Time Scales Analysis
Lecture 4

Functions and Backward Jump Operators. Induction Principle

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Definition

For a function $f : \mathbb{T} \to \mathbb{R}$ and $k, l \in \mathbb{N}_0$, define

$$f^{\rho^k}(t) = f\left(\rho^k(t)\right), \quad t \in \mathbb{T},$$

and

$$f^{\rho^k \sigma^l}(t) = f\left(\sigma^l\left(\rho^k(t)\right)\right), \quad t \in \mathbb{T},$$

and

$$f^{\sigma^l \rho^k}(t) = f\left(\rho^k\left(\sigma^l(t)\right)\right), \quad t \in \mathbb{T}.$$

Note that in the general case, we have

$$f^{\rho^k \sigma^l}(t) \neq f^{\sigma^l \rho^k}(t), \quad t \in \mathbb{T}.$$

Really, let $\mathbb{T} = \{-1, 0\} \cup \left\{\frac{1}{n}\right\}_{n \in \mathbb{N}}$, and $f(t) = t, \quad t \in \mathbb{T}$. Then

$$\rho(\sigma(0)) \;=\; \rho(0) = -1$$

and

$$\sigma(\rho(0)) \;=\; \sigma(-1) = 0.$$

Thus,

$$\rho(\sigma(0)) \neq \sigma(\rho(0)).$$

Example

Let $\mathbb{T} = \{-1, 0\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ and

$$f(t) = 1 + 4t^2, \quad t \in \mathbb{T}.$$

We will find

$$f^{\rho^2}(t), \quad f^{\sigma\rho\sigma}(t) \quad \text{and} \quad f^{\sigma^2\rho^3}(t) \quad \text{for} \quad t \in \mathbb{T}.$$

Firstly, we will determine the forward and backward jump operators for the time scale $\mathbb{T}$. We have the following cases.

Example

Let $t = \frac{1}{n}$, $n \in \mathbb{N}$, $n \geq 2$. Then

$$\begin{aligned} \sigma(t) &= \frac{1}{n-1} \\ &= \frac{1}{\frac{1}{t} - 1} \\ &= \frac{t}{1-t} \end{aligned}$$

and

$$\begin{aligned} \rho(t) &= \frac{1}{n+1} \\ &= \frac{1}{\frac{1}{t} + 1} \\ &= \frac{t}{1+t}. \end{aligned}$$

Example

Let $t = 1$. Then

$$\begin{aligned} \sigma(1) &= 1, \\ \rho(1) &= \frac{1}{2}. \end{aligned}$$

Let $t = 0$. Then

$$\begin{aligned} \sigma(0) &= 0, \\ \rho(0) &= -1. \end{aligned}$$

Example

Let $t = -1$. Then

$$\begin{aligned} \sigma(-1) &= 0, \\ \rho(-1) &= -1. \end{aligned}$$

Therefore

$$\sigma(t) = \begin{cases} \frac{t}{1-t} & \text{if} \quad t \in \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}, n\geq 2} \\ 1 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t = -1 \end{cases}$$

and

Example

$$\rho(t) = \begin{cases} \frac{t}{1+t} & \text{if} \quad t \in \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}, n\geq 2} \\ \frac{1}{-2} & \text{if} \quad t = 1 \\ -1 & \text{if} \quad t = 0 \\ -1 & \text{if} \quad t = -1. \end{cases}$$

Hence, we have the following cases.

Example

Let $t \in \left\{\frac{1}{n}\right\}_{n\in\mathbb{N},n\geq 2}$. Then

$$\begin{aligned}\rho^2(t) &= \rho(\rho(t))\\ &= \rho\left(\frac{t}{1+t}\right)\\ &= \frac{\rho(t)}{1+\rho(t)}\\ &= \frac{\frac{t}{1+t}}{1+\frac{t}{1+t}}\\ &= \frac{t}{1+2t}\end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\rho^2}(t) &= f\left(\rho^2(t)\right) \\
&= f\left(\frac{t}{1+2t}\right) \\
&= 1+4\frac{t^2}{(1+2t)^2} \\
&= \frac{(1+2t)^2+4t^2}{(1+2t)^2} \\
&= \frac{1+4t+4t^2+4t^2}{(1+2t)^2} \\
&= \frac{1+4t+8t^2}{(1+2t)^2}.
\end{aligned}
$$

Next,

Example

$$
\begin{aligned}
\sigma(\rho(\sigma(t))) &= \sigma\left(\rho\left(\frac{t}{1-t}\right)\right) \\
&= \sigma\left(\frac{\rho(t)}{1-\rho(t)}\right) \\
&= \sigma\left(\frac{\frac{t}{1+t}}{1-\frac{t}{1+t}}\right) \\
&= \sigma(t) \\
&= \frac{t}{1-t}
\end{aligned}
$$

and

Example

$$
\begin{aligned}
f^{\sigma\rho\sigma}(t) &= f(\sigma(\rho(\sigma(t)))) \\
&= f\left(\frac{t}{1-t}\right) \\
&= 1 + \frac{4t^2}{(1-t)^2} \\
&= \frac{(1-t)^2 + 4t^2}{(1-t)^2} \\
&= \frac{1 - 2t + t^2 + 4t^2}{(1-t)^2} \\
&= \frac{1 - 2t + 5t^2}{(1-t)^2}.
\end{aligned}
$$

Moreover,

Example

$$
\begin{aligned}
\rho^2\left(\sigma^2(t)\right) &= \rho^2(\sigma(\sigma(t))) \\
&= \rho^3\left(\sigma\left(\frac{t}{1-t}\right)\right) = \rho^3\left(\frac{\sigma(t)}{1-\sigma(t)}\right) \\
&= \rho^3\left(\frac{\frac{t}{1-t}}{1-\frac{t}{1-t}}\right) = \rho^3\left(\frac{t}{1-2t}\right) \\
&= \rho^2\left(\rho\left(\frac{t}{1-2t}\right)\right) = \rho^2\left(\frac{\rho(t)}{1-2\rho(t)}\right) \\
&= \rho^2\left(\frac{\frac{t}{1+t}}{1-\frac{2t}{1+t}}\right) = \rho^2\left(\frac{t}{1-t}\right) \\
&= \rho\left(\rho\left(\frac{t}{1-t}\right)\right) = \rho\left(\frac{\rho(t)}{1-\rho(t)}\right) \\
&= \rho\left(\frac{\frac{t}{1+t}}{1-\frac{t}{1+t}}\right) = \rho(t) = \frac{t}{t+1}
\end{aligned}
$$

