Time Scales Analysis
Lecture 8

Regulated Functions, Pre-Differentiable Functions, Chain Rules. One Sided Derivatives

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Definition

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Example

Let $\mathbb{T} = \mathbb{N}$ and

$$f(t) = \frac{t^2}{t-1}, \quad g(t) = \frac{t}{t+1}, \quad t \in \mathbb{T}.$$

We note that all points of $\mathbb{T}$ are right-scattered. The points $t \in \mathbb{T}$, $t \neq 1$, are left-scattered. Also, $\lim_{t\to 1-} f(t)$ is not finite and $\lim_{t\to 1-} g(t)$ exists and it is finite. Therefore, the function f is not regulated and the function g is regulated.

Example

Let $\mathbb{T} = \mathbb{R}$ and

$$f(t) = \begin{cases} \frac{1}{t} & \text{for} \quad t \in \mathbb{R} \setminus \{0\} \\ 0 & \text{for} \quad t = 0. \end{cases}$$

We have that all points of $\mathbb{T}$ are dense and $\lim_{t \to 0-} f(t)$, $\lim_{t \to 0+} f(t)$ are not finite. Therefore, the function f is not regulated.

Definition

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (region of differentiation) D, provided

1. $D \subset \mathbb{T}^\kappa$,
2. $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$,
3. f is differentiable at each $t \in D$.

Example

Let $\mathbb{T} = P_{a,b} = \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+a]$ for $a > b > 0$. Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+b] \\ t-(a+b)k-b & \text{if} \quad t \in [(a+b)k+b, (a+b)k+a]. \end{cases}$$

Then f is pre-differentiable with $D = \mathbb{T} \setminus \bigcup_{k=0}^{\infty}\{(a+b)k+b\}$.

Example

Let $\mathbb{T} = \mathbb{R}$ and

$$f(t) = \begin{cases} \frac{1}{t-3} & \text{if} \quad \mathbb{R} \setminus \{3\} \\ \\ 0 & \text{if} \quad t = 3. \end{cases}$$

Since $f : \mathbb{T} \to \mathbb{R}$ is not continuous at $t = 3$, the function f is not pre-differentiable.

Example

Let $\mathbb{T} = \mathbb{N}_0 \bigcup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \mathbb{N} \\ \\ t & \text{otherwise.} \end{cases}$$

Since f is not continuous at $t = 1$, it is not pre-differentiable.

Definition

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C_{rd}}$ or $\mathrm{C_{rd}}(\mathbb{T})$ or $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathrm{C^1_{rd}}(\mathbb{T})$.

Some results concerning rd-continuous and regulated functions are contained in the following theorem. Since its statements follow directly from the definitions, we leave the proofs to the reader.

Theorem

Assume $f : \mathbb{T} \to \mathbb{R}$*.*

1. *If* f *is continuous, then* f *is rd-continuous.*
2. *If* f *is rd-continuous, then* f *is regulated.*
3. *The jump operator* σ *is rd-continuous.*
4. *If* f *is regulated or rd-continuous, then so is* f^{σ}*.*
5. *Assume* f *is continuous. If* $g : \mathbb{T} \to \mathbb{R}$ *is regulated or rd-continuous, then* $f \circ g$ *has that property.*

Theorem

Every regulated function on a compact interval is bounded.

Proof.

Assume that $f : [a, b] \to \mathbb{R}$, $[a, b] \subset \mathbb{T}$, is unbounded. Then, for each $n \in \mathbb{N}$, there exists $t_n \in [a, b]$ such that $|f(t_n)| > n$. Because $\{t_n\}_{n\in\mathbb{N}} \subset [a, b]$, there exists a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}} \subset \{t_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} t_{n_k} = t_0.$$

Since $\mathbb{T}$ is closed, we have that $t_0 \in \mathbb{T}$. Also, t_0 is a right-dense point or a left-dense point. Using that f is regulated, we get

$$\left|\lim_{k\to\infty} f(t_{n_k})\right| \neq \infty,$$

which is a contradiction. □

Theorem (Mean Value Theorem)

If $f, g : \mathbb{T} \to \mathbb{R}$ are pre-differentiable with D, then

$$|f^{\Delta}(t)| \leq |g^{\Delta}(t)| \quad \textit{for all} \quad t \in D$$

implies

$$|f(s) - f(r)| \leq g(s) - g(r) \quad \textit{for all} \quad r, s \in \mathbb{T}, \quad r \leq s. \tag{1}$$

Proof.

