Time Scales Analysis
Lecture 3
Topology in Time Scales, Functions and Jump Operators, Induction Principle

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Definition

Let $a, b \in \mathbb{T}$, $a \leq b$. Define closed, half open and open time scales intervals as follows

$$
\begin{aligned}
[a, b]_{\mathbb{T}} &= \{x \in \mathbb{T} : a \leq x \leq b\}, \\
[a, b)_{\mathbb{T}} &= \{x \in \mathbb{T} : a \leq x < b\}, \\
(a, b]_{\mathbb{T}} &= \{x \in \mathbb{T} : a < x \leq b\}, \\
(a, b)_{\mathbb{T}} &= \{x \in \mathbb{T} : a < x < b\},
\end{aligned}
$$

respectively.

Example

Let $\mathbb{T} = 3\mathbb{Z}$. Then

$$\begin{aligned} [-3, 12]_{\mathbb{T}} &= \{-3, 0, 3, 6, 9, 12\}, \\ [-3, 12)_{\mathbb{T}} &= \{-3, 0, 3, 6, 9\}, \\ (-3, 12]_{\mathbb{T}} &= \{0, 3, 6, 9, 12\}, \\ (-3, 12)_{\mathbb{T}} &= \{0, 3, 6, 9\}. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then

$$
\begin{aligned}
[3, 243]_{\mathbb{T}} &= \{3, 9, 27, 81, 243\}, \\
[3, 243)_{\mathbb{T}} &= \{3, 9, 27, 81\}, \\
(3, 243]_{\mathbb{T}} &= \{9, 27, 81, 243\}, \\
(3, 243)_{\mathbb{T}} &= \{9, 27, 81\}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \mathbb{N}_0^3$. Then

$$\begin{aligned}
[1,27]_{\mathbb{T}} &= \Big\{1, 8, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, \\
&\quad (\sqrt[3]{6}+1)^3, (\sqrt[3]{7}+1)^3, 27\Big\}, \\
[1,27)_{\mathbb{T}} &= \Big\{1, 8, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, \\
&\quad (\sqrt[3]{6}+1)^3, (\sqrt[3]{7}+1)^3\Big\}, \\
(1,27]_{\mathbb{T}} &= \Big\{8, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, \\
&\quad (\sqrt[3]{6}+1)^3, (\sqrt[3]{7}+1)^3, 27\Big\}, \\
(1,27)_{\mathbb{T}} &= \Big\{8, 9, (\sqrt[3]{2}+1)^3, (\sqrt[3]{3}+1)^3, (\sqrt[3]{4}+1)^3, (\sqrt[3]{5}+1)^3, \\
&\quad (\sqrt[3]{6}+1)^3, (\sqrt[3]{7}+1)^3\Big\}.
\end{aligned}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Then

$$\begin{aligned}
\left[1, \frac{147}{60}\right]_{\mathbb{T}} &= \left\{1, \frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}\right\}, \\
\left[1, \frac{147}{60}\right)_{\mathbb{T}} &= \left\{1, \frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}\right\}, \\
\left(1, \frac{147}{60}\right]_{\mathbb{T}} &= \left\{\frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}\right\}, \\
\left(1, \frac{147}{60}\right)_{\mathbb{T}} &= \left\{\frac{3}{2}, \frac{1}{6}, \frac{25}{12}, \frac{137}{60}\right\}.
\end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$
\begin{aligned}
[0,12]_{\mathbb{T}} &= [0,1] \cup [4,5] \cup [8,9] \cup \{12\}, \\
[0,12)_{\mathbb{T}} &= [0,1] \cup [4,5] \cup [8,9]\,, \\
(0,12]_{\mathbb{T}} &= (0,1] \cup [4,5] \cup [8,9] \cup \{12\}, \\
(0,12)_{\mathbb{T}} &= (0,1] \cup [4,5] \cup [8,9]\,.
\end{aligned}
$$

Example

Let $\mathbb{T} = \left\{ \sum_{k=0}^{n} k : n \in \mathbb{N} \right\}$. Then

$$
\begin{aligned}
[0, 28]_{\mathbb{T}} &= \{0, 1, 3, 6, 10, 15, 21, 28\}, \\
[0, 28)_{\mathbb{T}} &= \{0, 1, 3, 6, 10, 15, 21\}, \\
(0, 28]_{\mathbb{T}} &= \{1, 3, 6, 10, 15, 21, 28\}, \\
(0, 28)_{\mathbb{T}} &= \{1, 3, 6, 10, 15, 21\}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$. Then

$$\begin{aligned}
\left[-\frac{1}{3}, 3\right]_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 3\right\} \cup \{0, 1, 2, 3\}, \\
\left[-\frac{1}{3}, 3\right)_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 3\right\} \cup \{0, 1, 2\}, \\
\left(-\frac{1}{3}, 3\right]_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 4\right\} \cup \{0, 1, 2, 3\}, \\
\left(-\frac{1}{3}, 3\right)_{\mathbb{T}} &= \left\{-\frac{1}{n} : n \in \mathbb{N}, \quad n \geq 4\right\} \cup \{0, 1, 2\}.
\end{aligned}$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$. Then

$$\begin{aligned} \left[0, \frac{1}{2}\right]_{\mathbb{T}} &= \left\{0, \frac{1}{16}, \frac{1}{4}, \frac{1}{2}\right\}, \\ \left[0, \frac{1}{2}\right)_{\mathbb{T}} &= \left\{0, \frac{1}{16}, \frac{1}{4}\right\}, \\ \left(0, \frac{1}{2}\right]_{\mathbb{T}} &= \left\{\frac{1}{16}, \frac{1}{4}, \frac{1}{2}\right\}, \\ \left(0, \frac{1}{2}\right)_{\mathbb{T}} &= \left\{\frac{1}{16}, \frac{1}{4}\right\}. \end{aligned}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\}.$$