Example

$$
\begin{aligned}
f^{\sigma^2\rho^3}(t) &= f\left(\rho^3\left(\sigma^2(t)\right)\right) \\
&= f\left(\frac{t}{1+t}\right) \\
&= 1+\frac{4t^2}{(1+t)^2} \\
&= \frac{(1+t)^2+4t^2}{(1+t)^2} \\
&= \frac{1+2t+t^2+4t^2}{(1+t)^2} \\
&= \frac{1+2t+5t^2}{(1+t)^2}.
\end{aligned}
$$

Example

Let $t = 1$. Then

$$\begin{aligned} \rho^2(1) &= \rho(\rho(1)) \\ &= \rho\left(\frac{1}{2}\right) \\ &= \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} f^{\rho^2}(1) &= f(\rho^2(1)) \\ &= f\left(\frac{1}{3}\right) \\ &= 1 + \frac{4}{9} \\ &= \frac{13}{9}. \end{aligned}$$

Next

Example

$$\begin{aligned} \sigma\rho\sigma(1) &= \sigma(\rho(\sigma(1))) \\ &= \sigma(\rho(1)) \\ &= \sigma\left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma\rho\sigma}(1) &= f(\sigma(\rho(\sigma(1)))) \\ &= f(1) \\ &= 1+4 \\ &= 5. \end{aligned}$$

Moreover,

Example

$$
\begin{aligned}
\rho^3\sigma^2(1) &= \rho^3(\sigma(\sigma(1))) \\
&= \rho^3(\sigma(1)) \\
&= \rho^3(1) \\
&= \rho(\rho(\rho(1))) \\
&= \rho\left(\rho\left(\frac{1}{2}\right)\right) \\
&= \rho\left(\frac{1}{3}\right) \\
&= \frac{1}{4}
\end{aligned}
$$

and

Example

$$
\begin{aligned}
f^{\sigma^2\rho^3}(1) &= f\left(\rho^3\sigma^2(1)\right) \\
&= f\left(\frac{1}{4}\right) \\
&= 1 + 4\cdot\frac{1}{16} \\
&= 1 + \frac{1}{4} \\
&= \frac{5}{4}.
\end{aligned}
$$

Example

Let $t = 0$. Then

$$\begin{aligned} \sigma\rho\sigma(0) &= \sigma\rho(\sigma(0)) \\ &= \sigma(\rho(0)) \\ &= \sigma(-1) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f^{\sigma\rho\sigma}(0) &= f(\sigma\rho\sigma(0)) \\ &= f(0) \\ &= 1. \end{aligned}$$

Next,

Example

$$\begin{aligned}\rho^2(0) &= \rho(\rho(0)) \\ &= \rho(-1) \\ &= -1\end{aligned}$$

and

$$\begin{aligned}f^{\rho^2}(0) &= f\left(\rho^2(0)\right) \\ &= f(-1) \\ &= 1+4 \\ &= 5.\end{aligned}$$

Moreover,

Example

$$
\begin{aligned}
\rho^3\sigma^2(0) &= \rho^3\sigma(\sigma(0)) \\
&= \rho^3(\sigma(0)) \\
&= \rho^3(0) \\
&= \rho(\rho(\rho(0)) \\
&= \rho(\rho(-1)) \\
&= \rho(-1) \\
&= -1
\end{aligned}
$$

and

Example

$$\begin{aligned} f^{\sigma^2\rho^3}(0) &= f\left(\rho^3\sigma^2(0)\right) \\ &= f(-1) \\ &= 1+4 \\ &= 5. \end{aligned}$$

Let $t=-1$. Then

$$\begin{aligned} \rho^2(-1) &= \rho(\rho(-1)) \\ &= \rho(-1) \\ &= -1 \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\rho^2}(-1) &= f\left(\rho^2(-1)\right) \\
&= f(-1) \\
&= 1+4 \\
&= 5.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\sigma\rho\sigma(-1) &= \sigma(\rho(\sigma(-1))) \\
&= \sigma(\rho(0)) \\
&= \sigma(-1) \\
&= 0
\end{aligned}
$$

and

Example

$$
\begin{aligned}
f^{\sigma\rho\sigma}(-1) &= f(\sigma\rho\sigma(-1)) \\
&= f(0) \\
&= 1.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\rho^3\sigma^2(-1) &= \rho^3\sigma(\sigma(-1)) \\
&= \rho^3(\sigma(0)) \\
&= \rho^3(0) \\
&= \rho^2(\rho(0)) \\
&= \rho^2(-1) = \rho(\rho(-1)) = \rho(-1) = -1
\end{aligned}
$$

and

Example

$$
\begin{aligned}
f^{\sigma^2\rho^3}(-1) &= f\left(\rho^3\sigma^2(-1)\right) \\
&= f(-1) \\
&= 1+4 \\
&= 5.
\end{aligned}
$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$ and

$$f(t) = \frac{2-t}{4+3t}, \quad t \in \mathbb{T}.$$

We will find $f^{\sigma\rho^2\sigma^4}(1)$ and $f^{\rho\sigma\rho}(27)$. By Example 57, we have $\sigma(t) = 3t, \quad t \in \mathbb{T}$, and by Example **??**, we find

$$\rho(t) = \begin{cases} 1 & \text{if} \quad t = 1 \\ \frac{t}{3} & \text{if} \quad t \in 3^{\mathbb{N}}. \end{cases}$$

Then

Example

$$\begin{aligned}\sigma^4\rho^2\sigma(1) &= \sigma^4\rho^2(\sigma(1)) = \sigma^4\rho^2(3)\\ &= \sigma^4\rho(\rho(3)) = \sigma^4(\rho(1)) = \sigma^4(1) = \sigma^3(\sigma(1))\\ &= \sigma^3(3) = \sigma^2(\sigma(3)) = \sigma^2(9) = \sigma(\sigma(9)) = \sigma(27) = 81\end{aligned}$$

and

$$\rho\sigma\rho(27) = \rho(\sigma(\rho(27) = \rho(\sigma(9)) = \rho(27) = 9.$$

Hence,

$$\begin{aligned}f^{\sigma\rho^2\sigma^4}(1) &= f\left(\sigma^4\rho^2\sigma(1)\right) = f(81)\\ &= \frac{2-81}{4+3\cdot 81} = \frac{-79}{4+243} = -\frac{79}{247}\end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\rho\sigma\rho}(27) &= f(\rho\sigma\rho(27)) \\
&= f(9) \\
&= \frac{2-9}{4+3\cdot 7} \\
&= \frac{-7}{4+27} \\
&= -\frac{7}{31}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$ and

$$f(t) = \sqrt{1 + t + t^2}, \quad t \in \mathbb{T}.$$

We will find

$$f^{\rho^2}(t), \quad f^{\sigma^3\rho^2}(t), \quad f^{\rho^2\sigma^3}(t), \quad t \in \mathbb{T}.$$

We have

$$\sigma(t) = \begin{cases} \sqrt{t} & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ 1 & \text{if} \quad t = \frac{1}{2} \\ 0 & \text{if} \quad t = 0 \\ 1 & \text{if} \quad t = 1 \end{cases}$$

and by Example **??**, we have

Example

$$\rho(t) = \begin{cases} t^2 & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \\ 0 & \text{if} \quad t = 0 \\ \frac{1}{2} & \text{if} \quad t = 1. \end{cases}$$

Thus, we have the following cases.

