Time Scales Analysis
Lecture 7

Nabla Derivative. Regulated and Predifferentiable Functions

Svetlin G. Georgiev

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Definition (The Nabla Derivative)

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be nabla differentiable at $t \in \mathbb{T}_\kappa$ if

1. f is defined in a neighbourhood U of t,
2. f is defined at $\rho(t)$,
3. there exists a unique real number $f^\nabla(t)$, called the nabla derivative of f at t, such that for each $\varepsilon > 0$, there exists a neighbourhood N of t with $N \subseteq U$ and

$$|f(\rho(t)) - f(s) - (\rho(t) - s)f^\nabla(t)| \leq \varepsilon|\rho(t) - s| \quad \text{for all} \quad s \in N.$$

The basic rules for nabla differentiation read as follows.

- Let $f, g : \mathbb{T} \to \mathbb{R}$ be functions and let $t \in \mathbb{T}_\kappa$. Then we have the following.
    - The nabla derivative is well defined.
    - If f is nabla differentiable at t, then f is continuous at t.
    - If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

    - If t is left-dense, then f is nabla differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

- If f is differentiable at t, then

$$f(\rho(t)) = f(t) + \nu(t)f^{\nabla}(t).$$

- If f and g are nabla differentiable at t, then
  - the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with

$$(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t).$$

  - For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with

$$(\alpha f)^{\nabla}(t) = \alpha f^{\nabla}(t).$$

  - The product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f(\rho(t))g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g(\rho(t)).$$

  - If $g(t)g(\rho(t)) \neq 0$, then $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g(\rho(t))}.$$

Definition

Let $f : \mathbb{T} \to \mathbb{R}$ and let $t \in (\mathbb{T}_\kappa)_\kappa = \mathbb{T}_{\kappa^2}$. We define the second nabla derivative of f at t, provided it exists, by

$$f^{\nabla\nabla} = \left(f^{\nabla}\right)^{\nabla} : \mathbb{T}_{\kappa^2} \to \mathbb{R}.$$

Similarly, we define higher-order nabla derivatives $f^{\nabla^n} : \mathbb{T}_{\kappa^n} \to \mathbb{R}$. Let $t \in \mathbb{T}_\kappa^\kappa$. We define the second mixed derivative of f at t, provided it exists, by

$$f^{\nabla\Delta} = \left(f^{\nabla}\right)^{\Delta} : \mathbb{T}_\kappa^\kappa \to \mathbb{R}$$

and

$$f^{\Delta\nabla} = \left(f^{\Delta}\right)^{\nabla} : \mathbb{T}_\kappa^\kappa \to \mathbb{R}.$$

Example

Let $\mathbb{T} = \mathbb{Z}$ and $f : \mathbb{T} \to \mathbb{R}$ be a function. Then

$$f^{\nabla}(t) = f(t) - f(t-1) \quad \text{for any} \quad t \in \mathbb{T}.$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and

$$f(t) = t^3 + t^2 + t + 1, \quad t \in \mathbb{T}_\kappa.$$

We will find $f^{\nabla}(t)$ for $t \in \mathbb{T}_\kappa$. We have that $\mathbb{T}_\kappa = \mathbb{T} \setminus \{1\}$ and $\rho(t) = \frac{t}{2}$, $t \in \mathbb{T}_\kappa$. Hence,

$$\begin{aligned} f^{\nabla}(t) &= (\rho(t))^2 + t\rho(t) + t^2 + \rho(t) + t + 1 \\ &= \frac{t^2}{4} + \frac{t^2}{2} + t^2 + \frac{t}{2} + t + 1 = \frac{7}{4}t^2 + \frac{3}{2}t + 1, \quad t \in \mathbb{T}_\kappa. \end{aligned}$$