Then

$$\begin{aligned}
\left[0, \frac{7}{8}\right]_{\mathbb{T}} &= \{0\} \cup U \cup \left\{ \frac{3}{4}, \frac{7}{8} \right\}, \\
\left[0, \frac{7}{8}\right)_{\mathbb{T}} &= \{0\} \cup U \cup \left\{ \frac{3}{4} \right\}, \\
\left(0, \frac{7}{8}\right]_{\mathbb{T}} &= U \cup \left\{ \frac{3}{4}, \frac{7}{8} \right\}, \\
\left(0, \frac{7}{8}\right)_{\mathbb{T}} &= U \cup \left\{ \frac{3}{4} \right\}.
\end{aligned}$$

Definition

Define the sets

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}]_{\mathbb{T}} & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} \quad \text{if} \quad \sup \mathbb{T} = \infty \end{cases}$$

and

$$\mathbb{T}_{\kappa} = \begin{cases} \mathbb{T}\backslash[\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} & \text{if} \quad \inf \mathbb{T} > -\infty \\ \mathbb{T} \quad \text{if} \quad \inf \mathbb{T} > -\infty. \end{cases}$$

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Then

$$\begin{aligned} \inf \mathbb{T} &= -\infty, \\ \sup \mathbb{T} &= \infty \end{aligned}$$

and

$$\begin{aligned} \mathbb{T}^{\kappa} &= \mathbb{T}, \\ \mathbb{T}_{\kappa} &= \mathbb{T}. \end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then

$$\sup \mathbb{T} = \infty, \quad \inf \mathbb{T} = 1, \quad \sigma(\inf \mathbb{T}) = \sigma(1) = 3.$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\mathbb{T}_{\kappa} = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [1, 3)_{\mathbb{T}} = \mathbb{T} \backslash \{1\} = 3^{\mathbb{N}}.$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$. Then

$$\sup \mathbb{T} = \infty, \quad \inf \mathbb{T} = 0,$$

$$\sigma(\inf \mathbb{T}) = \sigma(0) = 1.$$

Hence,

$$\mathbb{T}^\kappa = \mathbb{T}$$

and

$$\mathbb{T}_\kappa = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 1)_{\mathbb{T}} = \mathbb{T} \backslash \{0\} = \mathbb{N}^k.$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$\sup \mathbb{T} \;\; = \;\; \infty, \quad \inf \mathbb{T} = 0, \quad \sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\mathbb{T}_{\kappa} \;\; = \;\; \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. Then

$$\sup \mathbb{T} = 1, \quad \rho(\sup \mathbb{T}) = \rho(1) = 1,$$

$$\inf \mathbb{T} = 0, \quad \sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\mathbb{T}^\kappa = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] = \mathbb{T} \backslash (1, 1] = \mathbb{T} \backslash \emptyset = \mathbb{T}$$

and

$$\mathbb{T}_\kappa = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N}_0 \right\}$. Then

$$\sup \mathbb{T} = \infty, \quad \inf \mathbb{T} = \alpha_0, \quad \sigma(\inf \mathbb{T}) = \sigma(\alpha_0) = \alpha_0 + \alpha_1.$$

Hence,

$$\mathbb{T}^\kappa = \mathbb{T}$$

and

$$\begin{aligned} \mathbb{T}_\kappa &= \mathbb{T} \backslash \inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [\alpha_0, \alpha_0 + \alpha_1)_{\mathbb{T}} \\ &= \mathbb{T} \backslash \{\alpha_0\} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad n \geq 2, \quad \alpha_k > 0, \quad k \in \mathbb{N}_0 \right\}. \end{aligned}$$

Example

Let $\mathbb{T} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0$. Then

$$\sup \mathbb{T} = \infty, \quad \inf \mathbb{T} = 0,$$

$$\sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T}$$

and

$$\mathbb{T}_{\kappa} = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$. Then

$$\sup \mathbb{T} = 1, \quad \rho(\sup \mathbb{T}) = \rho(1) = \frac{1}{2}, \quad \inf \mathbb{T} = 0, \quad \sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\begin{aligned} \mathbb{T}^\kappa &= \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] \\ &= \mathbb{T} \backslash \left(\frac{1}{2}, 1\right] = \mathbb{T} \backslash \{1\} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0\}, \end{aligned}$$

and

$$\mathbb{T}_\kappa = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0, 0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{1,2\}.$$

Then

$$\sup \mathbb{T} = \frac{5}{2}, \quad \rho(\sup \mathbb{T}) = \rho\left(\frac{5}{2}\right) = \frac{9}{4}, \quad \inf \mathbb{T} = 0, \quad \sigma(\inf \mathbb{T}) = \sigma(0) = 0.$$

Hence,

$$\mathbb{T}^{\kappa} = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] = \mathbb{T} \backslash \left(\frac{9}{4}, \frac{5}{2}\right] = \mathbb{T} \backslash \left\{\frac{5}{2}\right\}$$

and

$$\mathbb{T}_{\kappa} = \mathbb{T} \backslash [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))_{\mathbb{T}} = \mathbb{T} \backslash [0,0)_{\mathbb{T}} = \mathbb{T} \backslash \emptyset = \mathbb{T}.$$