Example

Let $t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\}$. Then

$$\rho^2(t) = \rho(\rho(t)) = \rho\left(t^2\right) = (\rho(t))^2 = t^4$$

and

$$\begin{aligned}
\sigma^3\rho^2(t) &= \sigma^3(\rho^2(t)) = \sigma^3(t^4) \\
&= \sigma^2(\sigma(t^4)) = \sigma^2(t^2) = \sigma(\sigma(t^2)) = \sigma(t) = \sqrt{t},
\end{aligned}$$

and

$$\begin{aligned}
\rho^2\sigma^3(t) &= \rho^2\sigma^2(\sigma(t)) = \rho^2\sigma^2(\sqrt{t}) = \rho^2\sigma(\sigma(\sqrt{t})) \\
&= \rho^2(\sigma(\sqrt[4]{t})) = \rho^2(\sqrt[8]{t}) = \rho(\rho(\sqrt[8]{t})) = \rho(\sqrt[4]{t}) = \sqrt{t}.
\end{aligned}$$

Hence,

Example

$$f^{\rho^2}(t) = f(\rho^2(t)) = f(t^2) = \sqrt{1 + t^4 + t^8}$$

and

$$f^{\sigma^3\rho^2}(t) = f(\rho^2(\sigma^3(t))) = f(\sqrt{t}) = \sqrt{1 + \sqrt{t} + t},$$

and

$$f^{\rho^2\sigma^3}(t) = f(\sigma^3\rho^2(t)) = f(\sqrt{t}) = \sqrt{1 + \sqrt{t} + t}.$$

Example

Let $t = 0$. Then

$$\rho^2(0) = \rho(\rho(0)) = \rho(0) = 0$$

and

$$\begin{aligned}\sigma^3\rho^2(0) &= \sigma^3(\rho^2(0)) = \sigma^3(0) = \sigma^2(\sigma(0)) \\ &= \sigma^2(0) = \sigma(\sigma(0)) = \sigma(0) = 0,\end{aligned}$$

and

$$\begin{aligned}\rho^2\sigma^3(0) &= \rho^2\sigma^2(\sigma(0)) = \rho^2\sigma(\sigma(0)) \\ &= \rho^2(\sigma(0)) = \rho^2(0) = \rho(\rho(0)) = \rho(0) = 0.\end{aligned}$$

Hence,

$$f^{\rho^2}(0) = f(\rho^2(0)) = f(0) = 1$$

and

Example

$$f^{\sigma^2 3\rho^2}(0) = f(\rho^2\sigma^3(0)) = f(0) = 1,$$

and

$$f^{\rho^2\sigma^3}(0) = f(\sigma^3\rho^2(0)) = f(0) = 1.$$

Example

Let $t = \frac{1}{2}$. Then $\rho^2\left(\frac{1}{2}\right) = \rho\left(\rho\left(\frac{1}{2}\right)\right) = \rho\left(\frac{1}{4}\right) = \frac{1}{16}$ and

$$\begin{aligned}\sigma^3\rho^2\left(\frac{1}{2}\right) &= \sigma^3\left(\rho^2\left(\frac{1}{2}\right)\right) = \sigma^3\left(\frac{1}{16}\right) = \sigma^2\left(\sigma\left(\frac{1}{16}\right)\right) \\ &= \sigma^2\left(\frac{1}{4}\right) = \sigma\left(\sigma\left(\frac{1}{4}\right)\right) = \sigma\left(\frac{1}{2}\right) = 1,\end{aligned}$$

and

$$\begin{aligned}\rho^2\sigma^3\left(\frac{1}{2}\right) &= \rho^2\sigma^2\left(\sigma\left(\frac{1}{2}\right)\right) \\ &= \rho^2\sigma(\sigma(1)) = \rho^2\sigma(1) = \rho^2(\sigma(1)) \\ &= \rho^2(1) = \rho(\rho(1)) = \rho\left(\frac{1}{2}\right) = \frac{1}{4}.\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
f^{\rho^2}\left(\frac{1}{2}\right) &= f\left(\rho^2\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{16}\right) \\
&= \sqrt{1+\frac{1}{16}+\frac{1}{256}} = \sqrt{\frac{256+16+1}{256}} = \frac{\sqrt{273}}{16}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\sigma^3\rho^2}\left(\frac{1}{2}\right) &= f\left(\rho^2\sigma^3\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{4}\right) \\
&= \sqrt{1+\frac{1}{4}+\frac{1}{16}} = \sqrt{\frac{16+4+1}{16}} = \frac{\sqrt{21}}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\rho^2\sigma^3}\left(\frac{1}{2}\right) &= f\left(\sigma^3\rho^2\left(\frac{1}{2}\right)\right) \\
&= f(1) = \sqrt{1+1+1} = \sqrt{3}.
\end{aligned}
$$

Example

Let $t = 1$. Then $\rho^2(1) = \rho(\rho(1)) = \rho\left(\frac{1}{2}\right) = \frac{1}{4}$ and

$$\begin{aligned}\sigma^3\rho^2(1) &= \sigma^3(\rho^2(1)) = \sigma^3\left(\frac{1}{4}\right) = \sigma^2\left(\sigma\left(\frac{1}{4}\right)\right)\\ &= \sigma^2\left(\frac{1}{2}\right) = \sigma\left(\sigma\left(\frac{1}{2}\right)\right) = \sigma(1) = 1,\end{aligned}$$

and

$$\rho^2\sigma^3(1) = \rho^2\sigma^2(\sigma(1)) = \rho^2\sigma(\sigma(1)) = \rho^2(\sigma(1)) = \rho^2(1) = \frac{1}{4}.$$

Hence,

$$f^{\rho^2}(1) = f(\rho^2(1)) = f\left(\frac{1}{4}\right) = \sqrt{1 + \frac{1}{4} + \frac{1}{16}} = \frac{\sqrt{21}}{4}$$

and

Example

$$f^{\sigma^3\rho^2}(1) \;=\; f(\rho^2\sigma^3(1)) = f\left(\frac{1}{4}\right) = \frac{\sqrt{21}}{4},$$

and

$$f^{\rho^2\sigma^3}(1) \;=\; f(\sigma^3\rho^2(1)) = f(1) = \sqrt{1+1+1} = \sqrt{3}.$$

The classical mathematical induction is a concept that helps to prove mathematical results and theorems for all natural numbers. The principle of the classical mathematical induction is a specific technique that is used to prove certain statements in algebra which are formulated in terms of n, where n is a natural number. Any mathematical statement, expression is proved based on the premise that it is true for $n = 1$, $n = k$, and then it is proved for $n = k + 1$.

