Time Scales Analysis
Lecture 6

Rules for Delta Differentiation. Higher Order Delta Differentiation

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Theorem

Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$. Then we have the following.

1. *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t, and the "product rule"*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$$

   holds.

2. *If $g(t)g(\sigma(t)) \neq 0$, then the quotient $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is differentiable at t, and the "quotient rule"*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))} \quad \textit{holds.}$$

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Let also $[a, b] \subset \mathbb{T}$ be such that $t, \sigma(t) \in [a, b]$. Set

$$M = \max_{t \in [a,b]} |f(t)|$$

and

$$\varepsilon^* = \frac{\varepsilon}{1 + M + |g(\sigma(t))| + |g^{\Delta}(t)|}.$$

Since f is differentiable at t, there exists a neighbourhood U_1 of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon^* |\sigma(t) - s| \quad \text{for any} \quad s \in U_1.$$

$\square$

Proof.

Since g is differentiable at t, there exists a neighbourhood U_2 of t such that

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon^* |\sigma(t) - s| \quad \text{for any} \quad s \in U_2.$$

Since f is differentiable at t, f is continuous at t. Therefore, there exists a neighbourhood U_3 of t such that

$$|f(t) - f(s)| \leq \varepsilon^* \quad \text{for any} \quad s \in U_3.$$

Then, for any $s \in U_1 \cap U_2 \cap U_3 \cap [a, b]$, we get

$$\begin{aligned} & |f(\sigma(t))g(\sigma(t)) - f(s)g(s) - (f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t))(\sigma(t) - s)| \\ = \; & |f(\sigma(t))g(\sigma(t)) - f(s)g(\sigma(t)) - f^{\Delta}(t)g(\sigma(t))(\sigma(t) - s) \end{aligned}$$

□

Proof.

$$+f(s)g(\sigma(t)) + f^{\Delta}(t)g(\sigma(t))(\sigma(t)-s) + g^{\Delta}(t)f(s)(\sigma(t)-s)$$

$$-(f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t))(\sigma(t)-s) - f(s)g(s) - g^{\Delta}(t)f(s)(\sigma(t)$$

$$= |(f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t)-s))g(\sigma(t))$$

$$+f(s)(g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t)-s)) - (f(t)-f(s))g^{\Delta}(t)(\sigma(t)$$

$$\leq |g(\sigma(t))||f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t)-s)|$$

$$+|f(s)||g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t)-s)| + |f(t)-f(s)||g^{\Delta}(t)||\sigma(t)$$

$$\leq \varepsilon^{*}|g(\sigma(t))||\sigma(t)-s|$$

Proof.

$$
\begin{aligned}
& +\varepsilon^*|f(s)||\sigma(t)-s| + \varepsilon^*|g^{\Delta}(t)||\sigma(t)-s| \\
\leq \quad & \varepsilon^*(|g(\sigma(t))| + M + |g^{\Delta}(t)|)|\sigma(t)-s| \\
\leq \quad & \varepsilon|\sigma(t)-s|,
\end{aligned}
$$

which completes the proof.

1. Let $\varepsilon > 0$ be arbitrarily chosen. Since f and g are differentiable at t, they are continuous at t. Because $g(t) \neq 0$, there exists a neighbourhood U of t such that

$$|g(s)| \geq m_1 > 0 \quad \text{for all} \quad s \in U,$$

for some constant m_1.

Proof.

Let $[a, b] \subset \mathbb{T}$ be such that $\sigma(t), t \in [a, b]$. Set

$$M_1 = \sup_{t \in [a,b]} |f(t)|, \quad M_2 = \sup_{t \in [a,b]} |g(t)|$$

and

$$\varepsilon^* = \varepsilon \frac{m_1 |g(\sigma(t))g(t)|}{1 + 2M_1 M_2 + |g^{\Delta}(t)|}.$$

Since f and g are continuous at t, there exists a neighbourhood U_1 of t such that

$$|f(t)g(s) - f(s)g(t)| \leq \varepsilon^* \quad \text{and} \quad |g(s) - g(t)| \leq \varepsilon^*$$

for any $s \in U_1$. Since f is differentiable at t, there exists a neighbourhood U_2 of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon^* |\sigma(t) - s| \quad \text{for any} \quad s \in U_2.$$

□

Proof.

Since g is differentiable at t, there exists a neighbourhood U_3 of t such that

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon^* |\sigma(t) - s| \quad \text{for any} \quad s \in U_3.$$

Hence, for $s \in U_1 \cap U_2 \cap U_3 \cap U \cap [a, b]$, we get

$$\begin{aligned}
&\left| \frac{f(\sigma(t))}{g(\sigma(t))} - \frac{f(s)}{g(s)} - \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}(\sigma(t) - s) \right| \\
&= \frac{1}{|g(\sigma(t))g(s)g(t)|} \Big| f(\sigma(t))g(t)g(s) - f(s)g(t)g(\sigma(t)) \\
&\quad -(f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t))g(s)(\sigma(t) - s) \Big| \\
&= \frac{1}{|g(\sigma(t))g(s)g(t)|} \Big| (f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s))g(t)g(s)
\end{aligned}$$

Proof.

$$+f(s)g(t)g(s)+f^{\Delta}(t)g(t)g(s)(\sigma(t)-s)$$

$$-(g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s))f(s)g(t)$$

$$-g(s)f(s)g(t)-f(s)g^{\Delta}(t)g(t)(\sigma(t)-s)$$

$$-f^{\Delta}(t)g(t)g(s)(\sigma(t)-s)+f(t)g^{\Delta}(t)g(s)(\sigma(t)-s)\Big|$$

$$= \frac{1}{|g(\sigma(t))g(s)g(t)|}\Big|(f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s))g(t)g(s)$$

$$-(g(\sigma(t))-g(s)-g^{\Delta}(t)(\sigma(t)-s))f(s)g(t)$$

$$+g^{\Delta}(t)(f(t)g(s)-f(s)g(t))(\sigma(t)-s)\Big|$$

Proof.

$$\leq \frac{1}{|g(\sigma(t))g(s)g(t)|}\Big(|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)||g(t)||g(s)|$$

$$+|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)||f(s)||g(t)|$$

$$+|g^{\Delta}(t)||f(t)g(s) - f(s)g(t)||\sigma(t) - s|\Big)$$

$$\leq \frac{1}{|g(\sigma(t))g(s)g(t)|}(\varepsilon^* M_1 M_2|\sigma(t) - s| + \varepsilon^*|\sigma(t) - s|M_1 M_2$$

$$+|g^{\Delta}(t)|\varepsilon^*|\sigma(t) - s|)$$

$$\leq \varepsilon^* \frac{1}{|g(\sigma(t))g(s)g(t)|}(1 + 2M_1 M_2 + |g^{\Delta}(t)|)|\sigma(t) - s|$$

$\square$

Proof.

$$
\begin{aligned}
&\leq \quad \varepsilon^* \frac{1}{m_1|g(\sigma(t))g(t)|}(1 + 2M_1M_2 + |g^\Delta(t)|)|\sigma(t) - s| \\
&= \quad \varepsilon|\sigma(t) - s|,
\end{aligned}
$$

which completes the proof. □

Example

Let $\mathbb{T} = \left\{ t_n = -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \mathbb{N}_0$ and

$$f(t) = \frac{1+t^2}{1+7t}, \quad t \in \mathbb{T}.$$

We will find $f^{\Delta}(t)$, $t \in \mathbb{T}$. Note that any points $t \in \mathbb{T}$ are right-scattered. Let

$$g(t) = 1 + t^2, \quad h(t) = 1 + 7t, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned} g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\ &= \frac{1 + (\sigma(t))^2 - 1 - t^2}{\sigma(t) - t} = \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\ &= \frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} = \sigma(t) + t, \quad t \in \mathbb{T}, \end{aligned}$$