Definition

For a function $f : \mathbb{T} \to \mathbb{R}$ and a $k \in \mathbb{N}_0$, define

$$f^{\sigma^k}(t) = f\left(\sigma^k(t)\right), \quad t \in \mathbb{T}.$$

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$ and

$$f(t) = 1 + 2t + 3t^2, \quad t \in \mathbb{T}.$$

We will find $f^\sigma(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$. By Example 76, we have

$$\sigma(t) = t + h, \quad t \in \mathbb{T},$$

and

$$\begin{aligned} f^\sigma(t) &= f(\sigma(t)) \\ &= 1 + 2\sigma(t) + 3(\sigma(t))^2 \\ &= 1 + 2(t+h) + 3(t+h)^2 \\ &= 1 + 2t + 2h + 3t^2 + 6ht + 3h^2 \\ &= 1 + 2h + 3h^2 + 2(1+3h)t + 3t^2, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Hence,

$$
\begin{aligned}
f^{\sigma^2}(t) &= f\left(\sigma^2(t)\right) \\
&= f(\sigma(\sigma(t))) \\
&= f^{\sigma}(\sigma(t)) \\
&= 1 + 2h + 3h^2 + 2(1+3h)\sigma(t) + 3(\sigma(t))^2 \\
&= 1 + 2h + 3h^2 + 2(1+3h)(t+h) + 3(t+h)^2 \\
&= 1 + 2h + 3h^2 + 2h(1+3h) + 2(1+3h)t + 3t^2 + 6ht + 3h^2 \\
&= 1 + 2h + 6h^2 + 2h + 6h^2 + 2(1+3h+3h)t + 3t^2 \\
&= 1 + 4h + 12h^2 + 2(1+6h)t + 3t^2, \quad t \in \mathbb{T}.
\end{aligned}
$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$ and

$$f(t) = \frac{1+t}{2+t}, \quad t \in \mathbb{T}.$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$. By Example 77, we have

$$\sigma(t) = 3t, \quad t \in \mathbb{T},$$

and

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= \frac{1+\sigma(t)}{2+\sigma(t)} \\ &= \frac{1+3t}{2+3t}, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Hence,

$$\begin{aligned} f^{\sigma^2}(t) &= f\left(\sigma^2(t)\right) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= \frac{1+3\sigma(t)}{2+3\sigma(t)} \\ &= \frac{1+9t}{2+9t}, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^2$ and

$$f(t) = \frac{1+2t}{1+t^2}, \quad t \in \mathbb{T}.$$

We will find $f^\sigma(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$. By Example 78, we have

$$\sigma(t) = (\sqrt{t}+1)^2, \quad t \in \mathbb{T},$$

Then

$$\begin{aligned} f^\sigma(t) &= f(\sigma(t)) \\ &= \frac{1+2\sigma(t)}{1+(\sigma(t))^2} \\ &= \frac{1+2(\sqrt{t}+1)^2}{1+(\sqrt{t}+1)^4}, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Hence,

$$\begin{aligned}
f^{\sigma^2}(t) &= f(\sigma^2(t)) \\
&= f(\sigma(\sigma(t))) \\
&= f^{\sigma}(\sigma(t)) \\
&= \frac{1+2(\sqrt{\sigma(t)}+1)^2}{1+(\sqrt{\sigma(t)}+1)^4} \\
&= \frac{1+2\left(\sqrt{(\sqrt{t}+1)^2}+1\right)^2}{1+\left(\sqrt{(\sqrt{t}+1)^2}+1\right)^4} \\
&= \frac{1+2(\sqrt{t}+1+1)^2}{1+(\sqrt{t}+1+1)^4} \\
&= \frac{1+2(\sqrt{t}+2)^2}{1+(\sqrt{t}+2)^4}, \quad t \in \mathbb{T}.
\end{aligned}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers, and

$$f(t) = \frac{1-t}{1+4t}, \quad t \in \mathbb{T}.$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$. By Example 79, we have

$$\sigma(H_n) = H_{n+1}, \quad n \in \mathbb{N}_0.$$

Then

$$\begin{aligned} f^{\sigma}(H_n) &= f(\sigma(H_n)) \\ &= \frac{1-\sigma(H_n)}{1+4\sigma(H_n)} \\ &= \frac{1-H_{n+1}}{1+4H_{n+1}}, \quad n \in \mathbb{N}_0. \end{aligned}$$

Example

Hence,

$$\begin{aligned} f^{\sigma^2}(H_n) &= f(\sigma^2(H_n)) \\ &= f(\sigma(\sigma(H_n))) \\ &= f^{\sigma}(\sigma(H_n)) \\ &= \frac{1-\sigma(H_{n+1})}{1+4\sigma(H_{n+1})} \\ &= \frac{1-H_{n+2}}{1+4H_{n+2}}, \quad n \in \mathbb{N}_0. \end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$ and

$$f(t) = 1 - 2t - t^2, \quad t \in \mathbb{T}.$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$. By Example 80, we have

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup\limits_{k=0}^{\infty} [4k, 4k+1) \\ t+3 & \text{if} \quad t \in \bigcup\limits_{k=0}^{\infty} \{4k+1\}. \end{cases}$$

Then, we will consider the following cases.