Theorem (Induction Principle)

Let $t_0 \in \mathbb{T}$ *and assume that*

$$\{S(t) : \; t \in [t_0, \infty)\}$$

is a family of statements satisfying

- i $S(t_0)$ *is true.*
- ii *If* $t \in [t_0, \infty)$ *is right-scattered and* $S(t)$ *is true, then* $S(\sigma(t))$ *is true.*
- iii *If* $t \in [t_0, \infty)$ *is right-dense and* $S(t)$ *is true, then there exists a neighbourhood* U *of* t *such that* $S(s)$ *is true for all* $s \in U \cap (t, \infty)$.
- iv *If* $t \in (t_0, \infty)$ *is left-dense and* $S(s)$ *is true for* $s \in [t_0, t)$, *then* $S(t)$ *is true.*

Then $S(t)$ *is true for all* $t \in [t_0, \infty)$.

Proof.

Let

$$S^* = \{t \in [t_0, \infty) : S(t) \quad isnottrue\}.$$

We assume that $S^* \neq \emptyset$. Let $\inf S^* = t^*$. Because $\mathbb{T}$ is closed, we have $t^* \in \mathbb{T}$.

1. If $t^* = t_0$, then $S(t^*)$ is true.
2. If $t^* \neq t_0$ and $t^* = \rho(t^*)$, then, using *iv, we get that* $S(t^*)$ *is true.*
3. *If* $t^* \neq t_0$ *and* $\rho(t^*) < t^*$, *then* $\rho(t^*)$ *is right-scattered. Since* $S(\rho(t^*))$ *is true, we get that* $S(t^*)$ *is true.*

Consequently, $t^* \notin S^*$. □

Proof.

If we suppose that t^* is right-scattered, then, using that $S(t^*)$ is true and *ii, we conclude that* $S(\sigma(t^*))$ *is true, which is a contradiction. From the definition of* t^**, it follows that* $t^* \neq \sup \mathbb{T}$*. Since* t^* *is not right-scattered and* $t^* \neq \sup \mathbb{T}$*, we obtain that* t^* *is right-dense. Because* $S(t^*)$ *is true, using iii, there exists a neighbourhood* U *of* t^* *such that* $S(s)$ *is true for all* $s \in U \cap (t^*, \infty)$*, which is a contradiction. Consequently,* $S^* = \emptyset$*.* □

Theorem (Dual Version of Induction Principle)

Let $t_0 \in \mathbb{T}$ *and assume that*

$$\{S(t): \ t \in (-\infty, t_0]\}$$

is a family of statements satisfying

- *i* $S(t_0)$ *is true.*
- *ii If* $t \in (-\infty, t_0]$ *is left-scattered and* $S(t)$ *is true, then* $S(\rho(t))$ *is true.*
- *iii If* $t \in (-\infty, t_0]$ *is left-dense and* $S(t)$ *is true, then there exists a neighbourhood* U *of* t *such that* $S(s)$ *is true for all* $s \in U \cap (-\infty, t)$.
- *iv If* $t \in (-\infty, t_0)$ *is right-dense and* $S(s)$ *is true for* $s \in (t, t_0)$, *then* $S(t)$ *is true.*

Then $S(t)$ *is true for all* $t \in (-\infty, t_0]$.

Proof.

Let

$$S^* = \{t \in (-\infty, t_0] : \; S(t) \quad isnottrue\}.$$

Assume that $S^* \neq \emptyset$. Note that $t_0 \notin S^*$. Let

$$t^* = \sup S^*.$$

1. If $t^* = t_0$, then $S(t^*)$ is true.
2. If $t^* \neq t_0$ and $t^* = \sigma(t^*)$, then, using *iv, we obtain that $S(t^*)$ is true.*
3. *If $t^* \neq t_0$ and $t^* < \sigma(t^*)$, then $\sigma(t^*)$ is left-scattered, $\rho(\sigma(t^*)) = t^*$, and $S(\sigma(t^*))$ is true. Therefore, we conclude that $S(t^*)$ is true.*

Therefore, $t^* \notin S^*$. □

Proof.

If t^* is left-scattered, then, using *ii, we get that* $S(\rho(t^*))$ *is true, which is a contradiction. If* t^* *is left-dense, then, using iii, there exists a neighbourhood* U *of* t^* *such that* $S(s)$ *is true for all* $s \in U \cap (-\infty, t^*)$, *which is a contradiction. Consequently,* $S^* = \emptyset$. □

Example

Let $\mathbb{T} = 3^{\mathbb{N}}$. We will prove the inequality

$$\frac{1}{4}(s^2 - r^2) \leq \frac{1}{13}(s^3 - r^3) \quad \text{for any} \quad r \leq s, \quad r, s \in \mathbb{T}. \tag{1}$$

Fix $r, s \in \mathbb{T}$, $r \leq s$. Let

$$S(t) : \frac{1}{4}(t^2 - r^2) \leq \frac{1}{13}(t^3 - r^3), \quad t \in [r, s]_{\mathbb{T}}. \tag{2}$$

The statement $S(r)$ is true. Observe that any $t \in \mathbb{T}$ is right-scattered. Assume that $S(t)$ is true. Then $\sigma(t) = 3t$ and $t^2 \leq t^3$. Hence, using (2), we find $\frac{1}{4}(t^2 - r^2) + 2t^2 \leq \frac{1}{13}(t^3 - r^3) + 2t^3$, or $\frac{1}{4}(9t^2 - r^2) \leq \frac{1}{13}(27t^3 - r^3)$. Thus, $\frac{1}{4}((\sigma(t))^2 - r^2) \leq \frac{1}{3}((\sigma(t))^3 - r^3)$. Consequently $S(\sigma(t))$ is true. Applying the induction principle, we conclude that (2) is true for any $t \in [r, s]_{\mathbb{T}}$ and then (1) is true for any $r, s \in \mathbb{T}$, $r \leq s$.