Let $r, s \in \mathbb{T}$ with $r \leq s$. Assume also

$$[r, s) \setminus D = \{t_n : n \in \mathbb{N}\}.$$

We take $\varepsilon > 0$. We consider the statements

$$S(t) : |f(t) - f(r)| \leq g(t) - g(r) + \varepsilon \left(t - r + \sum_{t_n < t} 2^{-n} \right)$$

Proof.

We will prove, using the induction principle, that $S(t)$ is true for all $t \in [r, s]$.
Since $0 \leq \varepsilon \sum_{t_n < t} 2^{-n}$, the statement $S(r)$ is true.
Let $t \in [r, s]$ be right-scattered and assume that $S(t)$ is true. Then we have

$$\begin{aligned} |f(\sigma(t)) - f(r)| &= |f(t) + \mu(t) f^{\Delta}(t) - f(r)| \\ &\leq \mu(t)|f^{\Delta}(t)| + |f(t) - f(r)| \\ &\leq \mu(t)|f^{\Delta}(t)| + g(t) - g(r) + \varepsilon \left(t - r + \sum_{t_n < t} 2^{-n} \right) \\ &\leq \mu(t) g^{\Delta}(t) + g(t) - g(r) + \varepsilon \left(t - r + \sum_{t_n < t} 2^{-n} \right) \end{aligned}$$

Proof.

$$
\begin{aligned}
&\leq \quad g(\sigma(t)) - g(r) + \varepsilon\left(t - r + \sum_{t_n < t} 2^{-n}\right) \\
&< \quad g(\sigma(t)) - g(r) + \varepsilon\left(\sigma(t) - r + \sum_{t_n < \sigma(t)} 2^{-n}\right)
\end{aligned}
$$

since $t < \sigma(t)$. Hence, $S(\sigma(t))$ is true. Let $t \in [r, s)$ be right-dense. Let $t \in D$. Then f and g are differentiable at t. Hence, there exists a neighbourhood U of t such that

$$|f(t) - f(\tau) - f^{\Delta}(t)(t - \tau)| \leq \frac{\varepsilon}{2}|t - \tau|$$

and

$$|g(t) - g(\tau) - g^{\Delta}(t)(t - \tau)| \leq \frac{\varepsilon}{2}|t - \tau|$$

for all $\tau \in U$. $\square$

Proof.

Thus,

$$|f(t) - f(\tau)| \leq \left(|f^{\Delta}(t)| + \frac{\varepsilon}{2}\right) |t - \tau|$$

for all $\tau \in U$ and

$$g(\tau) - g(t) + g^{\Delta}(t)(t - \tau) \geq -\frac{\varepsilon}{2}|t - \tau|$$

for all $\tau \in U$, i.e.,

$$g(\tau) - g(t) - g^{\Delta}(t)(\tau - t) \geq -\frac{\varepsilon}{2}|t - \tau|$$

for all $\tau \in U$. Hence, for all $\tau \in U \cap (t, \infty)$, we have □

Proof.

$$
\begin{aligned}
|f(\tau) - f(r)| &= |f(\tau) - f(t) + f(t) - f(r)| \\
&\leq \; |f(\tau) - f(t)| + |f(t) - f(r)| \\
&\leq \; \left(|f^\Delta(t)| + \frac{\varepsilon}{2}\right)|t - \tau| + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n < t} 2^{-n}\right) \\
&\leq \; \left(g^\Delta(t) + \frac{\varepsilon}{2}\right)|t - \tau| + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n < t} 2^{-n}\right) \\
&= \; g^\Delta(t)(\tau - t) + \frac{\varepsilon}{2}(\tau - t) + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n < t} 2^{-n}\right)
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
&\leq g(\tau) - g(T) + \frac{\varepsilon}{2}|t-\tau| + \frac{\varepsilon}{2}(\tau - t) + g(t) - g(r) \\
&\quad +\varepsilon\left(t - r + \sum_{t_n<t} 2^{-n}\right) \\
&= g(\tau) - g(r) + \varepsilon(\tau - t) + \varepsilon\left(t - r + \sum_{t_n<t} 2^{-n}\right) \\
&= g(\tau) - g(r) + \varepsilon\left(\tau - r + \sum_{t_n<t} 2^{-n}\right).
\end{aligned}
$$

Therefore, $S(\tau)$ is true for all $\tau \in U \cap (t, \infty)$. □

Proof.