Example

$$
\begin{aligned}
h^{\Delta}(t) &= \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} \\
&= \frac{1 + 7\sigma(t) - 1 - 7t}{\sigma(t) - t} \\
&= \frac{7(\sigma(t) - t)}{\sigma(t) - t} \\
&= 7, \quad t \in \mathbb{T},
\end{aligned}
$$

and

$$h(\sigma(t)) = 1 + 7\sigma(t), \quad t \in \mathbb{T}.$$

Now, applying the rule for delta derivative of a quotient of two functions, we arrive at

Example

$$
\begin{aligned}
f^{\Delta}(t) &= \left(\frac{g}{h}\right)^{\Delta}(t) \\
&= \frac{g^{\Delta}(t)h(t) - g(t)h^{\Delta}(t)}{h(t)h(\sigma(t))} \\
&= \frac{(\sigma(t)+t)(1+7t) - (1+t^2)7}{(1+7t)(1+7\sigma(t))} \\
&= \frac{\sigma(t) + 7t\sigma(t) + t + 7t^2 - 7 - 7t^2}{(1+7t)(1+7\sigma(t))} \\
&= \frac{-7 + t + \sigma(t) + 7t\sigma(t)}{(1+7t)(1+7\sigma(t))}, \quad t \in \mathbb{T}.
\end{aligned}
$$

Then, we have the following cases.

Example

Let $t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$. Then

$$\sigma(t) = -\frac{t}{t-1}$$

and

$$\begin{aligned}
f^{\Delta}(t) &= \frac{-7 + t - \frac{t}{t-1} - \frac{7t^2}{t-1}}{(1+7t)\left(1 - \frac{7t}{t-1}\right)} \\
&= \frac{(-7+t)(t-1) - t - 7t^2}{(1+7t)(t-1-7t)} \\
&= \frac{-7t + 7 + t^2 - t - t - 7t^2}{-(1+7t)(1+6t)} \\
&= \frac{-6t^2 - 9t + 7}{-(1+7t)(1+6t)}
\end{aligned}$$

Example

Let $t \in \mathbb{N}_0$. Then

$$\sigma(t) = t + 1$$

and

$$\begin{aligned} f^{\Delta}(t) &= \frac{-7 + t + t + 1 + 7(t+1)t}{(1+7t)(1+7t+7)} \\ &= \frac{-6 + 2t + 7t^2 + 7t}{(1+7t)(8+7t)} \\ &= \frac{-6 + 9t + 7t^2}{(1+7t)(8+7t)}. \end{aligned}$$

Consequently

Example

$$f^{\Delta}(t) = \begin{cases} \frac{6t^2+9t-7}{(1+7t)(1+6t)} & \text{if} \quad t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \\ \\ \frac{7t^2+9t-6}{(1+7t)(8+7t)} & \text{if} \quad t \in \mathbb{N}_0. \end{cases}$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$ and

$$f(t) = \frac{1+7t}{4+5t}, \quad t \in \mathbb{T}.$$

We will find $f^{\Delta}(t)$, $t \in \mathbb{T}^{\kappa}$. Here $\mathbb{T}^{\kappa} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N}_0 \right\} \cup \{0\}$. We have the following cases. Let $t \in \mathbb{T}$ be right-scattered. Set

$$g(t) \;=\; 1 + 7t, \quad h(t) = 4 + 5t, \quad t \in \mathbb{T}^{\kappa}.$$

Then

Example

$$\begin{aligned} g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\ &= \frac{1 + 7\sigma(t) - 1 - 7t}{\sigma(t) - t} \\ &= \frac{7(\sigma(t) - t)}{\sigma(t) - t} \\ &= 7, \quad t \in \mathbb{T}^{\kappa}, \end{aligned}$$

and

$$\begin{aligned} h^{\Delta}(t) &= \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} = \frac{4 + 5\sigma(t) - 4 - 5t}{\sigma(t) - t} \\ &= \frac{5(\sigma(t) - t)}{\sigma(t) - t} = 5, \quad t \in \mathbb{T}^{\kappa}, \end{aligned}$$

$$h(\sigma(t)) = 4 + 5\sigma(t), \quad t \in \mathbb{T}^{\kappa}.$$

Example

Now, applying the rule for delta derivative of quotient of two functions, we arrive at

$$\begin{aligned} f^{\Delta}(t) &= \left(\frac{g}{h}\right)^{\Delta}(t) \\ &= \frac{g^{\Delta}(t)h(t) - g(t)h^{\Delta}(t)}{h(t)h(\sigma(t))} \\ &= \frac{7(4+5t) - 5(1+7t)}{(4+5t)(4+5\sigma(t))} \\ &= \frac{28+35t-5-35t}{(4+5t)(4+5\sigma(t))} = \frac{23}{(4+5t)(4+5\sigma(t))}. \end{aligned}$$

We have the following subcases. Let $t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\}$. Then

Example

$\sigma(t) = \sqrt{t}$ and

$$f^{\Delta}(t) = \frac{23}{(4+5t)(4+5\sqrt{t})}.$$

Let $t = \frac{1}{2}$. Then $\sigma\left(\frac{1}{2}\right) = 1$ and

$$\begin{aligned} f^{\Delta}\left(\frac{1}{2}\right) &= \frac{23}{\left(4+\frac{5}{2}\right)(4+5)} \\ &= \frac{23}{\frac{13}{2}\cdot 9} \\ &= b\frac{46}{117}. \end{aligned}$$

Example

Let $t \in \mathbb{T}$ be right-dense. Then $t = 0$. Hence,

$$\begin{aligned} f^{\Delta}(0) &= \lim_{s \to 0} \frac{f(s) - f(0)}{s} \\ &= \lim_{s \to 0} \frac{\frac{1+7s}{4+5s} - \frac{1}{4}}{s} \\ &= \lim_{s \to 0} \frac{4 + 28s - 4 - 5s}{4s(4+5s)} \\ &= \lim_{s \to 0} \frac{23s}{4s(4+5s)} \\ &= \lim_{s \to 0} \frac{23}{4(4+5s)} = \frac{23}{16}. \end{aligned}$$

Consequently

Example

$$f^{\Delta}(t) = \begin{cases} \frac{23}{(4+5t)(4+5\sqrt{t})} & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ \frac{46}{117} & \text{if} \quad t = \frac{1}{2} \\ \frac{23}{16} & \text{if} \quad t = 0. \end{cases}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = \{0\} \cup U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{1,2\}$$

and

$$f(t) = (2+t^2)(1-t^2), \quad t \in \mathbb{T}.$$

We will find $f^{\Delta}(t)$, $t \in \mathbb{T}^{\kappa}$. Here $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \left\{ \frac{5}{2} \right\}$. We have the following cases. Let $t \in \mathbb{T}^{\kappa}$ be right-scattered. Set

$$g(t) \;=\; 2 - t^2, \quad h(t) = 1 + t^2, \quad t \in \mathbb{T}^{\kappa}.$$

Then

Example

$$
\begin{aligned}
g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\
&= \frac{2 + (\sigma(t))^2 - 2 - t^2}{\sigma(t) - t} = \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\
&= \frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} = \sigma(t) + t, \quad t \in \mathbb{T}^{\kappa},
\end{aligned}
$$

and

$$
\begin{aligned}
h^{\Delta}(t) &= \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} \\
&= \frac{1 - (\sigma(t))^2 - 1 + t^2}{\sigma(t) - t} = -\frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\
&= -\frac{(\sigma(t) - t)(\sigma(t) + t)}{\sigma(t) - t} = -\sigma(t) - t, \quad t \in \mathbb{T}^{\kappa},
\end{aligned}
$$

and

Example

$$
\begin{aligned}
f^{\Delta}(t) &= (gh)^{\Delta}(t) \\
&= g^{\Delta}(t)h(\sigma(t)) + g(t)h^{\Delta}(t) \\
&= (\sigma(t_+t)\left(1-(\sigma(t))^2\right) + (2+t^2)(-\sigma(t)-t) \\
&= \sigma(t) - (\sigma(t))^3 + t - t(\sigma(t))^2 - 2\sigma(t) - 2t - t^2\sigma(t) \\
&= -(\sigma(t))^3 - \sigma(t) - t - t^3 - t(\sigma(t))^2 - t^2\sigma(t), \quad t \in \mathbb{T}^{\kappa}.
\end{aligned}
$$