Example

Let $\mathbb{T} = [-2, -1) \cup \left\{-\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{0\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{4}, \frac{11}{6}\right]$. We will prove

$$S(t) : \sqrt{t+2} > t \quad \text{for any} \quad t \in \mathbb{T}.$$

We have the following cases. For $t = -2$, we have $0 > -2$, i.e., the given inequality is true. Let $t \in \mathbb{T}$ be right-scattered. Then, we have the following cases. Let $t = -\frac{1}{n}$ for some $n \in \mathbb{N}$. Then $\sqrt{-\frac{1}{n}+2} > 0 > -\frac{1}{n}$, i.e., $S(t)$ is true. Note that $\sigma(t) = -\frac{1}{n+1}$ and $\sqrt{-\frac{1}{n+1}+2} > 0 > -\frac{1}{n+1}$, i.e., $S(\sigma(t))$ is true. Let $t = \frac{1}{n}$ for some $n \in \mathbb{N}$, $n \geq 2$. Then $\frac{1}{n} > \frac{1}{n^2}$ and $\frac{1}{n} + 2 > \frac{1}{n^2}$, whereupon $\sqrt{\frac{1}{n}+2} > \frac{1}{n}$, i.e., $S(t)$ is true. Note that $\sigma(t) = \frac{1}{n-1}$. Then $\frac{1}{n-1} > \frac{1}{(n-1)^2}$ and $\frac{1}{n-1} + 2 > \frac{1}{(n-1)^2}$, whereupon $\sqrt{\frac{1}{n-1}+2} > \frac{1}{n-1}$. Thus, $S(\sigma(t))$ is true.

Example

Let $t = \frac{3}{2}$. Then

$$\sqrt{\frac{3}{2}+2} \;=\; \sqrt{\frac{7}{2}} > \sqrt{\frac{9}{4}} = \frac{3}{2},$$

i.e., $S(t)$ is true. Note that $\sigma(t) = \frac{7}{4}$ and

$$\sqrt{\frac{7}{4}+2} \;=\; \sqrt{\frac{15}{4}} > \sqrt{\frac{49}{16}} = \frac{7}{4},$$

i.e., $S(\sigma(t))$ is true. Let t is right-dense. Then
$t \in [-2,-1) \cup \{0\} \cup \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{4}, \frac{11}{6}\right]$. We have the following cases. Let
$t \in [-2,-1) \cup \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{4}, \frac{11}{6}\right]$. Then
$t^2 - t - 2 < 0, \quad t \in \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{4}, \frac{11}{6}\right]$, and hence,
$t^2 < t + 2, \quad t \in \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{4}, \frac{11}{6}\right]$, whereupon
$\sqrt{t+2} > t, \quad t \in \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{4}, \frac{11}{6}\right]$. For $t \in [-2,-1)$ the inequality is true.

Example

Let $t = 0$. Then $\sqrt{2} > 0$ and by point 2 it follows that there is a neighbourhood U of 0 so that $S(s)$ is true for any $s \in U \cap \mathbb{T}$. Let t is left0dense. Then $t \in [-2, -1) \cup \{0\} \cup \left(1, \frac{3}{2}\right] \cup \left(\frac{7}{4}, \frac{11}{6}\right]$. By points 2 and 3, we get that if $s \in [t_0, t)_{\mathbb{T}}$ for some $t_0 \in \mathbb{T}$ and $S(s)$ is true, then $S(t)$ is true. By the induction principle, it follows that $S(t)$ is true for any $t \in \mathbb{T}$.

Example

Let $\mathbb{T} =$
$(-\infty, -8) \cup \left\{-7 - \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{-7\} \cup \left\{-7 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \{8\} \cup \left\{8 + \frac{1}{n}\right\}_{n\in\mathbb{N}} \cup 10^{\mathbb{N}}$.
We will prove

$$S(t) : \frac{2t - 15}{t + 5} \geq 0 \quad \text{for any} \quad t \in \mathbb{T}. \tag{3}$$

We will prove that (3) holds for any $t \in (-\infty, -6]_{\mathbb{T}}$ using the dual induction principle. We have

$$\frac{2 \cdot (-6) - 15}{-6 + 5} = \frac{-12 - 15}{-1} = 27 > 0,$$

i.e., $S(-6)$ is true. Let $t \in (-\infty, -6]_{\mathbb{T}}$ is left-scattered. Then we have the following cases. Let $t = -7 - \frac{1}{n}$ for some $n \in \mathbb{N}$, $n \geq 2$. Then $\rho(t) = -7 - \frac{1}{n-1}$. We have

$$\frac{2\left(-7 - \frac{1}{n}\right) - 15}{-7 - \frac{1}{n} + 5} = \frac{-14 - \frac{2}{n} - 15}{-2 - \frac{1}{n}} = \frac{29 + \frac{2}{n}}{2 + \frac{1}{n}} > 0,$$

Example

Let $t = -8$. Then t is left-dense and (3) holds for any $t \in (-\infty, -8]_{\mathbb{T}}$. Thus, there is a neighbourhood U of -8 such that (3) holds for any $t \in U \cap (-\infty, -6]_{\mathbb{T}}$. Let $t = -7 + \frac{1}{n}$ for some $n \in \mathbb{N}$. Then $\rho(t) = -7 + \frac{1}{n+1}$. We have

$$\frac{2\left(-7+\frac{1}{n}\right)-15}{-7+\frac{1}{n}+5} = \frac{-14+\frac{2}{n}-15}{-2+\frac{1}{n}} = \frac{29-\frac{2}{n}}{2-\frac{1}{n}} > 0,$$

because $29 - \frac{2}{n} > 0$ and $2 - \frac{1}{n} > 0$. Thus, $S(t)$ is true. Next,

$$\frac{2\left(-7+\frac{1}{n+1}\right)-15}{-7+\frac{1}{n+1}+5} = \frac{-14+\frac{2}{n+1}-15}{-2+\frac{1}{n+1}} = \frac{29-\frac{2}{n+1}}{2-\frac{1}{n+1}} > 0,$$

Example

$29 - \frac{2}{n+1} > 0$ and $2 - \frac{1}{n+1} > 0$. Therefore $S(\rho(t))$ is true. Let $t \in (-\infty, -6]_{\mathbb{T}}$ is left-dense. Then we have the following cases. Let $t \in (-\infty, -8)_{\mathbb{T}}$. Then $\frac{2t-15}{t+5} > 0$ and $\frac{2s-15}{s+5} > 0$ for any $s \in (-\infty, t)_{\mathbb{T}}$. Let $t = -7$. By 1.2.1, it follows that there is a neighbourhood U of -7 such that $S(s)$ holds for any $s \in U \cap (-\infty, -7)_{\mathbb{T}}$. Let $t \in (-\infty, -6]_{\mathbb{T}}$ is right-dense. Then we have the following cases. Let $t \in (-\infty, -8)_{\mathbb{T}}$. By the previous cases, we get that $S(s)$ is true for any $s \in (t, -6)_{\mathbb{T}}$ and $S(t)$ is true. Let $t = -7$. By 1.2.1, it follows that there is a neighbourhood U of -7 such that $S(s)$ holds for any $s \in U \cap (-\infty, -7)_{\mathbb{T}}$. Now, applying the dual induction principle, we conclude that $S(t)$ is true for any $t \in (-\infty, -6]_{\mathbb{T}}$.