Let $t \notin D$. Then $t = t_m$ for some $m \in \mathbb{N}$. Since f and g are pre-differentiable, they both are continuous. Therefore, there exists a neighbourhood U of t such that

$$|f(\tau) - f(t)| \leq \frac{\varepsilon}{2} 2^{-m} \quad \text{for all} \quad \tau \in U$$

and

$$|g(\tau) - g(t)| \leq \frac{\varepsilon}{2} 2^{-m} \quad \text{for all} \quad \tau \in U.$$

Therefore,

$$g(\tau) - g(t) \geq -\frac{\varepsilon}{2} 2^{-m} \quad \text{for all} \quad \tau \in U.$$

Consequently, □

Proof.

$$
\begin{aligned}
|f(\tau)-f(r)| &= |f(\tau)-f(t)+f(t)-f(r)| \\
&\leq |f(\tau)-f(t)|+|f(t)-f(r)| \\
&\leq \frac{\varepsilon}{2}2^{-m}+g(t)-g(r)+\varepsilon\left(t-r+\sum_{t_n<t}2^{-n}\right) \\
&\leq \frac{\varepsilon}{2}2^{-m}+g(\tau)+\frac{\varepsilon}{2}2^{-m}-g(r)+\varepsilon\left(t-r+\sum_{t_n<t}2^{-n}\right) \\
&= \varepsilon 2^{-m}+g(\tau)-g(r)+\varepsilon\left(t-r+\sum_{t_n<t}2^{-n}\right) \\
&\leq \varepsilon 2^{-m}+g(\tau)-g(r)+\varepsilon\left(\tau-r+\sum_{t_n<\tau}2^{-n}\right),
\end{aligned}
$$

Proof.

so $S(\tau)$ is true for all $\tau \in U \cap (t, \infty)$.
Let t be left-dense and assume that $S(\tau)$ is true for $\tau \in [r, t)$. Then

$$\begin{aligned}
\lim_{\tau \to t-} |f(\tau) - f(r)| &\leq \lim_{\tau \to t-} \left\{ g(\tau) - g(r) + \varepsilon \left(\tau - r + \sum_{t_n < \tau} 2^{-n} \right) \right\} \\
&\leq \lim_{\tau \to t-} \left\{ g(\tau) - g(r) + \varepsilon \left(\tau - r + \sum_{t_n < t} 2^{-n} \right) \right\}
\end{aligned}$$

implies that $S(t)$ is true since f and g are continuous at t.
Thus, using the induction principle, if follows that $S(t)$ is true for all $t \in [r, s]$. Consequently, (1) holds for all $r \leq s$, $r, s \in \mathbb{T}$. $\square$

Theorem

Suppose $f : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D. If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U^\kappa \cap D} |f^\Delta(t)| \right\} |s - r|.$$

Proof.

Without loss of generality, we suppose that $r \leq s$. We set

$$g(t) = \left\{ \sup_{\tau \in U^\kappa \cap D} |f^\Delta(\tau)| \right\} (t - r), \quad t \in \mathbb{T}.$$

Then □

Proof.

$$g^{\Delta}(t) = \left\{ \sup_{\tau \in U^{\kappa} \cap D} |f^{\Delta}(\tau)| \right\} \geq |f^{\Delta}(t)|$$

for all $t \in D \cap [r, s]^{\kappa}$. Thus, using Theorem 11, we obtain

$$|f(t) - f(r)| \leq g(t) - g(r) \quad \text{for all} \quad t \in [r, s],$$

whereupon

$$|f(s) - f(r)| \leq g(s) - g(r) = g(s) = \left\{ \sup_{t \in U^{\kappa} \cap D} |f^{\Delta}(t)| \right\} (s - r).$$

□

Theorem

Let f be pre-differentiable with D. If $f^{\Delta}(t) = 0$ for all $t \in D$, then f is a constant function.

Proof.

Let U be a compact interval with endpoints $r, s \in \mathbb{T}$. From Theorem 12, it follows that for all $r, s \in \mathbb{T}$,

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U^{\kappa} \cap D} |f^{\Delta}(t)| \right\} |s - r| = 0,$$

i.e., $f(s) = f(r)$. Therefore, f is a constant function. $\square$

Theorem

Let f and g be pre-differentiable with D. If $f^{\Delta}(t) = g^{\Delta}(t)$ for all $t \in D$, then

$$g(t) = f(t) + C \quad \textit{for all} \quad t \in \mathbb{T},$$

where C is a constant.

Proof.