Example

Let $\mathbb{T} = P_{1,3}$ and

$$f(t) = \frac{1-t}{1+t}, \quad t \in \mathbb{T}.$$

We will find $f^{\nabla}(t)$ and $t \in \mathbb{T}_\kappa$ and $f^{\Delta\nabla}(t)$ for $t \in \mathbb{T}_\kappa^\kappa$. Firstly, we will find $f^{\nabla}(t)$ for $t \in \mathbb{T}_\kappa$. We have the following cases. Let $t \in \mathbb{T}_\kappa$ be left-scattered. Then $\rho(t) = t - 3$ and

Example

$$
\begin{aligned}
f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\
&= \frac{f(t-3) - f(t)}{t-3-t} = \frac{\frac{1-t+3}{1+t-3} - \frac{1-t}{1+t}}{-3} = \frac{1}{3}\left(\frac{1-t}{1+t} - \frac{4-t}{t-2}\right) \\
&= \frac{1}{3}\left(\frac{(1-t)(t-2) - (1+t)(4-t)}{(t+1)(t-2)}\right) \\
&= \frac{1}{3}\left(\frac{t-2-t^2+2t-4+t-4t+t^2}{(t+1)(t-2)}\right) \\
&= \frac{1}{3}\left(-\frac{6}{(t+1)(t-2)}\right) = -\frac{2}{(t+1)(t-2)}.
\end{aligned}
$$

Example

Let $t \in \mathbb{T}_\kappa$ be left-dense. Then

$$
\begin{aligned}
f^{\nabla}(t) &= \lim_{s\to t} \frac{f(t)-f(s)}{t-s}\\
&= \lim_{s\to t} \frac{\frac{1-t}{1+t}-\frac{1-s}{1+s}}{t-s}\\
&= \lim_{s\to t} \frac{(1-t)(1+s)-(1-s)(1+t)}{(t-s)(1+t)(1+s)}\\
&= \lim_{s\to t} \frac{1+s-t-st-1-t+s+st}{(t-s)(1+t)(1+s)}\\
&= \lim_{s\to t} \frac{2s-2t}{(t-s)(1+t)(1+s)}\\
&= -2\lim_{s\to t} \frac{1}{(1+t)(1+s)}\\
&= -\frac{2}{(1+t)^2}.
\end{aligned}
$$

Example

Now, we will find $f^{\Delta\nabla}(t)$ for $t \in \mathbb{T}^{\kappa}_{\kappa}$. We have the following cases. Let $t \in \mathbb{T}^{\kappa}_{\kappa}$ be right-scattered. We have

$$f^{\Delta}(t) = -\frac{2}{(1+t)(4+t)}.$$

Let t be left-scattered. Then

$$\begin{aligned} f^{\Delta\nabla}(t) &= \frac{f^{\Delta}(\rho(t)) - f^{\Delta}(t)}{\rho(t) - t} \\ &= \frac{f^{\Delta}(t-3) - f^{\Delta}(t)}{t-3-t} \\ &= \frac{-\frac{2}{(1+t-3)(4+t-3)} + \frac{2}{(1+t)(4+t)}}{-3} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{-\frac{2}{(t-2)(t+1)} + \frac{2}{(t+1)(t+4)}}{-3} \\
&= \frac{2}{3}\left(\frac{t+4-t+2}{(t-2)(t+1)(t+4)}\right) \\
&= \frac{2}{3}\left(\frac{6}{(t-2)(t+1)(t+4)}\right) \\
&= \frac{4}{(t-2)(t+1)(t+4)}.
\end{aligned}
$$

Example

Let t be left-dense. Then

$$\begin{aligned}
f^{\Delta\nabla}(t) &= \lim_{s\to t} \frac{f^{\Delta}(t) - f^{\Delta}(s)}{t-s} \\
&= \lim_{s\to t} \frac{-\frac{2}{(1+t)(4+t)} + \frac{2}{(1+s)(4+s)}}{t-s} \\
&= -2\lim_{s\to t} \frac{\frac{1}{(1+t)(4+t)} - \frac{1}{(1+s)(4+s)}}{t-s} \\
&= -2\lim_{s\to t} \left(\frac{(1+s)(4+s)-(1+t)(4+t)}{(t-s)(1+t)(4+t)(1+s)(4+s)} \right)
\end{aligned}$$