Example

Now, we will prove that (3) holds for any $t \in [8, \infty)_{\mathbb{T}}$ using the induction principle. We have

$$\frac{2 \cdot 8 - 15}{8 + 5} = \frac{16 - 15}{13} = \frac{1}{13} > 0.$$

Thus, $S(8)$ is true. Let $t \in [8, \infty)_{\mathbb{T}}$ is right-scattered and $S(t)$ is true. Then we have the following cases. Let $t = 8 + \frac{1}{n}$ for some $n \in \mathbb{N}$, $n \geq 2$. Then $\sigma(t) = 8 + \frac{1}{n-1}$. We have

$$\frac{2\left(8 + \frac{1}{n}\right) - 15}{8 + \frac{1}{n} + 5} = \frac{16 + \frac{2}{n} - 15}{13 + \frac{1}{n}} = \frac{1 + \frac{2}{n}}{13 + \frac{1}{n}} > 0$$

and

Example

$$\frac{2\left(8+\frac{1}{n-1}\right)-15}{8+\frac{1}{n-1}+5} \quad = \quad \frac{16+\frac{2}{n-1}-15}{13+\frac{1}{n-1}} = \frac{1+\frac{2}{n-1}}{13+\frac{1}{n-1}} > 0,$$

i.e., $S(t)$ and $S(\sigma(t))$ is true. Let $t=9$. Then $\sigma(9)=10$. We have

$$\frac{2\cdot 9-15}{9+5} \quad = \quad \frac{18-15}{14} = \frac{3}{14} > 0$$

and

$$\frac{2\cdot 10-15}{10+5} \quad = \quad \frac{20-15}{15} = \frac{5}{15} = \frac{1}{3} > 0.$$

So, $S(9)$ and $S(10)$ are true. Let $t=10^n$ for some $n\in\mathbb{N}$. Then $\sigma(t)=10^{n+1}$. We have $\frac{2\cdot 10^n-15}{10^n+5}>0$ and $\frac{2\cdot 10^{n+1}-15}{10^{n+1}+5}>0$. Thus, $S(t)$ and $S(\sigma(t))$ are true.

Example

Let $t = 8$. We have

$$\frac{2 \cdot 8 - 15}{8 + 5} = \frac{16 - 15}{13} = \frac{1}{13} > 0,$$

i.e., $S(8)$ is true. By 2.2.1, it follows that there is a neighbourhood U of 8 such that $S(t)$ is true for any $t \in U \cap [8, \infty)_{\mathbb{T}}$. Note that there is no left-dense points in $(8, \infty)_{\mathbb{T}}$. Now, applying the induction principle, we conclude that $S(t)$ is true for any $t \in [8, \infty)_{\mathbb{T}}$.

Example

Let $\mathbb{T} = \{-1\} \cup \left\{-1 - \frac{1}{n^2}\right\}_{n\in\mathbb{N}} \cup \left\{\frac{1}{n^3}\right\}_{n\in\mathbb{N}} \cup (1, 2]$. We will prove that $S(t) : t^2 - t - 2 \leq 2$ for any $t \in \mathbb{T}$. We have

$$(-1)^2 - (1-1) - 2 = 1 + 1 - 2 = 0,$$

i.e., $S(-1)$ is true. Let $t \in \mathbb{T}$ be right-scattered. We have the following cases. Let $t = -1 + \frac{1}{n^2}$ for some $n \in \mathbb{N}$, $n \geq 2$. Then $\sigma(t) = -1 + \frac{1}{(n-1)^2}$. We have

$$\begin{aligned}\left(-1 + \frac{1}{n^2}\right)^2 - \left(-1 + \frac{1}{n^2}\right) - 2 &= 1 - \frac{2}{n^2} + \frac{1}{n^4} + 1 - \frac{1}{n^2} - 2 \\ &= \frac{1}{n^4} - \frac{3}{n^2} = \frac{1 - 3n^2}{n^4} \leq 0,\end{aligned}$$

i.e., $S(t)$ is true. Next,

Example

$$\left(-1+\frac{1}{(n-1)^2}\right)^2-\left(-1+\frac{1}{(n-1)^2}\right)-2$$

$$= \quad 1-\frac{2}{(n-1)^2}+\frac{1}{(n-1)^4}+1-\frac{1}{(n-1)^2}-2$$

$$= \quad \frac{1}{(n-1)^4}-\frac{3}{(n-1)^2}=\frac{1-3(n-1)^2}{(n-1)^4}\leq 0.$$

Thus, $S(\sigma(t))$ is true. Let $t=\frac{1}{n^3}$ for some $n\in\mathbb{N}$, ≥ 2. Then $\sigma(t)=\frac{1}{(n-1)^3}$. We have

Example

$$\left(-1+\frac{1}{n^3}\right)^2-\left(-1+\frac{1}{n^3}\right)-2 = 1-\frac{2}{n^3}+\frac{1}{n^6}+1-\frac{1}{n^3}-2$$
$$= \frac{1}{n^6}-\frac{3}{n^3}=\frac{1-3n^3}{n^6}\leq 0,$$

i.e., $S(t)$ is true. Moreover,

$$\left(-1+\frac{1}{(n-1)^3}\right)^2-\left(-1+\frac{1}{(n-1)^3}\right)-2$$

$$= 1-\frac{2}{(n-1)^3}+\frac{1}{(n-1)^6}+1-\frac{1}{(n-1)^3}-2$$
$$= \frac{1}{(n-1)^6}-\frac{3}{(n-1)^3}=\frac{1-3(n-1)^3}{(n-1)^6}\leq 0$$

and then $S(\sigma(t))$ is true.

