 6, 9\}. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then

$$
\begin{aligned}
[3, 243]_{\mathbb{T}} &= \{3, 9, 27, 81, 243\}, \\
[3, 243)_{\mathbb{T}} &= \{3, 9, 27, 81\}, \\
(3, 243]_{\mathbb{T}} &= \{9, 27, 81, 243\}, \\
(3, 243)_{\mathbb{T}} &= \{9, 27, 81\}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \mathbb{N}_0^3$. Then

$$[1,27]_{\mathbb{T}} = \Big\{1, 2, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, (\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3, 27\Big\},$$

$$[1,27)_{\mathbb{T}} = \Big\{1, 2, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, (\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3\Big\},$$

$$(1,27]_{\mathbb{T}} = \Big\{2, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, (\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3, 27\Big\},$$

$$(1,27)_{\mathbb{T}} = \Big\{2, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, (\sqrt[3]{6}+1)^3(\sqrt[3]{7}+1)^3\Big\}.$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Then

$$\begin{aligned}
\left[1, \frac{147}{60}\right]_{\mathbb{T}} &= \left\{1, \frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}\right\}, \\
\left[1, \frac{147}{60}\right)_{\mathbb{T}} &= \left\{1, \frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}\right\}, \\
\left(1, \frac{147}{60}\right]_{\mathbb{T}} &= \left\{\frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}\right\}, \\
\left(1, \frac{147}{60}\right)_{\mathbb{T}} &= \left\{\frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}\right\}.
\end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$\begin{aligned}
[0,12]_{\mathbb{T}} &= [0,1] \cup [4,5] \cup [8,9] \cup \{12\}, \\
[0,12)_{\mathbb{T}} &= [0,1] \cup [4,5] \cup [8,9], \\
(0,12]_{\mathbb{T}} &= (0,1] \cup [4,5] \cup [8,9] \cup \{12\}, \\
(0,12)_{\mathbb{T}} &= (0,1] \cup [4,5] \cup [8,9].
\end{aligned}$$

Example

Let $\mathbb{T} = \left\{ \sum_{k=0}^{n} k : n \in \mathbb{N} \right\}$. Then

$$\begin{aligned} [0,28]_{\mathbb{T}} &= \{0,1,3,6,10,15,21,28\}, \\ [0,28)_{\mathbb{T}} &= \{0,1,3,6,10,15,21\}, \\ (0,28]_{\mathbb{T}} &= \{1,3,6,10,15,21,28\}, \\ (0,28)_{\mathbb{T}} &= \{1,3,6,10,15,21\}. \end{aligned}$$

Example

Let $\mathbb{T} = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$. Then

$$\begin{aligned}
\left[-\frac{1}{3}, 3\right]_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 3\right\} \cup \{0, 1, 2, 3\}, \\
\left[-\frac{1}{3}, 3\right)_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 3\right\} \cup \{0, 1, 2\}, \\
\left(-\frac{1}{3}, 3\right]_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 4\right\} \cup \{0, 1, 2, 3\}, \\
\left(-\frac{1}{3}, 3\right)_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 4\right\} \cup \{0, 1, 2\}.
\end{aligned}$$

Example

Let $\mathbb{T} = \left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0, 1\}$. Then

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

Then

$$\begin{aligned} \left[0, \frac{7}{8}\right]_{\mathbb{T}} &= \{0\} \cup U \cup \left\{ \frac{3}{4}, \frac{7}{8} \right\}, \\ \left[0, \frac{7}{8}\right)_{\mathbb{T}} &= \{0\} \cup U \cup \left\{ \frac{3}{4} \right\}, \\ \left(0, \frac{7}{8}\right]_{\mathbb{T}} &= U \cup \left\{ \frac{3}{4}, \frac{7}{8} \right\}, \\ \left(0, \frac{7}{8}\right)_{\mathbb{T}} &= U \cup \left\{ \frac{3}{4} \right\}. \end{aligned}$$

Definition

Define the sets

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}]_{\mathbb{T}} & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} \quad \text{if} \quad \sup \mathbb{T} = \infty \end{cases}$$

and

$$\mathbb{T}_{\kappa} = \begin{cases} \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} & \text{if} \quad \inf \mathbb{T} > -\infty \\ \mathbb{T} \quad \text{if} \quad \inf \mathbb{T} > -\infty. \end{cases}$$