Example

$$
\begin{aligned}
&= -2\lim_{s\to t}\frac{4+s+4s+s^2-4-t-4t-t^2}{(t-s)(1+t)(4+t)(1+s)(4+s)}\\
&= -2\lim_{s\to t}\frac{5(s-t)+(s-t)(s+t)}{(t-s)(1+t)(4+t)(1+s)(4+s)}\\
&= 2\lim_{s\to t}\frac{5+s+t}{(1+t)(4+t)(1+s)(4+s)}\\
&= \frac{2(5+2t)}{(1+t)^2(4+t)^2}.
\end{aligned}
$$

Example

Let $t \in \mathbb{T}^{\kappa}_{\kappa}$ be right-dense. We have

$$f^{\Delta}(t) = -\frac{2}{(1+t)^2}.$$

Let t be left-scattered. Then

$$\begin{aligned} f^{\Delta\nabla}(t) &= \frac{f^{\Delta}(\rho(t)) - f^{\Delta}(t)}{\rho(t) - t} \\ &= \frac{f^{\Delta}(t-3) - f^{\Delta}(t)}{t-3-t} \\ &= \frac{-\frac{2}{(1+t-3)^2} + \frac{2}{(1+t)^2}}{-3} \\ &= \frac{2}{3}\left(\frac{1}{(t-2)^2} - \frac{1}{(t+1)^2}\right) \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{2}{3}\left(\frac{(t+1)^2-(t-2)^2}{(t+1)^2(t-2)^2}\right)\\
&= \frac{2}{3}\left(\frac{t^2+2t+1-t^2+4t-4}{(t+1)^2(t-2)^2}\right)\\
&= \frac{2}{3}\left(\frac{6t-3}{(t+1)^2(t-2)^2}\right)\\
&= \frac{2(2t-1)}{(t+1)^2(t-2)^2}.
\end{aligned}
$$

Example

Let t be left-dense. Then

$$
\begin{aligned}
f^{\Delta\nabla}(t) &= \lim_{s\to t} \frac{f^{\Delta}(t) - f^{\Delta}(s)}{t-s} \\
&= \lim_{s\to t} \frac{-\frac{2}{(1+t)^2} + \frac{2}{(1+s)^2}}{t-s} \\
&= \lim_{s\to t} \frac{(1+t)^2 - (1+s)^2}{(t-s)(1+t)^2(1+s)^2} \\
&= 2\lim_{s\to t} \frac{1+2t+t^2-1-2s-s^2}{(t-s)(1+t)^2(1+s)^2}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= 2\lim_{s\to t}\frac{2(t-s)+(t-s)(t+s)}{(t-s)(1+t)^2(1+s)^2} \\
&= 2\lim_{s\to t}\frac{2+t+s}{(1+t)^2(1+s)^2} \\
&= \frac{4(1+t)}{(1+t)^4} \\
&= \frac{4}{(1+t)^3}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \mathbb{N}_0$ and

$$f(t) = \frac{1+t^2}{1+7t}, \quad t \in \mathbb{T}.$$

We will find $f^{\nabla}(t)$, $t \in \mathbb{T}_{\kappa}$. We have the following cases. Let $t \in \left\{-\frac{1}{n} : n \in \mathbb{N}, n \geq 2\right\}$. Then