Example

Let $t \in \mathbb{T}$ be right-dense and $S(t)$ be true. We have the following cases. Let $t = -1$. Then, by 2.1, it follows that there is a neighbourhood U of -1 such that $S(s)$ is true for any $s \in U \cap \mathbb{T}$. Let $t = 0$. Then, by 2.2, it follows that there is a neighbourhood U of 0 such that $S(s)$ is true for any $s \in U \cap \mathbb{T}$. Let $t \in (1, 2]$. Then $t^2 - t - 2 \leq 0$ and there is a neighbourhood U of t such that $S(s)$ is true for any $s \in U \cap \mathbb{T}$. Let $t \in \mathbb{T}$ be left0dense and $S(s)$ is true for any $s \in [-1, t)$. We have that $t \in (1, 2]$. By the previous cases it follows that $S(t)$ is true. Now, applying the induction principle, we conclude that $S(t)$ is true for any $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = \mathbb{N}$. We will prove that

$$S(n) : 1^4+2^4+\cdots+n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) \quad \text{for any} \quad n \in \mathbb{T}.$$

We have

$$\frac{1}{30} \cdot 1 \cdot (1+1) \cdot (2 \cdot 1 + 1)(3 \cdot 1^2 + 3 \cdot 1 - 1) = \frac{1}{30} \cdot 2 \cdot 3 \cdot 5 = 1 = 1^4.$$

Thus, $S(1)$ is true. Firstly, note that any point of $\mathbb{T}$ is right-scattered. Assume that $S(n)$ is true for some $n \in \mathbb{T}$. We have $\sigma(n) = n+1$. Then

$$1^4 + 2^4 + \cdots + (n+1)^4 = 1^4 + 2^4 + \cdots + n^4 + (n+1)^4$$

$$\begin{aligned} &= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) + (n+1)^4 \\ &= (n+1)\left(\frac{1}{30}n(2n+1)(3n^2+3n-1) + (n+1)^3\right) \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{1}{30}(n+1)\Big(6n^4 + 6n^3 - 2n^2 + 3n^3 + 3n^2 \\
&\quad -n + 30n^3 + 90n^2 + 90n + 30\Big) \\
&= \frac{1}{30}(n+1)\left(6n^4 + 39n^3 + 91n^2 + 89n + 30\right) \\
&= \frac{1}{30}(n+1)(n+2)\left(6n^3 + 27n^2 + 37n + 15\right) \\
&= \frac{1}{30}(n+1)(n+2)(2n+3)(3n^2 + 9n + 5) \\
&= \frac{1}{30}(n+1)(n+2)(2n+3)(3n^2 + 6n + 3 + 3n + 3 - 1) \\
&= \frac{1}{30}(n+1)(n+2)(2n+3)\left(3(n+1)^2 + 3(n+1) - 1\right).
\end{aligned}
$$

Thus, $S(\sigma(n))$ is true. Applying the induction principle, we conclude that $S(n)$ is true for any $n \in \mathbb{N}$.

Appendix

We start by defining the forward jump operator.

Definition

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ in the following manner

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$. Then, $t = \sigma(t)$ if t is a maximum of $\mathbb{T}$.

Note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

Example

Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{Z}$ such that $t = hn$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned}
\sigma(t) &= \inf\{s = hp, p \in \mathbb{Z} : hp > hn\} \\
&= h(n+1) \\
&= hn + h \\
&= t + h.
\end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = 3^n$. Hence, applying the definition for forward jump operators, we find

$$\begin{aligned} \sigma(t) &= \inf \{3^s, s \in \mathbb{N}_0 : 3^s > 3^n\} \\ &= 3^{n+1} \\ &= 3 \cdot 3^n \\ &= 3t. \end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{N}_0^k$, where $k \in \mathbb{N}$ is fixed. Take $t \in \mathbb{T}$ arbitrarily. Then, there is a $n \in \mathbb{N}_0$ such that $t = n^k$. Hence, $n = \sqrt[k]{t}$. Now, applying the definition for forward jump operators, we arrive at

$$\begin{aligned} \sigma(t) &= \inf\{s^k, s \in \mathbb{N}_0 : s^k > n^k\} \\ &= (n+1)^k \\ &= \left(\sqrt[k]{t} + 1\right)^k. \end{aligned}$$

Example

Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n, $n \in \mathbb{N}_0$, are the harmonic numbers. Take $n \in \mathbb{N}_0$ arbitrarily. Then, applying the definition for forward jump operators, we find

$$\begin{aligned}
\sigma(H_n) &= \inf\{H_s, s \in \mathbb{N}_0 : H_s > H_n\} \\
&= \inf\left\{H_s, s \in \mathbb{N}_0 : \sum_{k=1}^{s} \frac{1}{k} > \sum_{k=1}^{n} \frac{1}{k}\right\} \\
&= \sum_{k=1}^{n+1} \frac{1}{k} \\
&= H_{n+1}.
\end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$. Then

$$\begin{aligned}\mathbb{T} &= \bigcup_{k=0}^{\infty}[4k, 4k+1] \\ &= [0,1] \cup [4,5] \cup [8,9] \cup [12,13] \cup \dots .\end{aligned}$$

If $t \in [0,1)$, then, applying the definition for forward jump operators, we find

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t.\end{aligned}$$

Example

If $t = 1$, then

$$\sigma(1) = \inf\{s \in \mathbb{T} : s > 1\}$$

$$= 4.$$

Let now, $k \in \mathbb{N}$ be arbitrarily chosen. If $t \in [4k, 4k+1)$, then we have

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

$$= t.$$

Example

If $t = 4k + 1$, then

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > 4k + 1\} \\ &= 4(k+1) \\ &= 4k + 4 \\ &= 4k + 1 + 3 \\ &= t + 3. \end{aligned}$$

Example

Therefore

$$\sigma(t) = \begin{cases} t \quad \text{if} \quad t \in \bigcup_{k=0}^{\infty} [4k, 4k+1) \\ \\ t+3 \quad \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{4k+1\}. \end{cases}$$

Example

Let $\mathbb{T} = C$, where C is the Cantor set. We will find $\sigma(t)$ for $t \in \mathbb{T}$. For this aim, let C_1 denote the set of all left-hand end points of the open intervals that are removed. Then

$$C_1 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \leq k \leq m \right\}.$$

With C_2 we will denote the set of all right-hand end points of the open intervals that are removed. We have

$$C_2 = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : m \in \mathbb{N}, \quad a_k \in \{0, 2\} \quad \text{for any} \quad 1 \leq k \leq m \right\}.$$

Take $t \in C$ arbitrarily. We have the following cases.
Let $t \in C_1$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}}.$$

Example

Hence, we obtain

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \\ &= \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} + \frac{1}{3^{m+1}} \\ &= t + \frac{1}{3^{m+1}}.\end{aligned}$$

Let $t \in C_2$. Then

$$t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}}.$$

Hence,

Example

Let $t \in \mathbb{T} \backslash (C_1 \cup C_2)$. Then

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\} \\ &= t. \end{aligned}$$

Consequently

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}} & \text{if} \quad t \in C_1, t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \\ t & \text{if} \quad t \in \mathbb{T} \backslash C_1. \end{cases}$$