Example

Let $t \in \bigcup_{k=0}^{\infty} [4k, 4k+1)$. Then $\sigma = I$, where I is the identity operator, and then

$$\begin{aligned} f^{\sigma}(t) &= f(t), \\ f^{\sigma^2}(t) &= f(t), \quad t \in \mathbb{T}. \end{aligned}$$

Let $t \bigcup_{k=0}^{\infty} \{4k+1\}$. Then

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= 1 - 2\sigma(t) - (\sigma(t))^2 \\ &= 1 - 2(t+2) - (y+3)^2 \\ &= 1 - 2t - 6 - t^2 - 6t - 9 \\ &= -14 - 8t - t^2, \quad t \in \mathbb{T}, \end{aligned}$$

and

Example

$$\begin{aligned} f^{\sigma^2}(t) &= f(\sigma^2(t)) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= -14 - 8\sigma(t) - (\sigma(t))^2 \\ &= -14 - 8(t+3) - (t+3)^2 \\ &= -14 - 8t - 24 - t^2 - 6t - 9 \\ &= -47 - 14t - t^2, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set, and

$$f(t) = t^3, \quad t \in \mathbb{T}.$$

We will find $f^{\sigma}(t)$ and $f^{\sigma^2}(t)$ for $t \in \mathbb{T}$. By Example 84, we have

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}}, \quad t \in C_1, \quad t = \sum\limits_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}, \\ \\ t, \quad t \in \mathbb{T} \backslash C_1. \end{cases}$$

Then, we have the following cases.

Example

Let $t \in C_1$ and

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}.$$

Then

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= (\sigma(t))^3 \\ &= \left(t + \frac{1}{3^{m+1}}\right)^3 \end{aligned}$$

and

Example

$$\begin{aligned} f^{\sigma^2}(t) &= f(\sigma^2(t)) \\ &= f(\sigma(\sigma(t))) \\ &= f^{\sigma}(\sigma(t)) \\ &= \left(\sigma(t) + \frac{1}{3^{m+1}}\right)^3 \\ &= \left(t + \frac{1}{3^{m+1}} + \frac{1}{3^{m+1}}\right)^3 \\ &= \left(t + \frac{2}{3^{m+1}}\right)^3. \end{aligned}$$

Example

If $t \in \mathbb{T} \backslash C_1$, then $\sigma(t) = t$ and

$$\begin{aligned} f^{\sigma}(t) &= f(t) \\ &= t^3 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= f(t) \\ &= t^3. \end{aligned}$$

Example

Let

$$\mathbb{T} = \left\{ \sum_{k=0}^{n-1} \alpha_k : n \in \mathbb{N}, \quad \alpha_k > 0, \quad k \in \mathbb{N}_0 \right\},$$

and

$$f(t) = \frac{1+t^3}{3+t^4}, \quad t \in \mathbb{T}.$$

We have $\sigma(t_n) = t_{n+1}, \quad n \in \mathbb{N}$. Then

$$\begin{aligned} f^\sigma(t_n) &= f(\sigma(t_n)) \\ &= \frac{1+(\sigma(t_n))^3}{3+(\sigma(t_n))^4} \\ &= \frac{1+t_{n+1}^3}{3+t_{n+1}^4}, \quad n \in \mathbb{N}, \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\sigma^2}(t_n) &= f(\sigma^2(t_n)) \\
&= f(\sigma(\sigma(t_n))) \\
&= f^{\sigma}(\sigma(t_n)) \\
&= \frac{1+(\sigma(t_{n+1}))^3}{3+(\sigma(t_{n+1}))^4} \\
&= \frac{1+t_{n+2}^3}{3+t_{n+2}^4}, \quad n \in \mathbb{N}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$ and

$$f(t) = 1 + t - t^2, \quad t \in \mathbb{T}.$$

By Example 88, we have

$$\sigma(t) = \begin{cases} -\frac{t}{t-1} & \text{if} \quad t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \\ t+1 & \text{if} \quad t \in \mathbb{N}_0. \end{cases}$$

Then, we will consider the following cases.

Example

Let $t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$. Then

$$\begin{aligned} f^\sigma(t) &= f(\sigma(t)) \\ &= 1+\sigma(t)-(\sigma(t))^2 \\ &= 1+\frac{t}{t-1}-\frac{t^2}{(t-1)^2} \\ &= \frac{(t-1)^2+t(t-1)-t^2}{(t-1)^2} \\ &= \frac{t^2-2t+1+t^2-t-t^2}{(t-1)^2} \\ &= \frac{t^2-3t+1}{(t-1)^2} \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\sigma^2}(t) &= f(\sigma^2(t)) = f(\sigma(\sigma(t))) \\
&= f^{\sigma}(\sigma(t)) \\
&= \frac{(\sigma(t))^2 - 3\sigma(t) + 1}{(\sigma(t) - 1)^2} \\
&= \frac{\frac{t^2}{(t-1)^2} + \frac{3t}{t-1} + 1}{\left(-\frac{t}{t-1} - 1\right)^2} \\
&= \frac{\frac{t^2+3t(t-1)+(t-1)^2}{(t-1)^2}}{\left(\frac{t+t-1}{t-1}\right)^2} \\
&= \frac{t^2 + 3t^2 - 3t + t^2 - 2t + 1}{(2t-1)^2} \\
&= \frac{5t^2 - 5t + 1}{(2t-1)^2}.
\end{aligned}
$$