Now, we will consider the following subcases. Let $t = \frac{1}{2}$. Then $\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$ and

Example

$$
\begin{aligned}
f^{\Delta}\left(\frac{1}{2}\right) &= -\left(\sigma\left(\frac{1}{2}\right)\right)^3 - \sigma\left(\frac{1}{2}\right) - \frac{1}{2} - \left(\frac{1}{2}\right)^3 - \frac{1}{2}\left(\sigma\left(\frac{1}{2}\right)\right)^2 - \left(\frac{1}{2}\right) \\
&= -\left(\frac{3}{4}\right)^3 - \frac{3}{4} - \frac{1}{2} - \frac{1}{8} - \frac{1}{2}\cdot\frac{3}{4} - \frac{1}{4}\cdot\frac{3}{4} \\
&= -\frac{27}{64} - \frac{3}{4} - \frac{1}{2} - \frac{1}{8} - \frac{3}{8} - \frac{3}{16} \\
&= -\frac{27}{64} - \frac{3}{4} - \frac{1}{2} - \frac{1}{2} - \frac{3}{16} \\
&= -\frac{27}{64} - \frac{3}{4} - 1 - \frac{3}{16} \\
&= \frac{-27 - 48 - 64 - 12}{64} \\
&= -\frac{151}{64}.
\end{aligned}
$$

Example

Let $t = \frac{3}{2}$. Then $\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$ and

$$
\begin{aligned}
f^{\Delta}\left(\frac{3}{2}\right) &= \left(\sigma\left(\frac{3}{2}\right)\right)^3 - \sigma\left(\frac{3}{2}\right) - \frac{3}{2} - \left(\frac{3}{2}\right)^3 - \frac{3}{2}\left(\sigma\left(\frac{3}{2}\right)\right)^2 - \left(\frac{3}{2}\right)^2 \sigma \\
&= -\left(\frac{7}{4}\right)^3 - \frac{7}{4} - \frac{3}{2} - \frac{27}{8} - \frac{3}{2}\cdot\frac{7}{4} - \frac{9}{4}\cdot\frac{7}{4} \\
&= -\frac{343}{64} - \frac{7}{4} - \frac{3}{2} - \frac{27}{8} - \frac{21}{8} - \frac{63}{16} \\
&= -\frac{343}{64} - \frac{7}{4} - \frac{3}{2} - \frac{49}{8} - \frac{63}{16} \\
&= \frac{-343 - 112 - 96 - 392 - 252}{64} \\
&= -\frac{1195}{64}.
\end{aligned}
$$

Example

Let $t \in U \setminus \left\{\frac{1}{2}\right\}$. Then $\sigma(t) = 2t$ and

$$\begin{aligned} f^{\Delta}(t) &= -(2t)^3 - 2t - t - t^3 - t(2t)^2 - t^2(2t) \\ &= -8t^3 - 3t - t^3 - 4t^3 - 2t^3 \\ &= -15t^3 - 3t. \end{aligned}$$

Example

Let $t \in (1-U)\setminus\left\{\frac{1}{2}\right\}$. Then $\sigma(t) = \frac{1+t}{2}$ and

$$
\begin{aligned}
f^{\Delta}(t) &= -\left(\frac{1+t}{2}\right)^3 - \frac{1+t}{2} - t - t^3 - t\left(\frac{1+t}{2}\right)^2 - t^2\frac{1+t}{2} \\
&= -\frac{(1+t)^3}{8} - \frac{1+t}{2} - t - t^3 - t\frac{(1+t)^2}{4} - t^2\frac{1+t}{2} \\
&= -\frac{(1+t)^3 + 4(1+t) + 8t + 8t^3 + 2t(1+t)^2 + 4t^2(1+t)}{8} \\
&= -\frac{t^3 + 3t^2 + 3t + 1 + 4t + 4 + 8t + 8t^3 + 2t(t^2+2t+1) + 4t^2}{8} \\
&= -\frac{13t^3 + 7t^2 + 15t + 5 + 2t^3 + 4t^2 + 2t}{8} \\
&= -\frac{15t^3 + 11t^2 + 17t + 5}{8}.
\end{aligned}
$$

Example

Let $t \in (1+U)\setminus\left\{\frac{3}{2}\right\}$. Then $\sigma(t) = 2t-1$ and

$$\begin{aligned}
f^{\Delta}(t) &= -(2t-1)^3 - (2t-1) - t - t^3 - t(2t-1)^2 - t^2(2t-1) \\
&= -8t^3 + 12t^2 - 6t + 1 - 2t + 1 - t - t^3 - t(4t^2 - 4t + 1) - 2t^3 \\
&= -11t^3 + 13t^2 - 9t + 2 - 4t^3 + 4t^2 - t \\
&= -15t^3 + 17t^2 - 10t + 2.
\end{aligned}$$

Example

Let $t \in (2-U) \setminus \left\{\frac{3}{2}\right\}$. Then $\sigma(t) = \frac{t+2}{2}$ and

$$
\begin{aligned}
f^{\Delta}(t) &= -\left(\frac{t+2}{2}\right)^3 - \frac{t+2}{2} - t - t^3 - t\left(\frac{t+2}{2}\right)^2 - t^2\left(\frac{t+2}{2}\right) \\
&= -\frac{(t+2)^3}{8} - \frac{t+2}{2} - t - t^3 - t\frac{(t+2)^2}{4} - t^2\frac{t+2}{2} \\
&= -\frac{(t+2)^3 + 4(t+2) + 8t + 8t^3 + 2t(t+2)^2 + 4t^2(t+2)}{8} \\
&= -\frac{t^3 + 6t^2 + 12t + 8 + 4t + 8 + 8t + 8t^3 + 2t(t^2+4t+4) + 4t}{8} \\
&= -\frac{13t^3 + 14t^2 + 24t + 8 + 2t^3 + 8t^2 + 8t}{8} \\
&= -\frac{15t^3 + 22t^2 + 32t + 8}{8}.
\end{aligned}
$$

Example

Let $t \in (2+U) \setminus \left\{ \frac{5}{2} \right\}$. Then $\sigma(t) = 2(t-1)$ and

$$\begin{aligned}
f^{\Delta}(t) &= -(2(t-1))^3 - 2(t-1) - t - t^3 - t(2(t-1))^2 - t^2(2(t-1)) \\
&= -8(t-1)^3 - 2(t-1) - t - t^3 - 4t(t-1)^2 - 2t^2(t-1) \\
&= -8t^3 + 24t^2 - 24t + 8 - 2t + 2 - t - t^3 - 4t(t^2 - 2t + 1) - 2t^3 \\
&= -11t^3 + 26t^2 - 27t + 10 - 4t^3 + 8t^2 - 4t \\
&= -15t^3 + 34t^2 - 31t + 10.
\end{aligned}$$

Example

Let $t \in \mathbb{T}^\kappa$ be right-dense. Then $t = 0$ or $t = 1$, or $t = 1$. Let $t = 0$. Then

$$\begin{aligned} f^\Delta(0) &= \lim_{s \to 0} \frac{f(s) - f(0)}{s} \\ &= \lim_{s \to 0} \frac{(2 + s^2)(1 - s^2) - 2}{s} \\ &= \lim_{s \to 0} \frac{2 - 2s^2 + s^2 - s^4 - 2}{s} \\ &= -\lim_{s \to 0} \frac{2(s + s^3)}{s} \\ &= -\lim_{s \to 0} (s + s^3) \\ &= 0. \end{aligned}$$