Let $h(t) = f(t) - g(t)$, $t \in \mathbb{T}$. Then

$$h^{\Delta}(t) = f^{\Delta}(t) - g^{\Delta}(t) = 0 \quad \text{for all} \quad t \in D.$$

Thus, using Theorem 13, it follows that h is a constant function. □

Theorem

Suppose $f_n : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D for each $n \in \mathbb{N}$. Assume that for each $t \in \mathbb{T}^\kappa$, there exists a compact interval neighbourhood $U(t)$ such that the sequence $\{f_n^\Delta\}_{n\in\mathbb{N}}$ converges uniformly on $U(t) \cap D$.

i *If $\{f_n\}_{n\in\mathbb{N}}$ converges at some $t_0 \in U(t)$ for some $t \in \mathbb{T}^\kappa$, then it converges uniformly on $U(t)$.*

ii *If $\{f_n\}_{n\in\mathbb{N}}$ converges at some $t_0 \in \mathbb{T}$, then it converges uniformly on $U(t)$ for all $t \in \mathbb{T}^\kappa$.*

iii *The limit mapping $f = \lim_{n\to\infty} f_n$ is pre-differentiable with D and*

$$f^\Delta(t) = \lim_{n\to\infty} f_n^\Delta(t) \quad \text{for all} \quad t \in D.$$

Proof.

i] Since $\{f_n^{\Delta}\}_{n\in\mathbb{N}}$ *converges uniformly on* $U(t)\cap D$*, there exists* $N\in\mathbb{N}$ *such that*

$$\sup_{s\in U(t)\cap D} |(f_m - f_n)^{\Delta}(s)|$$

is finite for all $m, n \in \mathbb{N}$*. Let* $m, n \geq N$ *and* $r \in U(t)$*. Then*

$$\begin{aligned} |f_n(r) - f_m(r)| &= |f_n(r) - f_m(r) - (f_n(t_0) - f_m(t_0)) + (f_n(t_0) - f_m(t_0))| \\ &\leq |f_n(t_0) - f_m(t_0)| + \left\{ \sup_{s\in U(t)\cap D} |(f_n - f_m)^{\Delta}(s)| \right\} |r - t_0|. \end{aligned}$$

Hence, $\{f_n\}_{n\in\mathbb{N}}$ *converges uniformly on* $U(t)$*, i.e.,* $\{f_n\}_{n\in\mathbb{N}}$ *is a locally uniformly convergent sequence.* □

ii.

Assume $\{f_n(t_0)\}_{n\in\mathbb{N}}$ converges for some $t_0 \in \mathbb{T}$. Consider the statement

$$S(t): \ \{f_n(t)\}_{n\in\mathbb{N}} \quad \text{converges.}$$

Since $\{f_n(t_0)\}$ converges, the statement $S(t_0)$ is true. Let t be right-scattered and assume that $S(t)$ is true. Then

$$f_n(\sigma(t)) = f_n(t) + \mu(t) f_n^{\Delta}(t)$$

converges by the assumption, i.e., $S(\sigma(t))$ is true. Let t be right-dense and assume that $S(t)$ is true. Then, by *i*, $\{f_n\}_{n\in\mathbb{N}}$ *converges on* $U(t)$, *and so* $S(r)$ *is true for all* $r \in U(t) \cap (t, \infty)$. *Let* t *be left-dense and assume that* $S(r)$ *is true for all* $t_0 \leq r < t$. *Since* $U(t) \cap [t_0, t) \neq \emptyset$, *using again part i, we have that* $\{f_n\}_{n\in\mathbb{N}}$ *converges on* $U(t)$; *in particular,* $S(t)$ *is true.* $\square$

Proof.

Consequently, $S(t)$ is true for all $t \in [t_0, \infty)$. Using the dual version of the induction principle for the negative direction, we have that $S(t)$ is also true for all $t \in (-\infty, t_0]$ (we note that the first part of this has already been shown, the second part follows by $f_n(\rho(t)) = f_n(t) - \mu(\rho(t)) f_n^{\Delta}(\rho(t))$, the third part and the fourth part follow again by *i). [iii] Let $t \in D$. Without loss of generality, we can assume that $\sigma(t) \in U(t)$. We take $\varepsilon > 0$ arbitrarily. Using i, there exists $N \in \mathbb{N}$ such that*

$$|(f_n - f_m)(r) - (f_n - f_m)(\sigma(t))| \leq \left\{ \sup_{s \in U(t) \cap D} |(f_n - f_m)^{\Delta}(s)| \right\} |r - \sigma(t)|$$

for all $r \in U(t)$ and all $m, n \geq N$. Since □

Proof.