Example

Let $t \in \mathbb{N}_0$. Then

$$\begin{aligned} f^\sigma(t) &= f(\sigma(t)) \\ &= 1+\sigma(t)-(\sigma(t))^2 \\ &= 1+t+1-(t+1)^2 \\ &= 2+t-t^2-2t-1 \\ &= -t^2-t-1 \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\sigma^2}(t) &= f(\sigma^2(t)) = f(\sigma(\sigma(t))) = f^{\sigma}(\sigma(t)) \\
&= -(\sigma(t))^2 - \sigma(t) + 1 \\
&= -(t+1)^2 - (t+1) + 1 \\
&= -t^2 - 2t - 1 - t - 1 + 1 \\
&= -t^2 - 3t - 1.
\end{aligned}
$$

Example

Let $\mathbb{T}\left\{\left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0\right\} \cup \{0,1\}$ and

$$f(t) = t - t^3, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned} f^\sigma(t) &= f(\sigma(t)) \\ &= \sigma(t) - (\sigma(t))^3, \quad t \in \mathbb{T}, \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= f(\sigma^2(t)) \\ &= \sigma^2(t) - (\sigma^2(t))^3, \quad t \in \mathbb{T}. \end{aligned}$$

We have the following cases.

Example

Let $t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\}$. Then

$$\begin{aligned} \sigma(t) &= \sqrt{t}, \\ \sigma^2(t) &= \sqrt{\sigma(t)} \\ &= \sqrt{\sqrt{t}} \\ &= \sqrt[4]{t}, \quad t \in \mathbb{T}. \end{aligned}$$

Hence,

$$f^{\sigma}(t) = \sqrt{t} - (\sqrt{t})^3$$

and

$$f^{\sigma^2}(t) = \sqrt[4]{t} - (\sqrt[4]{t})^3.$$

Example

Let $t = \frac{1}{2}$. Then $\sigma\left(\frac{1}{2}\right) = 1$ and

$$\begin{aligned}\sigma^2\left(\frac{1}{2}\right) &= \sigma\left(\sigma\left(\frac{1}{2}\right)\right) \\ &= \sigma(1) \\ &= 1.\end{aligned}$$

Hence,

Example

$$\begin{aligned} f^{\sigma}\left(\frac{1}{2}\right) &= \sigma\left(\frac{1}{2}\right) - \left(\sigma\left(\frac{1}{2}\right)\right)^3 \\ &= 1 - 1^3 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}\left(\frac{1}{2}\right) &= \sigma^2\left(\frac{1}{2}\right) - \left(\sigma^2\left(\frac{1}{2}\right)\right)^3 \\ &= 1 - 1^3 \\ &= 0. \end{aligned}$$

Example

Let $t = 0$. Then $\sigma(0) = 0$ and

$$\begin{aligned}\sigma^2(0) &= \sigma(\sigma(0)) \\ &= \sigma(0) \\ &= 0.\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\sigma}(0) &= \sigma(0) - (\sigma(0))^3 \\
&= 0 - 0^3 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^2}(0) &= \sigma^2(0) - (\sigma^2(0))^3 \\
&= 0 - 0^3 \\
&= 0.
\end{aligned}
$$

Example

Let $t = 1$. Then $\sigma(1) = 1$ and

$$\begin{aligned}\sigma^2(1) &= \sigma(\sigma(1)) \\ &= \sigma(1) \\ &= 1.\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\sigma}(1) &= \sigma(1) - (\sigma(1))^3 \\
&= 1 - 1^3 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^2}(1) &= \sigma^2(1) - (\sigma^2(1))^3 \\
&= 1 - 1^3 \\
&= 0.
\end{aligned}
$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{1, 2\},$$

and

$$f(t) = \frac{1+t}{1+4t}, \quad t \in \mathbb{T}.$$

We have

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma(t)) \\ &= \frac{1+\sigma(t)}{1+4\sigma(t)}, \quad t \in \mathbb{T}, \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= f(\sigma^2(t)) \\ &= \frac{1+\sigma^2(t)}{1+4\sigma^2(t)}, \quad t \in \mathbb{T}. \end{aligned}$$

Example

Let $t \in U \backslash \left\{ \frac{1}{2} \right\}$. Then $\sigma(t) = 2t$ and

$$\begin{aligned} \sigma^2(t) &= \sigma(\sigma(t)) \\ &= 2\sigma(t) \\ &= 2(2t) \\ &= 4t. \end{aligned}$$

Hence,

Example

$$f^{\sigma}(t) = \frac{1+2t}{1+4(2t)} = \frac{1+2t}{1+8t}$$

and

$$f^{\sigma^2}(t) = \frac{1+4t}{1+4(4t)} = \frac{1+4t}{1+16t}.$$

Example

Let $t \in (1-U) \setminus \left\{ \frac{1}{2} \right\}$. Then $\sigma(t) = \frac{1+t}{2}$ and

$$\begin{aligned}
\sigma^2(t) &= \sigma(\sigma(t)) \\
&= \frac{1\sigma(t)}{2} \\
&= \frac{1+\frac{1+t}{2}}{2} \\
&= \frac{2+1+t}{4} \\
&= \frac{3+t}{4}.
\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\sigma}(t) &= \frac{1+\frac{1+t}{2}}{1+4\frac{1+t}{2}} \\
&= \frac{2+1+t}{2+4+4t} \\
&= \frac{3+t}{6+4t}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^2}(t) &= \frac{1+\frac{3+t}{4}}{2+4\frac{3+t}{4}} \\
&= \frac{4+3+t}{8+12+4t} \\
&= \frac{7+t}{20+4t}.
\end{aligned}
$$