$$\rho(t) = \frac{t}{t+1}.$$

Hence,

$$\begin{aligned} f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\ &= \frac{\frac{1+(\rho(t))^2}{1+7\rho(t)} - \frac{1+t^2}{1+7t}}{\frac{t}{t+1} - t} = \frac{\frac{1+\frac{t^2}{(t+1)^2}}{1+7\frac{t}{t+1}} - \frac{1+t^2}{1+7t}}{t\left(\frac{1}{t+1} - 1\right)} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{\frac{(t+1)^2+t^2}{(t+1)(t+1+7t)} - \frac{1+t^2}{1+7t}}{t\left(\frac{1-t-1}{t+1}\right)}\\
&= \frac{\frac{2t^2+2t+1}{(t+1)(8t+1)} - \frac{1+t^2}{1+7t}}{-\frac{t^2}{t+1}}\\
&= -\frac{(2t^2+2t+1)(1+7t)-(1+t^2)(t+1)(8t+1)}{t^2(8t+1)(7t+1)}\\
&= -\frac{2t^2+14t^3+2t+14t^2+1+7t-(1+t^2)(8t^2+9t+1)}{t^2(8t+1)(7t+1)}\\
&= -\frac{14t^3+16t^2+9t+1-8t^2-9t-1-8t^4-9t^3-t^2}{t^2(8t+1)(7t+1)}\\
&= -\frac{-8t^4+5t^3+7t^2}{t^2(8t+1)(7t+1)} = \frac{8t^2-5t-7}{(8t+1)(7t+1)}.
\end{aligned}
$$

Example

Let $t = 0$. Then $\rho(0) = 0$ and

$$\begin{aligned} f^{\nabla}(t) &= \lim_{s \to 0} \frac{f(s) - f(0)}{s} \\ &= \lim_{s \to 0} \frac{\frac{1+s^2}{1+7s} - 1}{s} \\ &= \lim_{s \to 0} \frac{1 + s^2 - 1 - 7s}{s(1+7s)} \\ &= \lim_{s \to 0} \frac{s(s-7)}{s(1+7s)} \\ &= \lim_{s \to 0} \frac{s-7}{1+7s} = -7. \end{aligned}$$

Example

Let $t \in \mathbb{N}$. Then $\rho(t) = t - 1$ and

$$
\begin{aligned}
f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\
&= \frac{\frac{1+(t-1)^2}{1+7(t-1)} - \frac{1+t^2}{1+7t}}{t - 1 - t} \\
&= \frac{1+t^2}{1+7t} - \frac{1+(t-1)^2}{1+7(t-1)} \\
&= \frac{1+t^2}{1+7t} - \frac{t^2-2t+2}{7t-6} \\
&= \frac{(1+t^2)(7t-6) - (1+7t)(t^2-2t+2)}{(7t+1)(7t-6)} \\
&= \frac{7t - 6 + 7t^3 - 6t^2 - t^2 + 2t - 2 - 7t^3 + 14t^2 - 14t}{(7t+1)(7t-6)} \\
&= \frac{7t^2 - 5t - 8}{(7t-1)(7t+6)}.
\end{aligned}
$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$, $\mathbb{T} = \{0\} \cup U \cup (1 - U) \cup \{1\}$ and

$$f(t) = (2 + t^2)(1 - t^2), \quad t \in \mathbb{T}.$$

We will find $f^{\nabla}(t)$, $t \in \mathbb{T}_\kappa$ and $f^{\nabla^2}(t)$, $t \in \mathbb{T}_{\kappa^2}$. We have the following cases. Let $t \in U$. We have

$$\rho(t) = \frac{1}{2} t.$$

Hence,

$$\begin{aligned} f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\ &= \frac{\left(2 + \frac{t^2}{4}\right)\left(1 - \frac{[}{t}{}^{2}]4\right) - (2 + t^2)(1 - t^2)}{\frac{t}{2} - t} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{(2+t^2)(1-t^2) - \frac{1}{16}(8+t^2)(4-t^2)}{\frac{t}{2}} \\
&= \frac{16(2+gt^2)(1-t^2) - (8+t^2)(4-t^2)}{8t} \\
&= \frac{32 - 32t^2 + 16t^2 - 16t^4 - 32 + 8t^2 - 4t^2 + t^4}{8t} \\
&= \frac{-15t^4 - 12t^2}{8t} \\
&= -\frac{15}{8}t^3 - \frac{3}{2}t.
\end{aligned}
$$

