Time Scales Analysis
Lecture 9
Chain Rules. One Sided Derivatives

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Theorem (Chain Rule)

Assume $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^{\kappa}$*, and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then there exists* $c \in [t, \sigma(t)]$ *with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t). \tag{1}$$

Proof.

Fix $t \in \mathbb{T}^\kappa$. If t is right-scattered, then

$$(f \circ g)^\Delta(t) = \frac{f(g(\sigma(t))) - f(g(t))}{\mu(t)}.$$

If $g(t) = g(\sigma(t))$, then

$$(f \circ g)^\Delta(t) = 0 \quad \text{and} \quad g^\Delta(t) = 0,$$

and so (1) holds for any $c \in [t, \sigma(t)]$. Assume that $g(\sigma(t)) \neq g(t)$. Then, by the mean value theorem,

$$\begin{aligned} (f \circ g)^\Delta(t) &= \frac{f(g(\sigma(t))) - f(g(t))}{g(\sigma(t)) - g(t)} \frac{g(\sigma(t)) - g(t)}{\mu(t)} \\ &= f'(\xi) g^\Delta(t), \end{aligned}$$

where ξ is between $g(t)$ and $g(\sigma(t))$. $\square$

Proof.

Since $g : \mathbb{R} \to \mathbb{R}$ is continuous, there exists $c \in [t, \sigma(t)]$ such that $g(c) = \xi$. If t is right-dense, then

$$\begin{aligned}
(f \circ g)^{\Delta}(t) &= \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{t - s} \\
&= \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)} \frac{g(t) - g(s)}{t - s} \\
&= \lim_{s \to t} \left(f'(\xi_s) \frac{g(t) - g(s)}{t - s} \right),
\end{aligned}$$

where ξ_s is between $g(s)$ and $g(t)$. By the continuity of g, we get that $\lim_{s \to t} \xi_s = g(t)$. Therefore,

$$(f \circ g)^{\Delta}(t) = f'(g(t)) g^{\Delta}(t).$$

This completes the proof. □

Example

Let $\mathbb{T} = \mathbb{Z}$, $f(t) = t^3 + 1$, $g(t) = t^2$. We have that $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$, $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $\sigma(t) = t + 1$. Then

$$g^\Delta(t) = \sigma(t) + t$$

and

$$(f \circ g)^\Delta(1) = f'(g(c))g^\Delta(1) = 3g^2(c)(\sigma(1) + 1) = 9c^4. \tag{2}$$

Here, $c \in [1, \sigma(1)] = [1, 2]$.

Example

Also,

$$f \circ g(t) = f(g(t)) = g^3(t) + 1 = t^6 + 1$$

so that

$$(f \circ g)^{\Delta}(t) = (\sigma(t))^5 + t(\sigma(t))^4 + t^2(\sigma(t))^3 + t^3(\sigma(t))^2 + t^4\sigma(t) + t^5$$

and

$$(f \circ g)^{\Delta}(1) = \sigma^5(1) + \sigma^4(1) + \sigma^3(1) + \sigma^2(1) + \sigma(1) + 1 = 63.$$

By (2), we get

$$63 = 9c^4, \quad \text{so} \quad c^4 = 7, \quad \text{so} \quad c = \sqrt[4]{7} \in [1, 2].$$

Example

Let $\mathbb{T} = \{2^n : n \in \mathbb{N}_0\}$, $f(t) = t + 2$, $g(t) = t^2 - 1$. We note that $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. For $t \in \mathbb{T}$, $t = 2^n$, $n \in \mathbb{N}_0$, $n = \log_2 t$, we have

$$\sigma(t) = \inf\left\{2^l : 2^l > 2^n, l \in \mathbb{N}_0\right\} = 2^{n+1} = 2t > t.$$

Therefore, all points of $\mathbb{T}$ are right-scattered. Since $\sup \mathbb{T} = \infty$, we have that $\mathbb{T}^\kappa = \mathbb{T}$. Also, for $t \in \mathbb{T}$, we have

$$(f \circ g)(t) = f(g(t)) = g(t) + 2 = t^2 - 1 + 2 = t^2 + 1$$

and

$$(f \circ g)^\Delta(t) = \sigma(t) + t = 2t + t = 3t.$$

Hence,

Example

$$(f \circ g)^{\Delta}(2) = 6. \tag{3}$$

Now, using Theorem 1, we get that there exists $c \in [2, \sigma(2)] = [2, 4]$ such that

$$(f \circ g)^{\Delta}(2) = f'(g(c))g^{\Delta}(2) = g^{\Delta}(2) = \sigma(2) + 2 = 4 + 2 = 6. \tag{4}$$

