Time Scales Analysis
Lecture 5

Definition for Delta Derivative. Rules for Delta Differentiation

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Definition

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. We define $f^\Delta(t)$ to be the number, provided it exists, with the property that for any $\varepsilon > 0$, there exists a neighbourhood U of t, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the *delta* or *Hilger* derivative of f at t. We say that f is *delta* or *Hilger differentiable*, shortly *differentiable*, in $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T} \to \mathbb{R}$ is said to be the *delta derivative* or *Hilger derivative*, shortly *derivative*, of f in $\mathbb{T}^\kappa$.

If $\mathbb{T} = \mathbb{R}$, then the delta derivative coincides with the classical derivative.

Theorem

The delta derivative is well defined.

Proof.

Let $t \in \mathbb{T}^\kappa$. Suppose $f_1^\Delta(t)$ and $f_2^\Delta(t)$ are such that

$$|f(\sigma(t)) - f(s) - f_1^\Delta(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{2}|\sigma(t) - s|$$

and

$$|f(\sigma(t)) - f(s) - f_2^\Delta(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{2}|\sigma(t) - s|$$

for any $\varepsilon > 0$ and any s belonging to a neighbourhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$. Hence, if $s \neq \sigma(t)$, then □

Proof.

$$
\begin{aligned}
|f_1^{\Delta}(t) - f_2^{\Delta}(t)| &= \left| f_1^{\Delta}(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} + \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^{\Delta}(t) \right| \\
&\leq \quad \left| f_1^{\Delta}(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \right| + \left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^{\Delta}(t) \right| \\
&= \quad \frac{|f(\sigma(t)) - f(s) - f_1^{\Delta}(t)(\sigma(t) - s)|}{|\sigma(t) - s|} + \frac{|f(\sigma(t)) - f(s) - f_2^{\Delta}(t)(\sigma(t) -}{|\sigma(t) - s|} \\
&\leq \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$f_1^{\Delta}(t) = f_2^{\Delta}(t).$$

This completes the proof. $\square$

Remark

Let us assume that $\sup \mathbb{T} < \infty$ *and* $f^{\Delta}(t)$ *is defined at a point* $t \in \mathbb{T} \setminus \mathbb{T}^{\kappa}$ *with the same definition as given in Definition 1. Then the unique point* $t \in \mathbb{T} \setminus \mathbb{T}^{\kappa}$ *is* $\sup \mathbb{T}$*. Hence, for any* $\varepsilon > 0$*, there is a neighbourhood* $U = (t-\delta, t+\delta) \cap (\mathbb{T} \setminus \mathbb{T}^{\kappa})$*, for some* $\delta > 0$*, such that*

$$f(\sigma(t)) = f(s) = f(\sigma(\sup \mathbb{T})) = f(\sup \mathbb{T}), \quad s \in U.$$

Therefore, for any $\alpha \in \mathbb{R}$ *and* $s \in U$*, we have*

$$\begin{aligned} |f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| &= |f(\sup \mathbb{T}) - f(\sup \mathbb{T}) - \alpha(\sup \mathbb{T} - \sup \mathbb{T}) \\ &\leq \varepsilon |\sigma(t) - s|, \end{aligned}$$

i.e., any $\alpha \in \mathbb{R}$ *is the delta derivative of* f *at the point* $t \in \mathbb{T} \setminus \mathbb{T}^{\kappa}$*.*

Example

Let $f(t) = \alpha \in \mathbb{R}$. We will prove that $f^{\Delta}(t) = 0$ for any $t \in \mathbb{T}^{\kappa}$. Indeed, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0$, $s \in (t-1, t+1) \cap \mathbb{T}$ implies

$$|f(\sigma(t)) - f(s) - 0(\sigma(t) - s)| = |\alpha - \alpha| = 0 \leq \varepsilon |\sigma(t) - s|.$$

Example

Let $f(t) = t$, $t \in \mathbb{T}$. We will prove that $f^{\Delta}(t) = 1$ for any $t \in \mathbb{T}^{\kappa}$. Indeed, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0$, $s \in (t-1, t+1) \cap \mathbb{T}$ implies

$$\begin{aligned} |f(\sigma(t)) - f(s) - 1(\sigma(t) - s)| &= |\sigma(t) - s - (\sigma(t) - s)| \\ &= 0 \leq \varepsilon|\sigma(t) - s|. \end{aligned}$$