$$\{f_n^\Delta\}_{n\in\mathbb{N}}$$

converges uniformly on $U(t)\cap D$, there exists $N_1 \geq N$ such that

$$\sup_{s\in U(t)\cap D} |(f_n - f_m)^\Delta(s)| \leq \frac{\varepsilon}{3} \quad \text{for all} \quad m, n \geq N_1.$$

Hence,

$$|(f_n - f_m)(r) - (f_n - f_m)(\sigma(t))| \leq \frac{\varepsilon}{3}|r - \sigma(t)|$$

for all $r \in U(t)$ and for all $m, n \geq N_1$. Now, letting $m \to \infty$, we obtain

$$|(f_n - f)(r) - (f_n - f)(\sigma(t))| \leq \frac{\varepsilon}{3}|r - \sigma(t)|$$

for all $r \in U(t)$ and all $n \geq N_1$. Let

$$g = \lim_{n\to\infty} f_n^\Delta.$$

Then there exists $M \geq N_1$ such that

Proof.

$$|f_M^{\Delta}(t) - g(t)| \leq \frac{\varepsilon}{3},$$

and since f_M is differentiable at t, there also exists a neighbourhood W of t with

$$|f_M(\sigma(t)) - f_M(r) - f_M^{\Delta}(t)(\sigma(t) - r)| \leq \frac{\varepsilon}{3}|\sigma(t) - r|$$

for all $r \in W$. From here, for all $r \in U(t) \cap W$, we get

$$\begin{aligned} |f(\sigma(t)) - f(r) - g(t)|\sigma(t) - r|| &\leq |(f_M - f)(\sigma(t)) - (f_M - f)(r)| \\ &\quad + |f_M^{\Delta}(t) - g(t)||\sigma(t) - r| + |f_M(\sigma(t)) - f_M(r) - f_M^{\Delta}(t)|\sigma(t) - r|| \\ &\leq \frac{\varepsilon}{3}|\sigma(t) - r| + \frac{\varepsilon}{3}|\sigma(t) - r| + \frac{\varepsilon}{3}|\sigma(t) - r| \\ &= \varepsilon|\sigma(t) - r|. \end{aligned}$$

Consequently, f is differentiable at t with $f^{\Delta}(t) = g(t)$. □

Theorem

Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}$. Assume $f : \mathbb{T}^\kappa \to \mathbb{R}$ is regulated. Then there exists exactly one pre-differentiable function f with D satisfying

$$F^\Delta(t) = f(t) \quad \textit{for all} \quad t \in D, \quad F(t_0) = x_0.$$

Proof.

Let $n \in \mathbb{N}$ and consider the statement

$$S(t) : \begin{cases} \quad \text{there exists a pre-differentiable} \quad (F_{nt}, D_{nt}), \\ F_{nt} : [t_0, t] \to \mathbb{R} \quad \text{with} \quad F_{nt}(t_0) = x_0 \quad \text{and} \\ |F_{nt}^\Delta(s) - f(s)| \leq \frac{1}{n} \quad \text{for} \quad s \in D_{nt}. \end{cases}$$

$\square$

Proof.

Let $t = t_0$, $D_{nt_0} = \emptyset$ and $F_{nt_0}(t_0) = x_0$. Then the statement $S(t_0)$ is true. Let t be right-scattered and assume that $S(t)$ is true. Define

$$D_{n\sigma(t)} = D_{nt} \cup \{t\}$$

and $F_{n\sigma(t)}$ on $[t_0, \sigma(t)]$ by

$$F_{n\sigma(t)}(s) = \begin{cases} F_{nt}(s) & \text{if} \quad s \in [t_0, t] \\ \\ F_{nt}(t) + \mu(t) f(t) & \text{if} \quad s = \sigma(t). \end{cases}$$

Then

$$F_{n\sigma(t)}(t_0) = F_{nt}(t_0) = x_0,$$

$$\left| F^{\Delta}_{n\sigma(t)}(s) - f(s) \right| = \left| F^{\Delta}_{nt}(s) - f(s) \right| \leq \frac{1}{n} \quad \text{for} \quad s \in D_{nt},$$

and

Proof.

$$\begin{aligned}
\left|F^{\Delta}_{n\sigma(t)}(t) - f(t)\right| &= \left|\frac{F_{n\sigma(t)}(\sigma(t)) - F_{n\sigma(t)}(t)}{\mu(t)} - f(t)\right| \\
&= \left|\frac{F_{nt}(t) + \mu(t)f(t) - F_{n\sigma(t)}(t)}{\mu(t)} - f(t)\right| \\
&= \left|\frac{F_{nt}(t) + \mu(t)f(t) - F_{nt}(t)}{\mu(t)} - f(t)\right| \\
&= \left|\frac{\mu(t)f(t)}{\mu(t)} - f(t)\right| \\
&= 0 \\
&\le \frac{1}{n},
\end{aligned}$$

and therefore the statement $S(\sigma(t))$ is true. □

Proof.