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Then

$$\inf \mathbb{T} = -\infty,$$

$$\sup \mathbb{T} = \infty$$

and

$$\mathbb{T}^\kappa = \mathbb{T},$$

$$\mathbb{T}_\kappa = \mathbb{T}.$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then

$$\sup \mathbb{T} \;=\; \infty, \quad \inf \mathbb{T} = 1, \quad \sigma(\inf \mathbb{T}) = \sigma(1) = 3.$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\mathbb{T}_{\kappa} \;=\; \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [1, 3)_{\mathbb{T}} = \mathbb{T} \backslash \{1\} = 3^{\mathbb{N}}.$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, $k \in \mathbb{N}$. Then

$$\sup \mathbb{T} = \infty, \quad \inf \mathbb{T} = 0 = H_0,$$

$$\sigma(\inf \mathbb{T}) = \sigma(H_0) = H_1.$$

Hence,

$$\mathbb{T}^\kappa = \mathbb{T}$$

and

$$\mathbb{T}_\kappa = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_\mathbb{T} = \mathbb{T} \backslash [H_0, H_1)_\mathbb{T} = \mathbb{T} \backslash \{H_0\} = \mathbb{N}^k.$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$\sup \mathbb{T} \quad = \quad \infty, \quad \inf \mathbb{T} = 0, \quad \sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\mathbb{T}_{\kappa} \quad = \quad \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. Then

$$\begin{aligned} \sup \mathbb{T} &= 1, \rho(\sup \mathbb{T}) = \rho(1) = 1, \\ \inf \mathbb{T} &= 0, \sigma(\inf \mathbb{T}) = \sigma(0) = 0. \end{aligned}$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] = \mathbb{T} \backslash (1, 1] = \mathbb{T} \backslash \emptyset = \mathbb{T}$$

and

$$\mathbb{T}_{\kappa} = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $\mathbb{T} = \left\{ \sum_{k=1}^{n-1} \alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N}_0 \right\}$. Then

$$\sup \mathbb{T} = \infty, \inf \mathbb{T} = \alpha_0, \sigma(\inf \mathbb{T}) = \sigma(\alpha_0) = \alpha_1.$$

Hence,

$$\mathbb{T}^\kappa = \mathbb{T}$$

and

$$\begin{aligned} \mathbb{T}_\kappa &= \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [\alpha_0, \alpha_1)_{\mathbb{T}} \\ &= \mathbb{T} \backslash \{\alpha_0\} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad n \geq 2, \quad \alpha_k >, \quad k \in \mathbb{N}_0 \right\}. \end{aligned}$$

Example

Let $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$. Then

$$\sup \mathbb{T} = \infty, \inf \mathbb{T} = 0,$$

$$\sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\mathbb{T}^\kappa = \mathbb{T}$$

and

$$\mathbb{T}_\kappa = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_\mathbb{T} = \mathbb{T} \backslash [0, 0)_\mathbb{T} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$. Then

$$\sup \mathbb{T} = 1, \rho(\sup \mathbb{T}) = \rho(1) = \frac{1}{2}, \inf \mathbb{T} = 0, \sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\begin{aligned} \mathbb{T}^\kappa &= \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] \\ &= \mathbb{T} \backslash \left(\frac{1}{2}, 1\right] = \mathbb{T} \backslash \{1\} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0\}, \end{aligned}$$

and

$$\mathbb{T}_\kappa = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

Then

$$\sup \mathbb{T} = \frac{5}{2}, \rho(\sup \mathbb{T}) = \rho\left(\frac{5}{2}\right) = \frac{9}{4}, \inf \mathbb{T} = 0, \sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] = \mathbb{T} \backslash \left(\frac{9}{4}, \frac{5}{2}\right] = \mathbb{T} \backslash \left\{\frac{5}{2}\right\}$$

and

$$\mathbb{T}_{\kappa} = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Appendix

We start by defining the forward jump operator.

Definition

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ in the following manner

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$. Then, $t = \sigma(t)$ if t is a maximum of $\mathbb{T}$.