Example

Let $f(t) = t^2$, $t \in \mathbb{T}$. We will prove that $f^{\Delta}(t) = \sigma(t) + t$, $t \in \mathbb{T}^{\kappa}$. Indeed, for $t \in \mathbb{T}^{\kappa}$ and for any $\varepsilon > 0$, $s \in (t - \varepsilon, t + \varepsilon) \cap \mathbb{T}$ implies $|t - s| < \varepsilon$ and

$$|f(\sigma(t)) - f(s) - (\sigma(t) + t)(\sigma(t) - s)| = |(\sigma(t))^2 - s^2 - (\sigma(t) + t)(\sigma(t) -$$

$$= |(\sigma(t) - s)(\sigma(t) + s) - (\sigma(t) + t)(\sigma(t) - s)|$$

$$= |\sigma(t) - s||t - s| \leq \varepsilon|\sigma(t) - s|.$$

Theorem

Assume $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^\kappa$*. Then we have the following.*

1. *If* f *is differentiable at* t*, then* f *is continuous at* t*.*
2. *If* f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*
$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$
3. *If* t *is right-dense, then* f *is differentiable at* t *iff the limit* $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ *exists as a finite number. In this case,*
$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$
4. *If* f *is differentiable at* t*, then the "simple useful formula"*
$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$$
holds

Proof.

- Assume that f is differentiable at $t \in \mathbb{T}^{\kappa}$. Let $\varepsilon \in (0,1)$ be arbitrarily chosen. Set
$$\varepsilon^* = \frac{\varepsilon}{1 + |f^{\Delta}(t)| + 2\mu(t)}.$$
Since f is differentiable at t, there exists a neighbourhood U of t such that
$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon^* |\sigma(t) - s|.$$
Hence, for all $s \in U \cap (t - \varepsilon^*, t + \varepsilon^*)$, we have

□

Proof.

$$
\begin{aligned}
|f(t)-f(s)| &= |f(t)+f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s) \\
&\quad -f(\sigma(t))+f^{\Delta}(t)(\sigma(t)-s)| \\
&\leq |f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)| \\
&\quad +|f(\sigma(t))-f(t)-f^{\Delta}(t)(\sigma(t)-s)| \\
&\leq \varepsilon^{*}|\sigma(t)-s|+|f(\sigma(t))-f(t)-f^{\Delta}(t)(\sigma(t)-t)+f^{\Delta}(t)(s-t)| \\
&\leq \varepsilon^{*}|\sigma(t)-s|+|f(\sigma(t))-f(t)-f^{\Delta}(t)(\sigma(t)-t)| \\
&\quad +|f^{\Delta}(t)||s-t|
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&= \varepsilon^* \left(|\sigma(t) - s| + \mu(t) + |f^\Delta(t)| \right) \\
&= \varepsilon^* \left(|\sigma(t) - t + t - s| + \mu(t) + |f^\Delta(t)| \right) \\
&\leq \varepsilon^* \left(\sigma(t) - t + |t - s| + \mu(t) + |f^\Delta(t)| \right) \\
&= \varepsilon^* \left(2\mu(t) + |t - s| + |f^\Delta(t)| \right) \\
&\leq \varepsilon^* (2\mu(t) + \varepsilon^* + |f^\Delta(t)|) \leq \varepsilon^* (1 + 2\mu(t) + |f^\Delta(t)|) = \varepsilon,
\end{aligned}
$$

which completes the proof. □

Proof.