Example

Let $t = 1$. Then

$$\begin{aligned} f^{\Delta}(1) &= \lim_{s \to 1} \frac{f(s) - f(1)}{s - 1} \\ &= \lim_{s \to 1} \frac{(2 + s^2)(1 - s^2) - (2 + 1)(1 - 1)}{s - 1} \\ &= -\lim_{s \to 1} \frac{(2 + s^2)(s - 1)(s + 1)}{s - 1} \\ &= -\lim_{s \to 1} (2 + s^2)(s + 1) \\ &= -(2 + 1)(1 + 1) \\ &= -6. \end{aligned}$$

Example

Let $t = 2$. Then

$$
\begin{aligned}
f^{\Delta}(2) &= \lim_{s\to 2} \frac{f(s) - f(2)}{s-2} \\
&= \lim_{s\to 2} \frac{(2+s^2)(1-s^2) - (2+4)(1-4)}{s-2} \\
&= \lim_{s\to 2} \frac{2 - 2s^2 + -s^2 - s^4 + 18}{s-2} \\
&= \lim_{s\to 2} \frac{s^4 + s^2 - 20}{s-2} \\
&= -\lim_{s\to 2} \frac{(s-2)(s^3 + 2s^2 + 5s + 10)}{s-2} \\
&= -\lim_{s\to 2} (s^3 + 2s^2 + 5s + 10) \\
&= -(2^3 + 2\cdot 2^2 + 5\cdot 2 + 10) \\
&= -(8 + 8 + 10 + 10) \\
&= -36.
\end{aligned}
$$

Example

Consequently

$$f^{\Delta}(t)=\begin{cases} -\frac{151}{64} & \text{if} \quad t=\frac{1}{2} \\ -\frac{1195}{64} & \text{if} \quad t=\frac{3}{2} \\ -15t^3-3t & \text{if} \quad t\in U\setminus\left\{\frac{1}{2}\right\} \\ -\frac{15t^3+11t^2+17t+5}{8} & \text{if} \quad t\in(1-U)\setminus\left\{\frac{1}{2}\right\} \\ -15t^3+17t^2-10t+2 & \text{if} \quad t\in(1+U)\setminus\left\{\frac{3}{2}\right\} \end{cases}$$

Example

$$f^{\Delta}(t) = \begin{cases} -\frac{15t^3+22t^2+32t+8}{8} \quad \text{if} \quad t \in (2-U) \setminus \left\{\frac{3}{2}\right\} \\ -15t^3 + 34t^2 - 31t + 10 \quad \text{if} \quad t \in (2+U) \setminus \left\{\frac{5}{2}\right\} \\ 0 \quad \text{if} \quad t = 0 \\ -6 \quad \text{if} \quad t = 1 \\ -36 \quad \text{if} \quad t = 2. \end{cases}$$

Example

Let

$$f(t) = t^n, \quad t \in \mathbb{T}, \quad n \in \mathbb{N}.$$

We will prove that

$$f^{\Delta}(t) = (\sigma(t))^{n-1} + t(\sigma(t))^{n-2} + \cdots + t^{n-2}\sigma(t) + t^{n-1}$$

if $t \in \mathbb{T}$ is right-scattered. Really, let $t \in \mathbb{T}$ be right-scattered. Then

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\ &= \frac{(\sigma(t))^n - t^n}{\sigma(t) - t} \\ &= \frac{(\sigma(t) - t)\left((\sigma(t))^{n-1} + t(\sigma(t))^{n-2} + \cdots + t^{n-2}\sigma(t) + t^{n-1}\right)}{\sigma(t) - t} \\ &= (\sigma(t))^{n-1} + t(\sigma(t))^{n-2} + \cdots + t^{n-2}\sigma(t) + t^{n-1}. \end{aligned}$$

Example

Let $\mathbb{T} = \left\{ \frac{1}{2n+1} : n \in \mathbb{N}_0 \right\} \cup \{0\}$, $f(t) = \sigma(t)$, $t \in \mathbb{T}$. We will find $f^{\Delta}(t)$, $t \in \mathbb{T}$. For

$$t \in \mathbb{T}, \quad t = \frac{1}{2n+1}, \quad n = \frac{1-t}{2t}, \quad n \geq 1,$$

we have

$$\begin{aligned} \sigma(t) &= \inf \left\{ \frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > \frac{1}{2n+1}, l \in \mathbb{N}_0 \right\} = \frac{1}{2n-1} \\ &= \frac{1}{2\frac{1-t}{2t} - 1} = \frac{t}{1-2t} > t, \end{aligned}$$

i.e., any point $t = \frac{1}{2n+1}$, $n \geq 1$, is right-scattered. At these points

Example

$$
\begin{aligned}
f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\
&= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\
&= 2\frac{(\sigma(t))^2}{(1 - 2\sigma(t))(\sigma(t) - t)} \\
&= 2\frac{\left(\frac{t}{1-2t}\right)^2}{\left(1 - \frac{2t}{1-2t}\right)\left(\frac{t}{1-2t} - t\right)} \\
&= 2\frac{\frac{t^2}{(1-2t)^2}}{\frac{1-4t}{1-2t}\frac{2t^2}{1-2t}} \\
&= 2\frac{t^2}{2t^2(1-4t)} = \frac{1}{1-4t}.
\end{aligned}
$$

Let $n = 0$, i.e., $t = 1$. Then

Example

$$\sigma(1) = \inf\left\{\frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > 1, l \in \mathbb{N}_0\right\}$$
$$= \inf\emptyset = \sup\mathbb{T} = 1,$$

i.e., $t = 1$ is a right-dense point. Also,

$$\lim_{h\to 0}\frac{f(1+h)-f(h)}{h} = \lim_{h\to 0}\frac{\sigma(1+h)-\sigma(h)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1+h}{1-2(1+h)} - \frac{h}{1-2h}}{h}$$
$$= -\lim_{h\to 0}\frac{\frac{1+h}{1+2h} + \frac{h}{1-2h}}{h} = -\lim_{h\to 0}\frac{(1+h)(1-2h)+h(1+2h)}{(1+2h)(1-2h)h}$$
$$= -\lim_{h\to 0}\frac{1}{(1+2h)(1-2h)h} = -\infty.$$

Example

Let now $t = 0$. Then

$$\sigma(0) = \inf\left\{\frac{1}{2l+1}, 0 : \frac{1}{2l+1}, 0 > 0, l \in \mathbb{N}_0\right\} = 0.$$

Consequently, $t = 0$ is right-dense. Also,

$$\lim_{h\to 0}\frac{\sigma(h)-\sigma(0)}{h} = \lim_{h\to 0}\frac{\frac{h}{1-2h}-0}{h} = \lim_{h\to 0}\frac{1}{1-2h} = 1.$$

Therefore $\sigma^{\Delta}(0) = 1$.