From (3) and (4), we find that for every $c \in [2, 4]$,

$$(f \circ g)^{\Delta}(2) = f'(g(c))g^{\Delta}(2).$$

Example

Let $\mathbb{T} = \left\{3^{n^2} : n \in \mathbb{N}_0\right\}$, $f(t) = t^2 + 1$, $g(t) = t^3$. We note that $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. For $t \in \mathbb{T}$, $t = 3^{n^2}$, $n \in \mathbb{N}_0$, $n = (\log_3 t)^{\frac{1}{2}}$, we have

$$\begin{aligned}
\sigma(t) &= \inf\left\{3^{l^2} :\ 3^{l^2} > 3^{n^2},\ l \in \mathbb{N}_0\right\} \\
&= 3^{(n+1)^2} \\
&= 3 \cdot 3^{n^2} \cdot 3^{2n} \\
&= 3t3^{2(\log_3 t)^{\frac{1}{2}}} > t.
\end{aligned}$$

Consequently, all points of $\mathbb{T}$ are right-scattered. Also, $\sup \mathbb{T} = \infty$. Then $\mathbb{T}^\kappa = \mathbb{T}$. Hence, for $t \in \mathbb{T}$, we have

Example

$$
\begin{aligned}
(f \circ g)^{\Delta}(t) &= (t^6+1)^{\Delta} \\
&= (\sigma(t))^5 + t(\sigma(t))^4 + t^2(\sigma(t))^3 + t^3(\sigma(t))^2 + t^4\sigma(t) + t^5.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(f \circ g)^{\Delta}(1) =& \sigma^5(1) + \sigma^4(1) + \sigma^3(1) + \sigma^2(1) + \sigma(1) + 1 \\
=& 3^5 + 3^4 + 3^3 + 3^2 + 3 + 1 \qquad (5) \\
=& 364.
\end{aligned}
$$

Example

From Theorem 1, it follows that there exists $c \in [1, \sigma(1)] = [1, 3]$ such that

$$(f \circ g)^{\Delta}(1) = f'(g(c))g^{\Delta}(1) = 2g(c)g^{\Delta}(1) = 2c^3 g^{\Delta}(1). \tag{6}$$

Because all points of $\mathbb{T}$ are right-scattered, we have

$$g^{\Delta}(1) = \sigma^2(1) + \sigma(1) + 1 = 9 + 3 + 1 = 13.$$

By (6), we find

$$(f \circ g)^{\Delta}(1) = 26c^3.$$

From the last equation and from (5), we obtain

$$364 = 26c^3, \quad \text{so} \quad c^3 = \frac{364}{26} = 14, \quad \text{so} \quad c = \sqrt[3]{14}.$$

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and suppose* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable. Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable, and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t) \tag{7}$$

holds.

Proof.

Note that

$$
\begin{aligned}
f(g(\sigma(t))) - f(g(s)) &= \int_{g(s)}^{g(\sigma(t))} f'(y)dy \\
&= (g(\sigma(t)) - g(s)) \int_0^1 f'(h(g(\sigma(t))) + (1-h)g(s))
\end{aligned}
$$

Let $\varepsilon > 0$ and $t \in \mathbb{T}^\kappa$. Set

$$\varepsilon^* = \frac{\varepsilon}{1 + 2\int_0^1 |f'(hg(\sigma(t)) + (1-h)g(t))|dh}.$$

We choose $\varepsilon > 0$ so that

$$\frac{\varepsilon}{2(1 + \varepsilon^* + |g^\Delta(t)|)} < 1.$$

Since g is differentiable at t, there exists a neighbourhood U_1 of t such that

Proof.