- Assume that f is continuous at t and t is right-scattered. By continuity, we have

$$\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s} = \frac{f(\sigma(t))-f(t)}{\sigma(t)-t} = \frac{f(\sigma(t))-f(t)}{\mu(t)}.$$

Therefore, for any $\varepsilon > 0$, there exists a neighbourhood U of t such that

$$\left|\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}-\frac{f(\sigma(t))-f(t)}{\mu(t)}\right| \le \varepsilon$$

for all $s \in U$, i.e.,

$\square$

Proof.

$$\left| f(\sigma(t)) - f(s) - \frac{f(\sigma(t)) - f(t)}{\mu(t)}(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Hence,

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

- Assume that t is right-dense. Let $\varepsilon > 0$ be arbitrarily chosen. Then f is differentiable at t iff there is a neighbourhood U of t such that

  $$|f(t) - f(s) - f^{\Delta}(t)(t - s)| \leq \varepsilon |t - s| \quad \text{for all} \quad s \in U,$$

  i.e., iff

  $$\left| \frac{f(t) - f(s)}{t - s} - f^{\Delta}(t) \right| \leq \varepsilon \quad \text{for all} \quad s \in U,$$

  i.e., iff $\lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f^{\Delta}(t)$.

Proof.

- Assume that f is differentiable at t.
  1. If t is right-dense, then $\sigma(t) = t$, $\mu(t) = 0$ and
  $$f(\sigma(t)) = f(t) = f(t) + \mu(t)f^{\Delta}(t).$$
  2. If t is right-scattered, then
  $$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$
  whereupon
  $$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

This completes the proof. $\square$

Example

Let

$$\mathbb{T} = \left\{ \frac{1}{2n+1} : n \in \mathbb{N}_0 \right\} \cup \{0\}$$

and $f(t) = \sigma(t)$ for $t \in \mathbb{T}$. We will find $f^{\Delta}(t)$ for $t \in \mathbb{T}^{\kappa} = \mathbb{T} \setminus \{1\}$. For

$$t \in \mathbb{T}^{\kappa} \setminus \{0\}, \quad t = \frac{1}{2n+1}, \quad n = \frac{1-t}{2t}, \quad n \in \mathbb{N},$$

we have

$$\begin{aligned} \sigma(t) &= \inf\left\{ s \in \mathbb{T} : \ s > \frac{1}{2n+1} \right\} = \frac{1}{2n-1} \\ &= \frac{1}{2\frac{1-t}{2t} - 1} = \frac{t}{1-2t} > t, \end{aligned}$$

i.e., any point $t = \frac{1}{2n+1}$, $n \in \mathbb{N}$, is right-scattered. At these points,

Example

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\
&= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\
&= 2\frac{(\sigma(t))^2}{(1-2\sigma(t))(\sigma(t)-t)} \\
&= 2\frac{\left(\frac{t}{1-2t}\right)^2}{\left(1-\frac{2t}{1-2t}\right)\left(\frac{t}{1-2t}-t\right)} \\
&= 2\frac{\frac{t^2}{(1-2t)^2}}{\frac{1-4t}{1-2t}\frac{2t^2}{1-2t}} = 2\frac{t^2}{2t^2(1-4t)} = \frac{1}{1-4t}.
\end{aligned}
$$

Let now $t = 0$. Then

Example

$$\sigma(0) = \inf\{s \in \mathbb{T} : \ s > 0\} = 0.$$

Consequently, $t = 0$ is right-dense. Also,

$$\lim_{h\to 0} \frac{\sigma(h) - \sigma(0)}{h} = \lim_{h\to 0} \frac{\frac{h}{1-2h} - 0}{h} = \lim_{h\to 0} \frac{1}{1-2h} = 1.$$

Therefore, $f^{\Delta}(0) = 1$. Altogether, $f^{\Delta}(t) = \frac{1}{1-4t}$ for all $t \in \mathbb{T}^{\kappa}$.

Example

Let $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$ and $f(t) = t^2$, $g(t) = \sigma(t)$ for $t \in \mathbb{T}$. We will find $f^{\Delta}(t)$ and $g^{\Delta}(t)$ for $t \in \mathbb{T}^{\kappa} = \mathbb{T}$. For $t \in \mathbb{T}$, $t = n^2$, $n = \sqrt{t}$, $n \in \mathbb{N}_0$, we have

$$\sigma(t) = \inf\{l^2 : \ l^2 > n^2, \ l \in \mathbb{N}_0\} = (n+1)^2 = (1+\sqrt{t})^2 > t.$$

Therefore, all points of $\mathbb{T}$ are right-scattered. We note that f and g are continuous functions in $\mathbb{T}$. Hence,