Example

Let $t \in (1+U)\setminus\left\{\frac{3}{2}\right\}$. Then $\sigma(t) = 2t-1$ and

$$\begin{aligned}
\sigma^2(t) &= \sigma(\sigma(t)) \\
&= 2\sigma(t) - 1 \\
&= 2(2t-1) - 1 \\
&= 4t - 2 - 1 \\
&= 4t - 3.
\end{aligned}$$

Hence,

Example

$$\begin{aligned} f^{\sigma}(t) &= \frac{1+2t-1}{1+4(2t-1)} \\ &= \frac{2t}{1+8t-4} \\ &= \frac{2t}{8t-3} \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(t) &= \frac{1+4t-3}{1+4(4t-3)} \\ &= \frac{4t-2}{1+16t-12} \\ &= \frac{4t-2}{16t-11}. \end{aligned}$$

Example

Let $t \in (2-U)\setminus\left\{\frac{3}{2}\right\}$. Then $\sigma(t) = \frac{t+2}{2}$ and

$$\begin{aligned}
\sigma^2(t) &= \sigma(\sigma(t)) \\
&= \frac{\sigma(t)+2}{2} \\
&= \frac{\frac{t+2}{2}+2}{2} \\
&= \frac{t+2+4}{4} \\
&= \frac{t+6}{4}.
\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\sigma}(t) &= \frac{1+\frac{t+2}{2}}{1+4\frac{t+2}{2}} \\
&= \frac{2+t+2}{2(1+2t+4)} \\
&= \frac{t+4}{2(2t+5)}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^2}(t) &= \frac{1+\frac{t+6}{4}}{1+4\frac{t+6}{4}} \\
&= \frac{4+t+6}{4(t+6+1)} \\
&= \frac{t+10}{4(t+7)}.
\end{aligned}
$$

Example

Let $t \in (2+U)\setminus\left\{\frac{5}{2}\right\}$. Then $\sigma(t) = 2(t-1)$ and

$$\begin{aligned}
\sigma^2(t) &= \sigma(\sigma(t)) \\
&= 2(\sigma(t)-1) \\
&= 2(2(t-1)-1) \\
&= 2(2t-2-1) \\
&= 2(2t-3).
\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\sigma}(t) &= \frac{1+2(t-1)}{1+4(t-1)} \\
&= \frac{1+2t-2}{1+8t-8} \\
&= \frac{2t-1}{8t-7}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^2}(t) &= \frac{1+2(2t-3)}{1+4(2(2t-3))} \\
&= \frac{1+4t-6}{1+16t-24} \\
&= \frac{4t-5}{16t-23}.
\end{aligned}
$$

Example

Let $t = 0$. Then $\sigma(0) = 0$ and

$$\begin{aligned}\sigma^2(0) &= \sigma(\sigma(0)) \\ &= \sigma(0) \\ &= 0.\end{aligned}$$

Hence,

Example

$$f^{\sigma}(0) = \frac{1+0}{1+4\cdot 0} = 1$$

and

$$f^{\sigma^2}(0) = \frac{1+0}{1+4\cdot 0} = 1.$$

Example

Let $t = \frac{1}{2}$. Then $\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$ and

$$\begin{aligned} \sigma^2\left(\frac{1}{2}\right) &= \sigma\left(\sigma\left(\frac{1}{2}\right)\right) \\ &= \sigma\left(\frac{3}{4}\right) \\ &= \frac{7}{8}. \end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\sigma}\left(\frac{1}{2}\right) &= \frac{1+\frac{3}{4}}{1+4\cdot\frac{3}{4}} \\
&= \frac{4+3}{4(1+3)} \\
&= \frac{7}{16}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^2}\left(\frac{1}{2}\right) &= \frac{1+\frac{7}{8}}{1+4\cdot\frac{7}{8}} \\
&= \frac{8+7}{8+28} \\
&= \frac{15}{36} \\
&= \frac{5}{12}.
\end{aligned}
$$

Example

Let $t = 1$. Then $\sigma(1) = 1$ and

$$\begin{aligned}\sigma^2(1) &= \sigma(\sigma(1)) \\ &= \sigma(1) \\ &= 1.\end{aligned}$$

Hence,

Example

$$f^{\sigma}(1) = \frac{1+1}{1+4\cdot 1} = \frac{2}{5}$$

and

$$f^{\sigma^2}(1) = \frac{1+1}{1+4\cdot 1} = \frac{2}{5}.$$

Example

Let $t = \frac{3}{2}$. Then $\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$ and

$$\begin{aligned}\sigma^2\left(\frac{3}{2}\right) &= \sigma\left(\sigma\left(\frac{3}{2}\right)\right)\\ &= \sigma\left(\frac{7}{4}\right)\\ &= \frac{15}{8}.\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\sigma}\left(\frac{3}{2}\right) &= \frac{1+\frac{7}{4}}{1+4\cdot\frac{7}{4}} \\
&= \frac{4+7}{4(1+7)} \\
&= \frac{11}{32}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^2}\left(\frac{3}{2}\right) &= \frac{1+\frac{15}{8}}{1+4\cdot\frac{15}{8}} \\
&= \frac{8+15}{8+60} \\
&= \frac{23}{68}.
\end{aligned}
$$

Example

Let $t = 2$. Then $\sigma(2) = 2$ and

$$\begin{aligned}\sigma^2(2) &= \sigma(\sigma(2)) \\ &= \sigma(2) \\ &= 2.\end{aligned}$$