Then

Example

$$\begin{aligned} f^{\nabla^2}(t) &= \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t} \\ &= \frac{-\frac{15}{64}t^3 - \frac{3}{4}t}{\frac{t}{2} - t} \\ &= \frac{\frac{15}{64}t^3 + \frac{3}{4}t}{\frac{t}{2}} \\ &= \frac{15}{32}t^2 + \frac{3}{2}. \end{aligned}$$

Example

Let $t \in (1-U) \backslash \left\{\frac{1}{2}\right\}$. We get

$$\rho(t) = 2t - 1$$

and

$$\begin{aligned} f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\ &= \frac{(2+(2t-1)^2)(1-(2t-1)^2) - (2+t^2)(1-t^2)}{2t-1-t} \\ &= \frac{(4t^2-4t+3)(1-2t+1)(1+2t-1) - (2+t^2)(1-t)(1+t)}{t-1} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{-4(4t^2 - 4t + 3)t(t-1) + (2 + t^2)(t-1)(t+1)}{t-1} \\
&= -4t(4t^2 - 4t + 3) + (2 + t^2)(t+1) \\
&= -16tr^3 + 16t^2 - 12t + 2t + 2 + t^3 + t^2 \\
&= -15t^3 + 17t^2 - 10t + 2.
\end{aligned}
$$

Then

Example

$$f^{\nabla^2}(t) = \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t}$$

$$= \frac{-15(2t-1)^3 + 17(2t-1)^2 - 10(2t-1) + 2 + 15t^3 - 17t^2 + 10t - 2}{2t - 1 - t}$$

$$= \frac{-15(t-1)(4t^2 - 4t + 1 + 2t^2 - t + t^2) + 17(t-1)(2t-1+t)}{t-1}$$

$$- \frac{10(t-1)}{t-1}$$

$$= -15(7t^2 - 5t + 1) + 17(3t - 1) - 10$$

$$= -105t^2 + 75t - 15 + 51t - 17 - 10$$

$$= -105t^2 + 126t - 42.$$

Example

Let $t = \frac{1}{2}$. We have

$$\rho\left(\frac{1}{2}\right) = \frac{1}{4}$$

and then

$$\begin{aligned} f^{\nabla}\left(\frac{1}{2}\right) &= \frac{f\left(\rho\left(\frac{1}{2}\right)\right) - f\left(\frac{1}{2}\right)}{\rho\left(\frac{1}{2}\right) - \frac{1}{2}} \\ &= \frac{\left(2+\frac{1}{16}\right)\left(1-\frac{1}{16}\right) - \left(2+\frac{1}{4}\right)\left(1-\frac{1}{4}\right)}{\frac{1}{4}-\frac{1}{2}} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{\frac{33}{16} \cdot \frac{15}{16} - \frac{9}{4} \cdot \frac{3}{4}}{-\frac{1}{4}} \\
&= \frac{27 \cdot 16 - 33 \cdot 15}{4} \\
&= \frac{432 - 495}{4} \\
&= -\frac{63}{4}.
\end{aligned}
$$

Note that

Example

$$
\begin{aligned}
f^{\nabla^2}\left(\frac{1}{4}\right) &= \frac{15}{32}\cdot\frac{1}{4}+\frac{3}{2} \\
&= \frac{15+3\cdot 64}{128} \\
&= \frac{15+192}{128} \\
&= \frac{207}{128}.
\end{aligned}
$$