Note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{Z}$ such that $t = hn$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned} \sigma(t) &= \inf\{s = hp, p \in \mathbb{Z} : hp > hn\} \\ &= h(n+1) \\ &= hn + h \\ &= t + h. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = 3^n$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned} \sigma(t) &= \inf\left\{3^s, s \in \mathbb{N}_0 : 3^s > 3^n\right\} \\ &= 3^{n+1} \\ &= 3 \cdot 3^n \\ &= 3t. \end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$ is fixed. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = n^k$. Hence, $n = \sqrt[k]{t}$. Now, applying the definition for forward jump operators, we arrive at

$$\begin{aligned} \sigma(t) &= \inf\{s^k, s \in \mathbb{N}_0 : s^k > n^k\} \\ &= (n+1)^k \\ &= \left(\sqrt[k]{t} + 1\right)^k. \end{aligned}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$ arbitrarily. Then, applying the definition for forward jump operators, we find

$$\begin{aligned}
\sigma(H_n) &= \inf\{H_s, s \in \mathbb{N}_0 : H_s > H_n\} \\
&= \inf\left\{H_s, s \in \mathbb{N}_0 : \sum_{k=1}^{s} \frac{1}{k} > \sum_{k=1}^{n} \frac{1}{k}\right\} \\
&= \sum_{k=1}^{n+1} \frac{1}{k} \\
&= H_{n+1}.
\end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$\begin{aligned}\mathbb{T} &= \bigcup_{k=0}^{\infty} [4k, 4k+1] \\ &= [0,1] \cup [4,5] \cup [8,9] \cup [12,13] \cup \ldots .\end{aligned}$$

If $t \in [0,1)$, then, applying the definition for forward jump operators, we find

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t.\end{aligned}$$

Example

If $t = 1$, then

$$\begin{aligned} \sigma(1) &= \inf\{s \in \mathbb{T} : s > 1\} \\ &= 4. \end{aligned}$$

Let now, $k \in \mathbb{N}$ be arbitrarily chosen. If $t \in [4k, 4k+1)$, then we have

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t. \end{aligned}$$

Example

If $t = 4k + 1$, then

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > 4k + 1\} \\ &= 4(k+1) \\ &= 4k + 4 \\ &= 4k + 1 + 3 \\ &= t + 3. \end{aligned}$$

Example

Therefore

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty} [4k, 4k+1) \\ t+3 & \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{4k+1\}. \end{cases}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. We will find $\sigma(t)$ for $t \in \mathbb{T}$. For this aim, let C_1 denote the set of all left-hand end points of the open intervals that are removed. Then

$$C_1 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \leq k \leq m \right\}.$$

With C_2 we will denote the set of all right-hand end points of the open intervals that are removed. We have

$$C_2 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \leq k \leq m \right\}.$$

Take $t \in C$ arbitrarily. We have the following cases.
Let $t \in C_1$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}.$$

Example

Hence, we obtain

$$\begin{aligned}
\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\
&= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\
&= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} + \frac{1}{3^{m+1}} \\
&= t + \frac{1}{3^{m+1}}.
\end{aligned}$$

Let $t \in C_2$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}.$$

Hence,

Example

Let $t \in \mathbb{T} \backslash (C_1 \cup C_2)$. Then

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t. \end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}} & \text{if} \quad t \in C_1, t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ t \quad \text{if} \quad t \in \mathbb{T} \backslash C_1. \end{cases}$$

Example

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers with $\alpha_n > 0$, and

$$t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbb{N},$$

and

$$\mathbb{T} = \{t_n : n \in \mathbb{N}\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned} \sigma(t_n) &= \inf\left\{x \in \mathbb{T} : s = \sum_{k=0}^{n-1} \alpha_k, \quad s > t_n\right\} \\ &= \sum_{k=0}^{n} \alpha_k = \sum_{k=0}^{n-1} + \alpha_n = t_n + \alpha_n. \end{aligned}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$n = -\frac{1}{t_n}$$

and

$$\begin{aligned} \sigma(t_n) &= \inf \left\{ s \in \mathbb{T} : s = -\frac{1}{m}, m \in \mathbb{N}, s > t_0 \right\} \\ &= -\frac{1}{n+1} = -\frac{1}{-\frac{1}{t_n} + 1} = -\frac{t_n}{t_n - 1}. \end{aligned}$$

Example

Next, if $t \in \mathbb{N}_0$, then

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

$$= t + 1.$$

Consequently

$$\sigma(t) = \begin{cases} -\frac{t}{t-1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\}, \quad t = t_n \\ t + 1 & \text{if} \quad t \in \mathbb{N}_0. \end{cases}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned}
\sigma(t_n) &= \inf\{s \in \mathbb{T} : s > t_n\} \\
&= \left(\frac{1}{2}\right)^{2^{n-1}} = \left(\frac{1}{2}\right)^{2^n \cdot \frac{1}{2}} \\
&= \left(\left(\frac{1}{2}\right)^{2^n}\right)^{\frac{1}{2}} = \sqrt{t_n}.
\end{aligned}$$