Example

Let $f, g, h : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^\kappa$. Then

$$\begin{aligned}
(fgh)^\Delta(t) &= ((fg)h)^\Delta(t) \\
&= (fg)^\Delta(t)h(t) + (fg)(\sigma(t))h^\Delta(t) \\
&= (f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t))h(t) + f^\sigma(t)g^\sigma(t)h^\Delta(t) \\
&= f^\Delta(t)g(t)h(t) + f^\sigma(t)g^\Delta(t)h(t) + f^\sigma(t)g^\sigma(t)h^\Delta(t).
\end{aligned}$$

Example

Let $f : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^\kappa$. Then

$$\begin{aligned}
\left(f^2\right)^\Delta(t) &= (ff)^\Delta(t) \\
&= f^\Delta(t)f(t) + f(\sigma(t))f^\Delta(t) \\
&= f^\Delta(t)(f^\sigma(t) + f(t)).
\end{aligned}$$

Also,

$$\begin{aligned}
(f^3)^\Delta(t) &= (ff^2)^\Delta(t) \\
\\
&= f^\Delta(t)f^2(t) + f(\sigma(t))(f^2)^\Delta(t) \\
&= f^\Delta(t)f^2(t) + f^\sigma(t)f^\Delta(t)(f^\sigma(t) + f(t)) \\
&= f^\Delta(t)(f^2(t) + f(t)f^\sigma(t) + (f^\sigma)^2(t)).
\end{aligned}$$

We assume that

Example

$$(f^n)^\Delta(t) = f^\Delta(t)\sum_{k=0}^{n-1} f^k(t)(f^\sigma)^{n-1-k}(t)$$

for some $n \in \mathbb{N}$. We will prove that

$$(f^{n+1})^\Delta(t) = f^\Delta(t)\sum_{k=0}^{n} f^k(t)(f^\sigma)^{n-k}(t).$$

Indeed,

$$\begin{aligned}
(f^{n+1})^\Delta(t) &= (ff^n)^\Delta(t) \\
&= f^\Delta(t)f^n(t) + f^\sigma(t)(f^n)^\Delta(t) \\
&= f^\Delta(t)f^n(t) + f^\Delta(t)f^\sigma(t)(f^{n-1}(t) + f^{n-2}(t)f^\sigma(t)
\end{aligned}$$

Example

$$
\begin{aligned}
& + \cdots + f(t)(f^\sigma)^{n-2}(t) + (f^\sigma)^{n-1}(t))f^\sigma(t) \\
= \;& f^\Delta(t)\left(f^n(t) + f^{n-1}(t)f^\sigma(t) + f^{n-2}(t)(f^\sigma)^2(t) + \cdots + (f^\sigma)^n(t)\right) \\
= \;& f^\Delta(t)\sum_{k=0}^{n} f^k(t)(f^\sigma)^{n-k}(t).
\end{aligned}
$$

Example

Now we consider $f(t) = (t-a)^m$ for $a \in \mathbb{R}$ and $m \in \mathbb{N}$. We set

$$h(t) = (t-a).$$

Then

$$h^{\Delta}(t) = 1.$$

We get

$$\begin{aligned} (h^m)^{\Delta}(t) &= h^{\Delta}(t) \sum_{k=0}^{m-1} h^k(t)(h^{\sigma})^{m-1-k}(t) \\ &= \sum_{k=0}^{m-1} (t-a)^k (\sigma(t)-a)^{m-1-k}. \end{aligned}$$

Let now $g(t) = \dfrac{1}{f(t)}$. Then

Example

$$g^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(\sigma(t))f(t)},$$

whereupon

$$\begin{aligned}\left(\frac{1}{h^m}\right)^{\Delta}(t) &= -\frac{1}{(\sigma(t)-a)^m(t-a)^m}\sum_{k=0}^{m-1}(t-a)^k(\sigma(t)-a)^{m-1-k} \\ &= -\sum_{k=0}^{m-1}\frac{1}{(t-a)^{m-k}}\frac{1}{(\sigma(t)-a)^{k+1}}.\end{aligned}$$

Definition

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in (\mathbb{T}^{\kappa})^{\kappa} = \mathbb{T}^{\kappa^2}$. We define the second derivative of f at t, provided it exists, by

$$f^{\Delta^2} = \left(f^{\Delta}\right)^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}.$$

Similarly, we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$.

Example

Let $\mathbb{T} = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$ and

$$f(t) = \frac{1+t^2}{1+7t}, \quad t \in \mathbb{T}.$$

We will find $f^{\Delta^2}(t)$, $t \in \mathbb{T}$. We have the following cases. Let $t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$. Then $\sigma(t) = -\frac{t}{t-1}$. By Example 2, we have

$$f^{\Delta}(t) = \frac{6t^2+9t-7}{(1+7t)(1+6t)}.$$

Then

$$\begin{aligned} f^{\Delta}(\sigma(t)) &= \frac{6(\sigma(t))^2+9\sigma(t)-7}{(1+7\sigma(t))(1+6\sigma(t))} \\ &= \frac{6\frac{t^2}{(t-1)^2} - \frac{9t}{t-1} - 7}{\left(1-\frac{7t}{t-1}\right)\left(1-\frac{6t}{t-1}\right)} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{6t^2 - 9t(t-1) - 7(t-1)^2}{(y-1-7t)(t-1-6t)} \\
&= \frac{6t^2 - 9t^2 + 9t - 7t^2 + 14t - 7}{(1+6t)(1+5t)} \\
&= \frac{-10t^2 + 23t - 7}{(1+6t)(1+5t)}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{gf^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{1}{-\frac{t}{t-1} - t} \left(\frac{-10t^2 + 23t - 7}{(1+6t)(1+5t)} - \frac{6t^2 + 9t - 7}{(1+6t)(1+7t)} \right) \\
&= -\frac{(t-1)\left((-10t^2 + 23t - 7)(1+7t) - (6t^2 + 9t - 7)(1+5t)\right)}{t^2(1+5t)(1+6t)(1+7t)}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= -\frac{(t-1)(-10t^2 - 70t^3 + 23t + 161t^2 - 7 - 49t - 6t^2 - 30t^3 - 9t -}{t^2(1+5t)(1+6t)(1+7t)} \\
&= -\frac{(t-1)(-100t^3 + 100t^2)}{t^2(1+5t)(1+6t)(1+7t)} \\
&= \frac{100(t-1)^2}{(1+5t)(1+6t)(1+7t)}.
\end{aligned}
$$

Let $t \in \mathbb{N}_0$. Then $\sigma(t) = t + 1$. By Example 2, we have

$$f^{\Delta}(t) = \frac{7t^2 + 9t - 6}{(1+7t)(8+7t)}.$$

Hence,

Example

$$\begin{aligned} f^{\Delta}(\sigma(t)) &= \frac{7(\sigma(t))^2 + 9\sigma(t) - 6}{(1 + 7\sigma(t))(8 + 7\sigma(t))} \\ &= \frac{7(t+1)^2 + 9(t+1) - 6}{(1 + 7t + 7)(8 + 7t + t)} \\ &= \frac{7t^2 + 14t + 7 + 9t + 9 - 6}{(8 + 7t)(15 + 7t)} \\ &= \frac{7t^2 + 23t + 10}{(8 + 7t)(15 + 7t)} \end{aligned}$$

and

Example

$$f^{\Delta^2}(t) = \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t}$$

$$\begin{aligned}
&= \frac{7t^2 + 23t + 10}{(8+7t)(15+7t)} - \frac{7t^2 + 9t - 6}{(1+7t)(8+7t)} \\
&= \frac{(7t^2 + 23t + 10)(1+7t) - (7t^2 + 9t - 6)(15+7t)}{(1+7t)(8+7t)(15+7t)} \\
&= \frac{10(t+10)}{(1+7t)(8+7t)(15+7t)}.
\end{aligned}$$

Example

Consequently

$$f^{\Delta^2}(t) = \begin{cases} \dfrac{100(t-1)^2}{(1+5t)(1+6t)(1+7t)} & \text{if} \quad t \in \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \\[2ex] \dfrac{10(t+10)}{(1+7t)(8+7t)(15+7t)} & \text{if} \quad t \in \mathbb{N}_0. \end{cases}$$