Hence,

Example

$$
\begin{aligned}
f^{\nabla^2}\left(\frac{1}{2}\right) &= \frac{f^{\nabla}\left(\rho\left(\frac{1}{2}\right)\right) - f^{\nabla}\left(\frac{1}{2}\right)}{\rho\left(\frac{1}{2}\right) - \frac{1}{2}} \\
&= \frac{\frac{207}{128} + \frac{63}{4}}{\frac{1}{4} - \frac{1}{2}} \\
&= -\frac{207 + 63 \cdot 32}{32} \\
&= -\frac{207 + 2016}{32} \\
&= -\frac{2223}{32}.
\end{aligned}
$$

Example

Let $t = 1$. We have $\rho(1) = 1$ and then

$$
\begin{aligned}
f^{\nabla}(1) &= \lim_{s \to 1} \frac{f(s) - f(1)}{s - 1} \\
&= \lim_{s \to 1} \frac{(2 + s^2)(1 - s^2)}{s - 1} \\
&= -\lim_{s \to 1} (2 + s^2)(1 + s) \\
&= -6.
\end{aligned}
$$

Hence,

Example

$$
\begin{aligned}
f^{\nabla^2}(1) &= \lim_{s\to 1} \frac{f^{\nabla}(s) - f^{\nabla}(1)}{s-1} \\
&= \lim_{s\to 1} \frac{-15s^3 + 17s^2 - 10s + 2 + 6}{s-1} \\
&= -\lim_{s\to 1} \frac{15s^3 - 17s^2 + 10s - 8}{s-1} \\
&= -\lim_{s\to 1} \frac{(15s^2 - 2s + 8)(s-1)}{s-1} \\
&= -\lim_{s\to 1} (15s^2 - 2s + 8) \\
&= -21.
\end{aligned}
$$

Example

Let $\mathbb{T} = (-\mathbb{N}_0) \cup [1, 2] \cup 4^{\mathbb{N}}$ and

$$f(t) = t^3 + t, \quad t \in \mathbb{T}.$$

We will find $f^{\nabla}(t)$, $f^{\nabla^2}(t)$, $f^{\nabla^3}(t)$, $t \in \mathbb{T}$. We have the following cases. Let $t \in (-\mathbb{N}_0)$. Then

$$\rho(t) = t - 1$$

and

$$\begin{aligned} f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\ &= \frac{f(t-1) - f(t)}{t - 1 - t} \end{aligned}$$

Example

$$\begin{aligned} &= t^3 + t - (t-1)^3 - (t-1) \\ &= t^3 - t^3 + 3t^2 - 3t + 1 + t - t + 1 \\ &= 3t^2 - 3t + 2 \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\nabla^2}(t) &= \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t} \\
&= \frac{f^{\nabla}(t-1) - f^{\nabla}(t)}{t - 1 - t} \\
&= 3t^2 - 3t + 2 - 3(t-1)^2 + 3(t-1) - 2 \\
&= 3t^2 - 3t^2 + 6t - 3 - 3t + 3t - 3 \\
&= 6t - 6,
\end{aligned}
$$

and

Example

$$
\begin{aligned}
f^{\nabla^3}(t) &= \frac{f^{\nabla^2}(\rho(t)) - f^{\nabla^2}(t)}{\rho(t) - t} \\
&= \frac{f^{\nabla^2}(t-1) - f^{\nabla^2}(t)}{t-1-t} \\
&= 6t - 6 - 6(t-1) + 6 \\
&= 6t - 6t + 6 \\
&= 6.
\end{aligned}
$$

Example

Let $t = 1$. Then $\rho(1) = 0$ and

$$\begin{aligned} f^{\nabla}(1) &= \frac{f(\rho(1)) - f(1)}{\rho(1) - 1} \\ &= \frac{f(0) - f(1)}{0 - 1} \\ &= f(1) - f(0) \\ &= 2. \end{aligned}$$