Example

Next,

$$t_0 = \frac{1}{2} \quad \text{and} \quad \sigma(t_0) = 1$$

and

$$\sigma(0) = 0, \quad \sigma(1) = 1.$$

Consequently

$$\sigma(t) = \begin{cases} \sqrt{t} \quad \text{if} \quad t \in \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ 1 \quad \text{if} \quad t = \frac{1}{2} \\ 0 \quad \text{if} \quad t = 0 \\ 1 \quad \text{if} \quad t = 1. \end{cases}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{0, 1, 2\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. We have the following cases.
Let $t = 0$. Then

$$\sigma(0) = 0.$$

Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}.$$

Let $t = 1$. Then

$$\sigma(1) = 1.$$

Example

Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}.$$

Let $t = 2$. Then

$$\sigma(2) = 2.$$

Let $t = \frac{5}{2}$. Then

$$\sigma\left(\frac{5}{2}\right) = \frac{5}{2}.$$

Example

Let $t \in U \setminus \left\{ \frac{1}{2} \right\}$. Then

$$t = \frac{1}{2^n}$$

and

$$\sigma(t) = \frac{1}{2^{n-1}} = \frac{2}{2^n} = 2t.$$

Let $t \in (1-U) \setminus \left\{ \frac{1}{2} \right\}$. Then $t = 1 - \frac{1}{2^n}$ and $\frac{1}{2^n} = 1 - t$. Hence,

$$\sigma(t) = 1 - \frac{1}{2^{n+1}} = 1 - \frac{1}{2} \cdot \frac{1}{2^n} = 1 - \frac{1-t}{2} = \frac{1+t}{2}.$$

Example

Let $t \in (1+U)\setminus\left\{\frac{3}{2}\right\}$. Then

$$t = 1 + \frac{1}{2^n}.$$

Hence,

$$\frac{1}{2^n} = t - 1$$

and

$$\sigma(t) \quad = \quad 1 + \frac{1}{2^{n-1}} = 1 + \frac{2}{2^n} = 1 + 2(t-1) = 2t - 1.$$

Example

Let $t \in (2-U)\setminus\left\{\frac{3}{2}\right\}$. Then

$$t = 2 - \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = 2 - t.$$

Hence, $\sigma(t) = 2 - \frac{1}{2^{n+1}} = 2 - \frac{1}{2}\cdot\frac{1}{2^n} = 2 - \frac{2-t}{2} = \frac{t+2}{2}$.

Example

Let $t \in (2+U)\setminus\left\{\frac{5}{2}\right\}$. Then

$$t = 2 + \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = t - 2.$$

Hence,

$$\sigma(t) \;=\; 2 + \frac{1}{2^{n-1}} = 2 + \frac{2}{2^n} = 2 + 2(t-2) = 2(t-1).$$

Example

Consequently

$$\sigma(t)=\begin{cases} 0 & \text{if} \quad t=0 \\ \frac{3}{4} & \text{if} \quad t=\frac{1}{2} \\ 1 & \text{if} \quad t=1 \\ \frac{7}{4} & \text{if} \quad t=\frac{3}{2} \\ 2 & \text{if} \quad t=2 \\ \frac{5}{2} & \text{if} \quad t=\frac{5}{2} \\ 2t & \text{if} \quad t\in U\backslash\left\{\frac{1}{2}\right\} \\ \frac{1+t}{2} & \text{if} \quad t\in(1-U)\backslash\left\{\frac{1}{2}\right\} \end{cases}$$

Example

$$\sigma(t)=\begin{cases} 2t-1 \quad \text{if} \quad t\in(1+U)\setminus\left\{\frac{3}{2}\right\} \\ \frac{t+2}{2} \quad \text{if} \quad t\in(2-U)\setminus\left\{\frac{3}{2}\right\} \\ 2(t-1) \quad \text{if} \quad t\in(2+U)\setminus\left\{\frac{5}{2}\right\}. \end{cases}$$

Exercise

Find $\sigma(t)$, $t \in \mathbb{T}$, where

1. $\mathbb{T} = h\mathbb{Z} + k$, $h > 0$, $k \in \mathbb{R}$.
2. $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}$.
3. $\mathbb{T} = P_{3,7} \cup [4, 6]$.
4. $\mathbb{T} = 11^{\mathbb{N}_0} \cup \{0\}$.
5. $\mathbb{T} = [1, 2] \cup [3, 4] \cup [7, 8] \cup 9^{\mathbb{N}}$.