Example

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers with $\alpha_n > 0$, and

$$t_n = \sum_{k=0}^{n-1} \alpha_k, \quad n \in \mathbb{N},$$

and

$$\mathbb{T} = \{t_n : n \in \mathbb{N}\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned} \sigma(t_n) &= \inf\left\{x \in \mathbb{T} : s = \sum_{k=0}^{n-1} \alpha_k, \quad s > t_n\right\} \\ &= \sum_{k=0}^{n} \alpha_k = \sum_{k=0}^{n-1} + \alpha_n = t_n + \alpha_n. \end{aligned}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$n = -\frac{1}{t_n}$$

and

$$\begin{aligned} \sigma(t_n) &= \inf \left\{ s \in \mathbb{T} : s = -\frac{1}{m}, m \in \mathbb{N}, s > t_0 \right\} \\ &= -\frac{1}{n+1} = -\frac{1}{-\frac{1}{t_n}+1} = -\frac{t_n}{t_n - 1}. \end{aligned}$$

Example

Next, if $t \in \mathbb{N}_0$, then

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

$$= t + 1.$$

Consequently

$$\sigma(t) = \begin{cases} -\frac{t}{t-1} & \text{if} \quad t \in \left\{t_n = -\frac{1}{n} : n \in \mathbb{N}\right\}, \quad t = t_n \\ t+1 & \text{if} \quad t \in \mathbb{N}_0. \end{cases}$$

Example

Let

$$\mathbb{T} = \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. Take $n \in \mathbb{N}$ arbitrarily. Then

$$\begin{aligned} \sigma(t_n) &= \inf\{s \in \mathbb{T} : s > t_n\} \\ &= \left(\frac{1}{2}\right)^{2^{n-1}} = \left(\frac{1}{2}\right)^{2^n \cdot \frac{1}{2}} \\ &= \left(\left(\frac{1}{2}\right)^{2^n}\right)^{\frac{1}{2}} = \sqrt{t_n}. \end{aligned}$$

Example

Next,

$$t_0 = \frac{1}{2} \quad \text{and} \quad \sigma(t_0) = 1$$

and

$$\sigma(0) \;=\; 0, \quad \sigma(1) = 1.$$

Consequently

$$\sigma(t) = \begin{cases} \sqrt{t} \quad \text{if} \quad t \in \left\{ t_n = \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ 1 \quad \text{if} \quad t = \frac{1}{2} \\ 0 \quad \text{if} \quad t = 0 \\ 1 \quad \text{if} \quad t = 1. \end{cases}$$

Example

Let $U = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$ and

$$\mathbb{T} = U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{0,1,2\}.$$

We will find $\sigma(t)$, $t \in \mathbb{T}$. We have the following cases.
Let $t = 0$. Then

$$\sigma(0) = 0.$$

Let $t = \frac{1}{2}$. Then

$$\sigma\left(\frac{1}{2}\right) = \frac{3}{4}.$$

Let $t = 1$. Then

$$\sigma(1) = 1.$$

Example

Let $t = \frac{3}{2}$. Then

$$\sigma\left(\frac{3}{2}\right) = \frac{7}{4}.$$

Let $t = 2$. Then

$$\sigma(2) = 2.$$

Let $t = \frac{5}{2}$. Then

$$\sigma\left(\frac{5}{2}\right) = \frac{5}{2}.$$

Example

Let $t \in U \setminus \left\{\frac{1}{2}\right\}$. Then

$$t = \frac{1}{2^n}$$

and

$$\sigma(t) = \frac{1}{2^{n-1}} = \frac{2}{2^n} = 2t.$$

Let $t \in (1-U) \setminus \left\{\frac{1}{2}\right\}$. Then $t = 1 - \frac{1}{2^n}$ and $\frac{1}{2^n} = 1 - t$. Hence,

$$\sigma(t) = 1 - \frac{1}{2^{n+1}} = 1 - \frac{1}{2} \cdot \frac{1}{2^n} = 1 - \frac{1-t}{2} = \frac{1+t}{2}.$$

Example

Let $t \in (1+U) \setminus \left\{ \frac{3}{2} \right\}$. Then

$$t = 1 + \frac{1}{2^n}.$$

Hence,

$$\frac{1}{2^n} = t - 1$$

and

$$\sigma(t) = 1 + \frac{1}{2^{n-1}} = 1 + \frac{2}{2^n} = 1 + 2(t-1) = 2t - 1.$$

Example

Let $t \in (2 - U) \setminus \left\{ \frac{3}{2} \right\}$. Then

$$t = 2 - \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = 2 - t.$$

Hence,

$$\sigma(t) \quad = \quad 2 - \frac{1}{2^{n+1}} = 2 - \frac{1}{2} \cdot \frac{1}{2^n} = 2 - \frac{2-t}{2} = \frac{t+2}{2}.$$

Example

Let $t \in (2+U) \setminus \left\{ \frac{5}{2} \right\}$. Then

$$t = 2 + \frac{1}{2^n}$$

and

$$\frac{1}{2^n} = t - 2.$$

Hence,

$$\sigma(t) = 2 + \frac{1}{2^{n-1}} = 2 + \frac{2}{2^n} = 2 + 2(t-2) = 2(t-1).$$

Example

Consequently

$$\sigma(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \frac{3}{4} & \text{if} \quad t = \frac{1}{2} \\ 1 & \text{if} \quad t = 1 \\ \frac{7}{4} & \text{if} \quad t = \frac{3}{2} \\ 2 & \text{if} \quad t = 2 \\ \frac{5}{2} & \text{if} \quad t = \frac{5}{2} \\ 2t & \text{if} \quad t \in U \setminus \left\{ \frac{1}{2} \right\} \\ \frac{1+t}{2} & \text{if} \quad t \in (1-U) \setminus \left\{ \frac{1}{2} \right\} \end{cases}$$

Example

$$\sigma(t) = \begin{cases} 2t - 1 & \text{if} \quad t \in (1 + U) \setminus \left\{ \frac{3}{2} \right\} \\ \frac{t+2}{2} & \text{if} \quad t \in (2 - U) \setminus \left\{ \frac{3}{2} \right\} \\ 2(t-1) & \text{if} \quad t \in (2 + U) \setminus \left\{ \frac{5}{2} \right\}. \end{cases}$$

Exercise

Find $\sigma(t)$, $t \in \mathbb{T}$, where

1. $\mathbb{T} = h\mathbb{Z} + k$, $h > 0$, $k \in \mathbb{R}$.
2. $\mathbb{T} = (-2\mathbb{N}_0) \cup 3^{\mathbb{N}_0}$.
3. $\mathbb{T} = P_{3,7} \cup [4, 6]$.
4. $\mathbb{T} = 11^{\mathbb{N}_0} \cup \{0\}$.
5. $\mathbb{T} = [1, 2] \cup [3, 4] \cup [7, 8] \cup 9^{\mathbb{N}}$.