Example

Let $\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^{2^n}; n \in \mathbb{N}_0 \right\} \cup \{0, 1\}$ and

$$f(t) = \frac{1+7t}{4+5t}, \quad t \in \mathbb{T}.$$

We will find $f^{\Delta^2}(t)$, $t \in \mathbb{T}^{\kappa^2}$. Here

$$\mathbb{T}^{\kappa^2} = \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

We have the following cases. Let $t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\}$. Then $\sigma(t) = \sqrt{t}$. By Example 8, we have

$$f^{\Delta}(t) = \frac{23}{(4+5t)(4+5\sqrt{t})}.$$

Then

Example

$$
\begin{aligned}
f(\sigma(t)) &= \frac{23}{(4+5\sigma(t))(4+5\sqrt{\sigma(t)})} \\
&= \frac{23}{(4+5\sqrt{t})(4+5\sqrt[4]{t})}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t)-t} \\
&= \frac{1}{\sqrt{t}-t}\left(\frac{23}{(4+5\sqrt{t})(4+5\sqrt[4]{t})} - \frac{23}{(4+5t)(4+5\sqrt{t})}\right) \\
&= \frac{23(4+5t-4-5\sqrt[4]{t})}{\sqrt{t}(1-\sqrt{t})(4+5t)(4+5\sqrt{t})(4+5\sqrt[4]{t})} \\
&= \frac{115(t-\sqrt[4]{t})}{\sqrt{t}(1-\sqrt{t})(4+5t)(4+5\sqrt{t})(4+5\sqrt[4]{t})}.
\end{aligned}
$$

Example

Let $t=0$. Then $\sigma(0)=0$. By Example 8, we have $f^{\Delta}(0)=\frac{23}{16}$. Hence,

$$\begin{aligned}
f^{\Delta^2}(0) &= \lim_{s\to 0}\frac{f^{\Delta}(s)-f^{\Delta}(0)}{s}\\
&= \lim_{s\to 0}\frac{\frac{23}{(4+5s)(4+5\sqrt{s})}-\frac{23}{16}}{s}\\
&= \lim_{s\to 0}\frac{23\left(16-(4+5s)(4+5\sqrt{s})\right)}{s(4+5s)(4+5\sqrt{s})}\\
&= \lim_{s\to 0}\frac{23(16-16-20\sqrt{s}-20s-25s^{\frac{3}{2}})}{s(4+5s)(4+5\sqrt{s})}\\
&= -\lim_{s\to 0}\frac{115(5\sqrt{s}+5s+5s^{\frac{3}{2}})}{s(4+5s)(4+5\sqrt{s})}\\
&= -\lim_{s\to 0}\frac{115(5+5\sqrt{s}+5s)}{\sqrt{s}(4+5s)(4+5\sqrt{s})}\\
&= -\infty.
\end{aligned}$$

Example

Consequently

$$f^{\Delta^2}(t) = \begin{cases} \frac{115(t-\sqrt[4]{t})}{\sqrt{t}(1-\sqrt{t})(4+5t)(4+5\sqrt{t})(4+5\sqrt[4]{t})} & \text{if} \quad t \in \left\{ \left(\frac{1}{2}\right)^{2^n} : n \in \mathbb{N} \right\} \\ \text{does not exist if} \quad t = 0. \end{cases}$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$ and

$$\mathbb{T} = U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{0,1,2\},$$

and

$$f(t) = (2+t^2)(1-t^2), \quad t \in \mathbb{T}.$$

We will find $f^{\Delta^2}(t)$, $t \in \mathbb{T}^{\kappa^2}$. Here $\mathbb{T}^{\kappa^2} = \mathbb{T} \setminus \left\{ \frac{5}{2}, \frac{9}{4} \right\}$. We have the following cases. Let $t \in U \setminus \left\{ \frac{1}{2} \right\}$. Then $\sigma(t) = 2t$. By Example 14, we have

$$f^{\Delta}(t) = -15t^3 - 3t.$$

Then

Example

$$
\begin{aligned}
f^{\Delta}(\sigma(t)) &= -15(\sigma(t))^3 - 3\sigma(t) \\
&= -15(8t^3) - 3(2t) \\
&= -120t^3 - 6t
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{-120t^3 - 6t + 15t^3 + 2t}{2t - t} \\
&= \frac{-105t^3 - 3t}{t} = -105t^2 - 3.
\end{aligned}
$$

Example

Let $t = \frac{1}{2}$. Then $\sigma\left(\frac{1}{2}\right) = \frac{3}{4}$. By Example 14, we have

$$f^{\Delta}\left(\frac{1}{2}\right) = -\frac{151}{64}$$

and

$$\begin{aligned} f^{\Delta}\left(\sigma\left(\frac{1}{2}\right)\right) &= f^{\Delta}\left(\frac{3}{4}\right) \\ &= -\frac{15\left(\frac{3}{4}\right)^3 + 11\left(\frac{3}{4}\right)^2 + 17\left(\frac{3}{4}\right) + 5}{8} \\ &= -\frac{15\left(\frac{27}{64}\right) + 11\left(\frac{9}{16}\right) + \frac{51}{4} + 5}{8} \\ &= -\frac{15 \cdot 27 + 11 \cdot 36 + 51 \cdot 16 + 5 \cdot 64}{512} \\ &= -\frac{405 + 396 + 816 + 320}{512} = -\frac{1937}{512}. \end{aligned}$$

Hence

Example

$$
\begin{aligned}
f^{\Delta^2}\left(\frac{1}{2}\right) &= \frac{f^{\Delta}\left(\sigma\left(\frac{1}{2}\right)\right) - f^{\Delta}\left(\frac{1}{2}\right)}{\sigma\left(\frac{3}{4}\right) - \frac{1}{2}} \\
&= \frac{f^{\Delta}\left(\frac{3}{4}\right) - f^{\Delta}\left(\frac{1}{2}\right)}{\frac{3}{4} - \frac{1}{2}} \\
&= \frac{-\frac{1937}{512} + \frac{151}{64}}{\frac{1}{4}} \\
&= \frac{-1937 + 1208}{128} \\
&= -\frac{729}{128}.
\end{aligned}
$$

Example

Let $t \in (1-U) \setminus \left\{ \frac{1}{2} \right\}$. Then $\sigma(t) = \frac{1+t}{2}$. By Example 14, we have

$$f^{\Delta}(t) = -\frac{15t^3 + 11t^2 + 17t + 5}{8}.$$

Then

$$\begin{aligned} f^{\Delta}(\sigma(t)) &= -\frac{15\left(\frac{1+t}{2}\right)^3 + 11\left(\frac{1+t}{2}\right)^2 + 17\left(\frac{1+t}{2}\right) + 5}{8} \\ &= -\frac{15(1+t)^3 + 22(1+t)^2 + 68(1+t) + 40}{64} \\ &= -\frac{15 + 45t^2 + 45t + 15t^3 + 22 + 44t + 22t^2 + 68 + 68t + 40}{64} \\ &= -\frac{15t^3 + 67t^2 + 157t + 77}{64} \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{1}{\frac{1+t}{2} - t}\left(-\frac{15t^3 + 67t^2 + 157t + 77}{64} + \frac{15t^3 + 11t^2 + 17t + 5}{8}\right) \\
&= \frac{2}{1-t}\left(\frac{-15t^3 - 67t^2 - 157t - 77 + 120t^3 + 88t^2 + 136t + 40}{64}\right) \\
&= \frac{105t^3 + 21t^2 - 21t - 37}{32(1-t)}.
\end{aligned}
$$

Let $t \in (1+U) \setminus \left\{\frac{3}{2}\right\}$. Then $\sigma(t) = 2t - 1$. By Example 14, we have $f^{\Delta}(t) = -15t^3 + 17t^2 - 10y + 2$. Then

Example

$$
\begin{aligned}
f^{\Delta}(\sigma(t)) &= -15(\sigma(t))^3 + 17(\sigma(t)0^2 - 10\sigma(t) + 2 \\
&= -15(2t-1)^3 + 17(2t-1)^2 - 10(2t-1) + 2 \\
&= -15(8t^3 - 12t^2 + 6t - 1) + 17(4t^2 - 4t + 1) - 20t + 10 + 2 \\
&= -120t^3 + 180t^2 - 90t + 15 + 68t^2 - 68t + 17 - 20t + 12 \\
&= -120t^3 + 248t^2 - 178t + 44
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{-120t^3 + 248t^2 - 178t + 44 + 15t^3 - 17t^2 + 10t - 2}{2t - 1 - t} \\
&= \frac{-105t^3 + 231t^2 - 168t + 42}{t - 1}.
\end{aligned}
$$