Let t be right-dense and assume that $S(t)$ is true. Since t is right-dense and f is regulated,

$$f(t^+) = \lim_{s \to t, s > t} f(s) \quad \text{exists.}$$

Hence, there exists a neighbourhood U of t with

$$|f(s) - f(t^+)| \leq \frac{1}{n} \quad \text{for all} \quad s \in U \cap (t, \infty). \tag{2}$$

Let $r \in U \cap (t, \infty)$. Define $D_{nr} = (D_{nt} \setminus \{t\}) \cup [t, r]^\kappa$ and F_{nr} on $[t_0, r]$ by

$$F_{nr}(s) = \begin{cases} F_{nt}(s) & \text{if} \quad s \in [t_0, t] \\ \\ F_{nt}(t) + f(t^+)(s - t) & \text{if} \quad s \in (t, r]. \end{cases}$$

$\square$

Proof.

Then F_{nr} is continuous at t and hence on $[t_0, r]$. Also, F_{nr} is differentiable on $(t, r]^{\kappa}$ with

$$F_{nr}^{\Delta}(s) = f(t^{+}) \quad \text{for all} \quad s \in (t, r]^{\kappa}.$$

Hence, F_{nr} is pre-differentiable on $[t_0, t)$. Since t is right-dense, we have that F_{nr} is pre-differentiable with D_{nr}. From here and from (2), we also have

$$|F_{nr}^{\Delta}(s) - f(s)| \leq \frac{1}{n} \quad \text{for all} \quad s \in D_{nr}.$$

Therefore, the statement $S(r)$ is true for all $r \in U \cap (t, \infty)$. Now, let t be left-dense and suppose that $S(r)$ is true for $r < t$. Since f is regulated,

$$f(t^{-}) = \lim_{s \to t, s < t} f(s) \quad \text{exists.} \tag{3}$$

□

Proof.

Hence, there exists a neighbourhood U of t with

$$|f(s) - f(t^-)| \leq \frac{1}{n} \quad \text{for all} \quad s \in U \cap (-\infty, t).$$

Fix some $r \in U \cap (-\infty, t)$ and define

$$D_{nt} = \begin{cases} D_{nr} \cup (r, t) & \text{if } r \text{ is right-dense} \\ \\ D_{nr} \cup [r, t) & \text{if } r \text{ is right-scattered} \end{cases}$$

and F_{nt} on $[t_0, t]$ by

$$F_{nt}(s) = \begin{cases} F_{nr}(s) & \text{if} \quad s \in (t_0, r] \\ \\ F_{nr}(t) + f(t^-)(s - r) & \text{if} \quad s \in (r, t]. \end{cases}$$

□

Proof.

We note that F_{nt} is continuous at r and hence in $[t_0, t]$. Since

$$F_{nt}^{\Delta}(s) = f(t^-) \quad \text{for all} \quad s \in (r, t],$$

F_{nt} is pre-differentiable with D_{nt} and

$$|F_{nt}^{\Delta}(s) - f(s)| \leq \frac{1}{n} \quad \text{for all} \quad s \in D_{nt}.$$

Hence, the statement $S(t)$ is true.
By the induction principle, it follows that $S(t)$ is true for all $t \geq t_0$, $t \in \mathbb{T}$. □

Proof.

Similarly, we can show that $S(t)$ is valid for $t \leq t_0$. Hence, F_n is pre-differentiable with D_n, $F_n(t_0) = x_0$, and

$$|F_n^{\Delta}(t) - f(t)| \leq \frac{1}{n} \quad \text{for all} \quad t \in D_n.$$

Now, let

$$F = \lim_{n \to \infty} F_n \quad \text{and} \quad D = \bigcap_{n \in \mathbb{N}} D_n.$$

Then $F(t_0) = x_0$, f is pre-differentiable on D, and, using Theorem 15,

$$F^{\Delta}(t) = \lim_{n \to \infty} F_n^{\Delta}(t) = f(t) \quad \text{for all} \quad t \in D.$$

$\square$