Hence,

Example

$$\begin{aligned} f^{\sigma}(2) &= \frac{1+2}{1+4\cdot 2} \\ &= \frac{3}{9} \\ &= \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} f^{\sigma^2}(2) &= \frac{1+2}{1+4\cdot 2} \\ &= \frac{1}{3}. \end{aligned}$$

Example

Let $t = \frac{5}{2}$. Then $\sigma\left(\frac{5}{2}\right) = \frac{5}{2}$ and

$$\begin{aligned}\sigma^2\left(\frac{5}{2}\right) &= \sigma\left(\sigma\left(\frac{5}{2}\right)\right)\\ &= \sigma\left(\frac{5}{2}\right)\\ &= \frac{5}{2}.\end{aligned}$$

Hence,

Example

$$f^{\sigma}\left(\frac{5}{2}\right) = \frac{1+\frac{5}{2}}{1+4\cdot\frac{5}{2}} = \frac{2+5}{2(1+10)} = \frac{7}{22}$$

and

$$f^{\sigma^2}\left(\frac{5}{2}\right) = \frac{1+\frac{5}{2}}{1+4\cdot\frac{5}{2}} = \frac{7}{22}.$$

Appendix

We start by defining the forward jump operator.

Definition

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ in the following manner

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$. Then, $t = \sigma(t)$ if t is a maximum of $\mathbb{T}$.

Note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{Z}$ such that $t = hn$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned}
\sigma(t) &= \inf\{s = hp, p \in \mathbb{Z} : hp > hn\} \\
&= h(n+1) \\
&= hn + h \\
&= t + h.
\end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = 3^n$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned}\sigma(t) &= \inf\{3^s, s \in \mathbb{N}_0 : 3^s > 3^n\}\\&= 3^{n+1}\\&= 3 \cdot 3^n\\&= 3t.\end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$ is fixed. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = n^k$. Hence, $n = \sqrt[k]{t}$. Now, applying the definition for forward jump operators, we arrive at

$$\begin{aligned} \sigma(t) &= \inf\{s^k, s \in \mathbb{N}_0 : s^k > n^k\} \\ &= (n+1)^k \\ &= \left(\sqrt[k]{t} + 1\right)^k. \end{aligned}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$ arbitrarily. Then, applying the definition for forward jump operators, we find

$$\begin{aligned}
\sigma(H_n) &= \inf\{H_s, s \in \mathbb{N}_0 : H_s > H_n\} \\
&= \inf\left\{H_s, s \in \mathbb{N}_0 : \sum_{k=1}^{s} \frac{1}{k} > \sum_{k=1}^{n} \frac{1}{k}\right\} \\
&= \sum_{k=1}^{n+1} \frac{1}{k} \\
&= H_{n+1}.
\end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$\begin{aligned} \mathbb{T} &= \bigcup_{k=0}^{\infty} [4k, 4k+1] \\ &= [0,1] \cup [4,5] \cup [8,9] \cup [12,13] \cup \ldots . \end{aligned}$$

If $t \in [0,1)$, then, applying the definition for forward jump operators, we find

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t. \end{aligned}$$

Example

If $t = 1$, then

$$\begin{aligned}\sigma(1) &= \inf\{s \in \mathbb{T} : s > 1\} \\ &= 4.\end{aligned}$$

Let now, $k \in \mathbb{N}$ be arbitrarily chosen. If $t \in [4k, 4k+1)$, then we have

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t.\end{aligned}$$

Example

If $t = 4k + 1$, then

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > 4k + 1\} \\ &= 4(k+1) \\ &= 4k + 4 \\ &= 4k + 1 + 3 \\ &= t + 3. \end{aligned}$$

Example

Therefore

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty} [4k, 4k+1) \\ t+3 & \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{4k+1\}. \end{cases}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. We will find $\sigma(t)$ for $t \in \mathbb{T}$. For this aim, let C_1 denote the set of all left-hand end points of the open intervals that are removed. Then

$$C_1 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \le k \le m \right\}.$$

With C_2 we will denote the set of all right-hand end points of the open intervals that are removed. We have

$$C_2 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \le k \le m \right\}.$$

Take $t \in C$ arbitrarily. We have the following cases.
Let $t \in C_1$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}.$$

Example

Hence, we obtain

$$\begin{aligned}
\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\
&= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\
&= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} + \frac{1}{3^{m+1}} \\
&= t + \frac{1}{3^{m+1}}.
\end{aligned}$$

Let $t \in C_2$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}.$$

Hence,

Example

Let $t \in \mathbb{T} \backslash (C_1 \cup C_2)$. Then

$$
\begin{aligned}
\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\
&= t.
\end{aligned}
$$

Consequently

$$
\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}} & \text{if} \quad t \in C_1, t = \sum\limits_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ t \quad \text{if} \quad t \in \mathbb{T} \backslash C_1. \end{cases}
$$

Example

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers with $\alpha_n > 0$, and

$$t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbb{N},$$

and

$$\mathbb{T} = \{t_n : n \in \mathbb{N}\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned} \sigma(t_n) &= \inf\left\{x \in \mathbb{T} : s = \sum_{k=0}^{n-1} \alpha_k, \quad s > t_n\right\} \\ &= \sum_{k=0}^{n} \alpha_k = \sum_{k=0}^{n-1} + \alpha_n = t_n + \alpha_n. \end{aligned}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$n = -\frac{1}{t_n}$$

and

$$\begin{aligned} \sigma(t_n) &= \inf \left\{ s \in \mathbb{T} : s = -\frac{1}{m}, m \in \mathbb{N}, s > t_0 \right\} \\ &= -\frac{1}{n+1} = -\frac{1}{-\frac{1}{t_n} + 1} = -\frac{t_n}{t_n - 1}. \end{aligned}$$