Example

Let $t = \frac{3}{2}$. Then $\sigma\left(\frac{3}{2}\right) = \frac{7}{4}$. By Example 14, we have $f^{\Delta}\left(\frac{3}{2}\right) = -\frac{1195}{64}$ and

$$\begin{aligned} f^{\Delta}\left(\sigma\left(\frac{3}{2}\right)\right) &= f^{\Delta}\left(\frac{7}{4}\right) \\ &= -\frac{15\left(\frac{7}{4}\right)^3 + 22\left(\frac{7}{4}\right)^2 + 32\left(\frac{7}{4}\right) + 8}{8} \\ &= -\frac{15\left(\frac{343}{64}\right) + 22\left(\frac{49}{16}\right) + 32\left(\frac{7}{4}\right) + 8}{8} \\ &= -\frac{15 \cdot 343 + 176 \cdot 49 + 32 \cdot 112 + 8 \cdot 64}{512} \\ &= -\frac{5145 + 8624 + 3584 + 512}{512} = -\frac{17865}{512}. \end{aligned}$$

Hence,

Example

$$\begin{aligned} f^{\Delta^2}\left(\frac{3}{2}\right) &= \frac{f^{\Delta}\left(\sigma\left(\frac{3}{2}\right)\right) - f^{\Delta}\left(\frac{3}{2}\right)}{\sigma\left(\frac{3}{2}\right) - \frac{3}{2}} \\ &= \frac{-\frac{17865}{512} + \frac{1195}{64}}{\frac{7}{4} - \frac{3}{2}} \\ &= \frac{-17865 + 9560}{128} \\ &= -\frac{8305}{128}. \end{aligned}$$

Let $t \in (2-U)\backslash\left\{\frac{3}{2}\right\}$. Then $\sigma(t) = \frac{t+2}{2}$. By Example 14, we have

$$f^{\Delta}(t) = -\frac{15t^3 + 22t^2 + 32t + 8}{8}.$$

Then

Example

$$
\begin{aligned}
f^{\Delta^2}(\sigma(t)) &= -\frac{15(\sigma(t))^3 + 22(\sigma(t))^2 + 32\sigma(t) + 8}{8} \\
&= -\frac{15\left(\frac{t+2}{2}\right)^3 + 22\left(\frac{t+2}{2}\right)^2 + 32\left(\frac{t+2}{2}\right) + 8}{8} \\
&= -\frac{15(t+2)^3 + 88(t+2)^2 + 128(t+2) + 64}{64} \\
&= -\frac{15t^3 + 90t^2 + 180t + 120 + 88t^2 + 352t + 352 + 128t + 2}{64} \\
&= -\frac{15t^3 + 178t^2 + 660t + 792}{64}
\end{aligned}
$$

and

$$
f^{\Delta^2}(t) = \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t}
$$

Example

$$
\begin{aligned}
&= \frac{1}{\frac{t+2}{2}-t}\left(-\frac{15t^3+178t^2+660t+792}{64}+\frac{15t^3+22t^2+32t+8}{8}\right)\\
&= \frac{-15t^3-178t^2-660t-792+120t^3+176t^2+256t+64}{32(2-t)}\\
&= \frac{105t^3-2t^2-404t-728}{32(2-t)}.
\end{aligned}
$$

Let $t \in (2+U)\setminus\left\{\frac{5}{2}\right\}$. Then $\sigma(t) = 2(t-1)$. By Example 14, we have

$$f^{\Delta}(t) = -15t^3 + 34t^2 - 31t + 10.$$

Example

Then

$$\begin{aligned} f^{\Delta}(\sigma(t)) &= -15(\sigma(t)0^3 + 34(\sigma(t))^2 - 31\sigma(t) + 10 \\ &= -120(t-1)^3 + 136(t-1)^2 - 62(t-1) + 10 \\ &= -120t^3 + 360t^2 - 360t - 120 + 136t^2 - 272t + 136 - 62t + \\ &= -120t^3 + 496t^2 - 694t + 88 \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\Delta^2}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\sigma(t) - t} \\
&= \frac{-120t^3 + 496t^2 - 694t + 88 + 15t^3 - 34t^2 + 31t - 10}{t-2} \\
&= \frac{-105t^3 + 461t^2 - 663t + 78}{t-2}.
\end{aligned}
$$

Let $t = 0$. Then $\sigma(0) = 0$. By Example 14, we have $f^{\Delta}(0) = 0$. Hence,

$$
\begin{aligned}
f^{\Delta^2}(0) &= \lim_{s \to 0} \frac{f^{\Delta}(s) - f^{\Delta}(0)}{s} \\
&= \lim_{s \to 0} \frac{-15s^3 - 3s}{s} = \lim_{s \to 0} (-15s^2 - 3) = -3.
\end{aligned}
$$

Example

Let $t = 1$. Then $\sigma(1) = 1$. By Example 14, we have $f^{\Delta}(1) = -6$. Hence,

$$\begin{aligned}
\lim_{s\to 1, s>1} \frac{f^{\Delta}(s) - f^{\Delta}(1)}{s-1} &= \lim_{s\to 1, s>1} \frac{-15s^3 + 17s^2 - 10s + 2 + 6}{s-1} \\
&= \lim_{s\to 1, s>1} \frac{15s^3 + 17s^2 - 10s + 8}{s-1} \\
&= \lim_{s\to 1, s>1} \frac{(s-1)(-15s^2 + 2s - 8)}{s-1} \\
&= \lim_{s\to 1, s>1} (-15s^2 + 2s - 8) \\
&= -21
\end{aligned}$$

and

Example

$$
\begin{aligned}
\lim_{s\to 1, s<1} \frac{f^{\Delta}(s) - f^{\Delta}(1)}{s-1} &= \lim_{s\to 1, s<1} \frac{-\frac{15s^3+11s^2+17s+5}{8} + 6}{s-1} \\
&= \lim_{s\to 1, s<1} \frac{-15s^3 - 11s^2 - 17s - 5 + 48}{8(s-1)} \\
&= \lim_{s\to 1, s<1} \frac{-15s^3 - 11s^2 - 17s + 43}{8(s-1)} \\
&= \lim_{s\to 1, s<1} \frac{(s-1)(-15s^2 - 26s - 43)}{8(s-1)} \\
&= \lim_{s\to 1, s<1} \frac{-15s^2 - 26s - 43}{8} \\
&= -\frac{21}{2}.
\end{aligned}
$$

Example

Since

$$\lim_{s\to 1, s<1} \frac{f^{\Delta}(s) - f^{\Delta}(1)}{s-1} \neq \lim_{s\to 1, s>1} \frac{f^{\Delta}(s) - f^{\Delta}(1)}{s-1},$$

we conclude that $f^{\Delta^2}(1)$ does not exist. Let $t = 2$. Then $\sigma(2) = 2$. By Example 14, we have $f^{\Delta}(2) = -36$. Then

$$\begin{aligned}
\lim_{s\to 2, s<2} \frac{f^{\Delta}(s) - f^{\Delta}(1)}{s-2} &= \lim_{s\to 2, s<2} \frac{1}{s-2} \left(-\frac{15s^3 + 22s^2 + 32s + 8}{8} + 36 \right) \\
&= \lim_{s\to 2, s<2} \frac{1}{s-2} \left(\frac{-15s^3 - 22s^2 - 32s - 8 + 288}{8} \right) \\
&= \lim_{s\to 2, s<2} \frac{-15s^3 - 22s^2 - 32s + 280}{8(s-2)} \\
&= -\infty.
\end{aligned}$$

Therefore $f^{\Delta^2}(2)$ does not exist.