Example

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\
&= \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\
&= \sigma(t) + t \\
&= \left(1 + \sqrt{t}\right)^2 + t \\
&= t + 2\sqrt{t} + 1 + t \\
&= 1 + 2\sqrt{t} + 2t
\end{aligned}
$$

and

Example

$$
\begin{aligned}
g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\
&= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\
&= \frac{(1 + \sqrt{\sigma(t)})^2 - \sigma(t)}{\sigma(t) - t} \\
&= \frac{\sigma(t) + 2\sqrt{\sigma(t)} + 1 - \sigma(t)}{\sigma(t) - t} \\
&= \frac{1 + 2\sqrt{\sigma(t)}}{\sigma(t) - t} \\
&= \frac{1 + 2(1 + \sqrt{t})}{(1 + \sqrt{t})^2 - t} = \frac{3 + 2\sqrt{t}}{1 + 2\sqrt{t}}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \left\{\sqrt[4]{2n+1} : n \in \mathbb{N}_0\right\}$ and $f(t) = t^4$ for $t \in \mathbb{T}$. We will find $f^{\Delta}(t)$ for $t \in \mathbb{T}$. For $t \in \mathbb{T}$, $t = \sqrt[4]{2n+1}$, $n = \frac{t^4-1}{2}$, $n \in \mathbb{N}_0$, we have

$$\begin{aligned}\sigma(t) &= \inf\{\sqrt[4]{2l+1} : \sqrt[4]{2l+1} > \sqrt[4]{2n+1},\ l \in \mathbb{N}_0\} \\ &= \sqrt[4]{2n+3} = \sqrt[4]{t^4+2} > t.\end{aligned}$$

Therefore, every point of $\mathbb{T}$ is right-scattered. We note that the function f is continuous in $\mathbb{T}$. Hence,

Example

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\
&= \frac{(\sigma(t))^4 - t^4}{\sigma(t) - t} \\
&= (\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3 \\
&= \sqrt[4]{(t^4+2)^3} + t^2\sqrt[4]{t^4+2} + t\sqrt{t^4+2} + t^3.
\end{aligned}
$$

Example

Let $\mathbb{T} = \mathbb{Z}$ and f be differentiable at t. Note that all points of t are right-scattered and $\sigma(t) = t + 1$. Therefore,

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\
&= \frac{f(t+1) - f(t)}{t + 1 - t} \\
&= f(t+1) - f(t) \\
&= \Delta f(t),
\end{aligned}
$$

where Δ is the usual forward difference operator.

Theorem

Assume $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^\kappa$. *Then we have the following.*

1. *The sum* $f + g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with* $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.
2. *For any constant* α, $\alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with* $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$.

Proof.

1. Let $\varepsilon > 0$ be arbitrarily chosen. Since f and g are differentiable at t, there exist neighbourhoods U_1 and U_2 of t so that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{2}|\sigma(t) - s| \quad \text{for all} \quad s \in U_1$$

and

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{2}|\sigma(t) - s| \quad \text{for all} \quad s \in U_2.$$

Hence, for $s \in U_1 \cap U_2$, we have

$$\begin{aligned} &|f(\sigma(t)) + g(\sigma(t)) - f(s) - g(s) - (f^{\Delta}(t) + g^{\Delta}(t))(\sigma(t) - s)| \\ &\quad \leq \quad |f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \\ &\qquad + |g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)| \end{aligned}$$

Proof.

$$\leq \frac{\varepsilon}{2}|\sigma(t) - s| + \frac{\varepsilon}{2}|\sigma(t) - s|$$

$$= \varepsilon|\sigma(t) - s|,$$

which completes the proof.

1. Let $\alpha \neq 0$. Assume that $\varepsilon > 0$ is arbitrarily chosen. Since f is differentiable at t, there exists a neighbourhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{|\alpha|}|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

Hence, for $s \in U$, we have

$$|\alpha f(\sigma(t)) - \alpha f(s) - \alpha f^{\Delta}(t)(\sigma(t) - s)|$$

$$= |\alpha||f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)|$$

Proof.

$$\leq \quad |\alpha|\frac{\varepsilon}{|\alpha|} = \varepsilon,$$

which completes the proof. $\square$