Example

Next, if $t \in \mathbb{N}_0$, then

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t + 1. \end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} -\frac{t}{t-1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\}, \quad t = t_n \\ t + 1 & \text{if} \quad t \in \mathbb{N}_0. \end{cases}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned} \sigma(t_n) &= \inf\{s \in \mathbb{T} : s > t_n\} \\ &= \left(\frac{1}{2}\right)^{2^{n-1}} = \left(\frac{1}{2}\right)^{2^n \cdot \frac{1}{2}} \\ &= \left(\left(\frac{1}{2}\right)^{2^n}\right)^{\frac{1}{2}} = \sqrt{t_n}. \end{aligned}$$

Example

Next,

$$t_0 = \frac{1}{2} \quad \text{and} \quad \sigma(t_0) = 1$$

and

$$\sigma(0) \;=\; 0, \quad \sigma(1) = 1.$$

Consequently

$$\sigma(t) = \begin{cases} \sqrt{t} \quad \text{if} \quad t \in \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ 1 \quad \text{if} \quad t = \frac{1}{2} \\ 0 \quad \text{if} \quad t = 0 \\ 1 \quad \text{if} \quad t = 1. \end{cases}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = U \cup (1 - U) \cup (1 + U) \cup (2 - U) \cup (2 + U) \cup \{0, 1, 2\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. We have the following cases.
Let $t = 0$. Then

$$\sigma(0) = 0.$$

Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}.$$

Let $t = 1$. Then

$$\sigma(1) = 1.$$

Example

Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}.$$

Let $t = 2$. Then

$$\sigma(2) = 2.$$

Let $t = \frac{5}{2}$. Then

$$\sigma\left(\frac{5}{2}\right) = \frac{5}{2}.$$

Example

Let $t \in U \setminus \left\{ \frac{1}{2} \right\}$. Then

$$t = \frac{1}{2^n}$$

and

$$\sigma(t) = \frac{1}{2^{n-1}} = \frac{2}{2^n} = 2t.$$

Let $t \in (1-U) \setminus \left\{ \frac{1}{2} \right\}$. Then $t = 1 - \frac{1}{2^n}$ and $\frac{1}{2^n} = 1 - t$. Hence,

$$\sigma(t) = 1 - \frac{1}{2^{n+1}} = 1 - \frac{1}{2} \cdot \frac{1}{2^n} = 1 - \frac{1-t}{2} = \frac{1+t}{2}.$$

Example

Let $t \in (1+U) \setminus \left\{\frac{3}{2}\right\}$. Then

$$t = 1 + \frac{1}{2^n}.$$

Hence,

$$\frac{1}{2^n} = t - 1$$

and

$$\sigma(t) = 1 + \frac{1}{2^{n-1}} = 1 + \frac{2}{2^n} = 1 + 2(t-1) = 2t - 1.$$

Example

Let $t \in (2-U)\setminus\left\{\frac{3}{2}\right\}$. Then

$$t = 2 - \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = 2 - t.$$

Hence,

$$\sigma(t) \;=\; 2 - \frac{1}{2^{n+1}} = 2 - \frac{1}{2} \cdot \frac{1}{2^n} = 2 - \frac{2-t}{2} = \frac{t+2}{2}.$$

Example

Let $t \in (2+U)\setminus\left\{\frac{5}{2}\right\}$. Then

$$t = 2 + \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = t - 2.$$

Hence,

$$\sigma(t) \;=\; 2 + \frac{1}{2^{n-1}} = 2 + \frac{2}{2^n} = 2 + 2(t-2) = 2(t-1).$$

Example

Consequently

$$\sigma(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \frac{3}{4} & \text{if} \quad t = \frac{1}{2} \\ 1 & \text{if} \quad t = 1 \\ \frac{7}{4} & \text{if} \quad t = \frac{3}{2} \\ 2 & \text{if} \quad t = 2 \\ \frac{5}{2} & \text{if} \quad t = \frac{5}{2} \\ 2t & \text{if} \quad t \in U \backslash \left\{ \frac{1}{2} \right\} \\ \frac{1+t}{2} & \text{if} \quad t \in (1 - U) \backslash \left\{ \frac{1}{2} \right\} \end{cases}$$

Example

$$\sigma(t) = \begin{cases} 2t-1 & \text{if} \quad t \in (1+U) \setminus \left\{ \frac{3}{2} \right\} \\ \frac{t+2}{2} & \text{if} \quad t \in (2-U) \setminus \left\{ \frac{3}{2} \right\} \\ 2(t-1) & \text{if} \quad t \in (2+U) \setminus \left\{ \frac{5}{2} \right\}. \end{cases}$$

Exercise

Find $\sigma(t)$, $t \in \mathbb{T}$, where

1. $\mathbb{T} = h\mathbb{Z} + k$, $h > 0$, $k \in \mathbb{R}$.
2. $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}$.
3. $\mathbb{T} = P_{3,7} \cup [4, 6]$.
4. $\mathbb{T} = 11^{\mathbb{N}_0} \cup \{0\}$.
5. $\mathbb{T} = [1, 2] \cup [3, 4] \cup [7, 8] \cup 9^{\mathbb{N}}$.