Example

Consequently

$$f^{\Delta^2}(t) = \begin{cases} -105t^2 - 3 \quad \text{if} \quad t \in U \backslash \left\{\frac{1}{2}\right\} \\ -\frac{729}{128} \quad \text{if} \quad t = \frac{1}{2} \\ \frac{105t^3+21t^2-21t-37}{32(1-t)} \quad \text{if} \quad t \in (1-U) \backslash \left\{\frac{1}{2}\right\} \\ \frac{-105t^3+231t^2-168t+42}{t-1} \quad \text{if} \quad t \in (1+U) \backslash \left\{\frac{3}{2}\right\} \\ -\frac{8305}{128} \quad \text{if} \quad t = \frac{3}{2} \\ \frac{105t^3-2t^2-404t-728}{32(2-t)} \quad \text{if} \quad t \in (2-U) \backslash \left\{\frac{3}{2}\right\} \\ \frac{-105t^3+461t^2-663t+78}{t-2} \quad \text{if} \quad t \in (2+U) \backslash \left\{\frac{5}{2}\right\} \\ -3 \quad \text{if} \quad t = 0 \\ \text{does not exist if} \quad t = 1 \\ \text{does not exist if} \quad t = 2. \end{cases}$$

Example (Leibniz Formula)

Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n-k$ times Δ. Let also,

$$f^{\Lambda} \quad \text{exists} \quad \text{for} \quad \text{all} \quad \Lambda \in S_k^{(n)}.$$

We will prove that

$$(fg)^{\Delta^n} = \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k}. \tag{1}$$

For this aim, we will use the classical induction principle. Let $n = 1$. Then

$$S_0^{(1)} = 0, \quad S_1^{(1)} = \sigma.$$

Hence,

Example

$$\sum_{k=0}^{1}\left(\sum_{\Lambda\in S_k^{(1)}} f^{\Lambda}\right) g^{\Delta^2} = \sum_{\Lambda\in S_0^{(1)}} f^{\Lambda} g + \sum_{\Lambda\in S_1^{(1)}} f^{\Lambda} g^{\Delta}$$
$$= f^{\Delta} g + f^{\sigma} g^{\Delta}$$
$$= (fg)^{\Delta},$$

i.e., the assertion holds for $n = 1$. Assume that the assertion is valid for some $n \in \mathbb{N}$.
We will prove that

$$(fg)^{\Delta^{n+1}} = \left(\sum_{\Lambda\in S_k^{(n+1)}} f^{\Lambda}\right) g^{\Delta^k}.$$

We have

Example

$$
\begin{aligned}
(fg)^{\Delta^{n+1}} &= \left((fg)^{\Delta^n}\right)^{\Delta} \\
&= \left(\sum_{k=0}^{n}\left(\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda}\right) g^{\Delta^k}\right)^{\Delta} \\
&= \sum_{k=0}^{n}\left(\left(\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda}\right) g^{\Delta^k}\right)^{\Delta}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \sum_{k=0}^{n}\left(\left(\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda\sigma}\right) g^{\Delta^{k+1}} + \left(\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda\Delta}\right) g^{\Delta^{k}}\right)\\
&= \sum_{k=0}^{n}\left(\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda\sigma}\right) g^{\Delta^{k+1}} + \sum_{k=0}^{n}\left(\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda\Delta}\right) g^{\Delta^{k}}\\
&= \sum_{k=1}^{n+1}\left(\sum_{\Lambda\in S_{k-1}^{(n)}} f^{\Lambda\sigma}\right) g^{\Delta^{k}} + \sum_{k=0}^{n}\left(\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda\Delta}\right) g^{\Delta^{k}}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \sum_{k=1}^{n}\left(\sum_{\Lambda\in S_{k-1}^{(n)}} f^{\Lambda\sigma}\right) g^{\Delta^k} + \left(\sum_{\Lambda\in S_{k}^{(n)}} f^{\Lambda\sigma}\right) g^{\Delta^{n+1}} \\
&\quad + \sum_{k=1}^{n}\left(\sum_{\Lambda\in S_{k}^{(n)}} f^{\Lambda\Delta}\right) g^{\Delta^k} + \left(\sum_{\Lambda\in S_{0}^{(n)}} f^{\Lambda\Delta}\right) g \\
&= \sum_{k=1}^{n}\left(\sum_{\Lambda\in S_{k-1}^{(n)}} f^{\Lambda\sigma} + \sum_{\Lambda\in S_{k}^{(n)}} f^{\Lambda\Delta}\right) g^{\Delta^k}
\end{aligned}
$$

Example

$$
\begin{aligned}
& + \left(\sum_{\Lambda \in S_n^{(n)}} f^{\Lambda\sigma} + \sum_{\Lambda \in S_0^{(n)}} f^{\Lambda\Delta} \right) g \\
= \; & \sum_{k=1}^{n} \left(\sum_{\Lambda \in S_k^{(n+1)}} f^{\Lambda} \right) g^{\Delta^k} + \left(\sum_{\Lambda \in S_{n+1}^{(n+1)}} f^{\Lambda} \right) g^{\Delta^{n+1}} + \left(\sum_{\Lambda \in S_0^{(n+1)}} f^{\Lambda} \right) g \\
= \; & \sum_{k=0}^{n+1} \left(\sum_{\Lambda \in S_k^{(n+1)}} f^{\Lambda} \right) g^{\Delta^k}.
\end{aligned}
$$

Example

Let f and μ be differentiable in $\mathbb{T}^{\kappa}$. Then

$$f^{\Delta\sigma} = \frac{f^{\sigma^2} - f^{\sigma}}{\mu^{\sigma}} \quad \text{and} \quad f^{\sigma\Delta} = \frac{f^{\sigma^2} - f^{\sigma}}{\mu}.$$

Therefore

$$\begin{aligned} f^{\sigma\Delta} &= \frac{f^{\Delta\sigma}\mu^{\sigma}}{\mu} \\ &= \frac{f^{\Delta\sigma}\mu(1+\mu^{\Delta})}{\mu} \\ &= (1+\mu^{\Delta})f^{\Delta\sigma}. \end{aligned}$$

Example

Also,

$$f^{\sigma^2\Delta} = \frac{f^{\sigma^3} - f^{\sigma^2}}{\mu} \quad \text{and} \quad f^{\sigma\Delta\sigma} = \frac{f^{\sigma^3} - f^{\sigma^2}}{\mu^\sigma}.$$

Therefore

$$\begin{aligned} f^{\sigma\Delta\sigma} &= \frac{f^{\sigma^2\Delta}\mu}{\mu^\sigma} \\ &= \frac{f^{\sigma^2\Delta}\mu}{\mu(1+\mu^\Delta)} \\ &= \frac{f^{\sigma^2\Delta}}{1+\mu^\Delta}, \end{aligned}$$

whereupon

Example

$$f^{\sigma^2\Delta} = (1+\mu^{\Delta})f^{\sigma\Delta\sigma}.$$

We have

$$\begin{aligned} f^{\Delta\sigma^2} &= \frac{f^{\sigma^3} - f^{\sigma^2}}{\mu^{\sigma^2}} \\ &= \frac{f^{\sigma^2\Delta}\mu}{\mu^{\sigma^2}} \\ &= \frac{\mu f^{\sigma^2\Delta}}{\mu(1+\mu^{\Delta})(1-\mu^{\Delta\sigma})} \\ &= \frac{f^{\sigma^2\Delta}}{(1+\mu^{\Delta})(1+\mu^{\Delta\sigma})}, \end{aligned}$$

from where

$$f^{\sigma^2\Delta} = (1+\mu^{\Delta})(1+\mu^{\Delta\sigma})f^{\Delta\sigma^2}.$$