Note that

$$f^{\nabla}(0) = 1.$$

Then

Example

$$
\begin{aligned}
f^{\nabla^2}(1) &= \frac{f^{\nabla}(\rho(1)) - f^{\nabla}(1)}{\rho(1) - 1} \\
&= \frac{f^{\nabla}(0) - f^{\nabla}(1)}{0 - 1} \\
&= 2 - 1 \\
&= 1.
\end{aligned}
$$

Next,

$$f^{\nabla^2}(0) = -6$$

and

Example

$$\begin{aligned} f^{\nabla^3}(1) &= \frac{f^{\nabla^2}(\rho(1)) - f^{\nabla^2}(1)}{\rho(1) - 1} \\ & \frac{f^{\nabla^2}(0) - f^{\nabla^2}(1)}{0 - 1} \\ &= 1 - (-6) \\ &= 7. \end{aligned}$$

Let $t \in (1, 2]$. Then

$$\begin{aligned} f^{\nabla}(t) &= 3t^2 + 1, \\ f^{\nabla^2}(t) &= 6t, \\ f^{\nabla^3}(t) &= 6. \end{aligned}$$

Example

Let $t = 4$. Then $\rho(4) = 2$ and

$$\begin{aligned} f^{\nabla}(4) &= \frac{f(\rho(4)) - f(4)}{\rho(4) - 4} \\ &= \frac{f(2) - f(4)}{2 - 4} \\ &= \frac{68 - 10}{2} \\ &= 29. \end{aligned}$$

Note that

$$\begin{aligned} f^{\nabla}(2) &= 3 \cot 4 + 1 \\ &= 13 \end{aligned}$$

and then

Example

$$\begin{aligned} f^{\nabla^2}(4) &= \frac{f^{\nabla}(\rho(4)) - f^{\nabla}(4)}{\rho(4) - 4} \\ &= \frac{f^{\nabla}(2) - f^{\nabla}(4)}{2-4} \\ &= \frac{29-13}{2} \\ &= 8. \end{aligned}$$

Next,

$$f^{\nabla^2}(2) = 12$$

and

Example

$$\begin{aligned} f^{\nabla^3}(4) &= \frac{f^{\nabla^2}(\rho(4)) - f^{\nabla^2}(4)}{\rho(4) - 4} \\ &= \frac{f^{\nabla^2}(2) - f^{\nabla^2}(4)}{2 - 4} \\ &= \frac{12 - 8}{2} \\ &= 2. \end{aligned}$$

Let $t \in 4^{\mathbb{N}}$, $t \geq 16$. Then

$$\rho(t) = \frac{t}{4}$$

and

Example

$$\begin{aligned} f^{\nabla}(t) &= \frac{f(\rho(t)) - f(t)}{\rho(t) - t} \\ &= \frac{t^3 + t - \frac{t^3}{64} - \frac{t}{4}}{t - \frac{t}{4}} \\ &= \frac{\frac{63}{64}t^3 + \frac{3}{4}t}{\frac{3}{4}t} \\ &= \frac{21}{16}t^2 + 1 \end{aligned}$$

and

Example

$$
\begin{aligned}
f^{\nabla^2}(t) &= \frac{f^{\nabla}(\rho(t)) - f^{\nabla}(t)}{\rho(t) - t} \\
&= \frac{\frac{21}{16}t^2 + 1 - \frac{21}{256}t^2 - 1}{t - \frac{t}{4}} \\
&= \frac{\frac{315}{256}t^2}{\frac{3}{4}t} \\
&= \frac{105}{64}t,
\end{aligned}
$$

and

Example

$$
\begin{aligned}
f^{\nabla^3}(t) &= \frac{f^{\nabla^2}(\rho(t)) - f^{\nabla^2}(t)}{\rho(t) - t} \\
&= \frac{\frac{105}{64}t - \frac{105}{256}t}{t - \frac{t}{4}} \\
&= \frac{\frac{315}{256}t}{\frac{3}{4}t} \\
&= \frac{105}{64}.
\end{aligned}
$$