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)| \le \frac{\varepsilon^*}{2}|\sigma(t) - s| \quad \text{for all} \quad s \in U_1.$$

Since f' is continuous on $\mathbb{R}$, using that

$$|hg(\sigma(t)) + (1-h)g(s) - hg(\sigma(t)) - (1-h)g(t)| = (1-h)|g(t) - g(s)|$$

for $h \in (0,1)$, there exists a neighbourhood U_2 of t such that

$$|f'(hg(\sigma(t)) + (1-h)g(s)) - f'(hg(\sigma(t)) + (1-h)g(t))| \le \frac{\varepsilon}{2(1 + \varepsilon^* + |g^{\Delta}(t)|)}$$

for all $s \in U_2$. Let

$$U = U_1 \cap U_2, \quad s \in U.$$

We put

$$\alpha = hg(\sigma(t)) + (1-h)g(s) \quad \text{and} \quad \beta = hg(\sigma(t)) + (1-h)g(t).$$

Proof.

$$\left|(f\circ g)(\sigma(t))-(f\circ g)(s)-(\sigma(t)-s)g^{\Delta}(t)\int_0^1 f'(\beta)dh\right|$$

$$\begin{aligned}
&= \Big|(g(\sigma(t))-g(s))\int_0^1 f'(hg(\sigma(t))+(1-h)g(s))dh \\
&\quad -(\sigma(t)-s)g^{\Delta}(t)\int_0^1 f'(\beta)dh\Big| \\
&= \Big|(g(\sigma(t))-g(s)-(\sigma(t)-s)g^{\Delta}(t))\int_0^1 f'(\alpha)dh \\
&\quad +(\sigma(t)-s)g^{\Delta}(t)\int_0^1 (f'(\alpha)-f'(\beta))dh\Big| \\
&\leq |g(\sigma(t))-g(s)-(\sigma(t)-s)g^{\Delta}(t)|\int_0^1 |f'(\alpha)|dh
\end{aligned}$$

Proof.

$$
\begin{aligned}
& +(\varepsilon^* + |g^\Delta(t)|)|\sigma(t) - s| \int_0^1 |f'(\alpha) - f'(\beta)| dh \\
\leq\ & |\sigma(t) - s| \frac{\varepsilon}{2\left(1 + 2\int_0^1 |f'(\beta)| dh\right)} \int_0^1 |f'(\alpha)| dh \\
& +(\varepsilon^* + |g^\Delta(t)|)|\sigma(t) - s| \frac{\varepsilon}{2(\varepsilon^* + |g^\Delta(t)|)} \\
\leq\ & |\sigma(t) - s| \frac{\varepsilon}{2\left(1 + 2\int_0^1 |f'(\beta)| dh\right)} \left(1 + \int_0^1 |f'(\beta)| dh\right) + \frac{\varepsilon}{2}|\sigma(t) - s| \\
\leq\ & \frac{\varepsilon}{2}|\sigma(t) - s| + \frac{\varepsilon}{2}|\sigma(t) - s| \\
=\ & \varepsilon|\sigma(t) - s|.
\end{aligned}
$$

Therefore, $f \circ g$ is differentiable at t, and its derivative satisfies (7).

Example

Let $\mathbb{T} = \mathbb{Z}$, $f(t) = \frac{1}{1+t^2}$, $g(t) = t + 1$. Note that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. We have

$$f'(t) = -\frac{2t}{(1+t^2)^2}, \quad \mu(t) = 1, \quad g^{\Delta}(t) = 1$$

and

$$g(t) + h\mu(t)g^{\Delta}(t) = t + 1 + h.$$

Hence,

$$\begin{aligned} f'(g(t) + h\mu(t))g^{\Delta}(t) &= f'(t + 1 + h) \\ &= -\frac{2(t+1+h)}{(1+(t+1+h)^2)^2}. \end{aligned}$$

Example

We conclude that$f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and

$$\begin{aligned}
(f \circ g)^{\Delta}(t) &= -\int_0^1 \frac{2(t+1+h)}{(1+(t+1+h)^2)^2} dh \\
&= -\int_0^1 \frac{d(t+1+h)^2}{(1+(t+1+h)^2)^2} \\
&= \left. \frac{1}{1+(t+1+h)^2} \right|_{h=0}^{h=1} \\
&= \frac{1}{1+(t+2)^2} - \frac{1}{1+(t+1)^2}
\end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{(t+1)^2-(t+2)^2}{(t^2+4t+5)(t^2+2t+3)} \\
&= \frac{t^2+2t+1-t^2-4t-4}{(t^2+4t+5)(t^2+2t+3)} \\
&= \frac{-2t-3}{(t^2+4t+5)(t^2+2t+3)}.
\end{aligned}
$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $f(t) = \sin t$, $g(t) = t^2 + 1$. We have that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, $\sigma(t) = 2t$, $\mu(t) = t$, and

$$g^{\Delta}(t) = \sigma(t) + t = 2t + t = 3t$$

so that

$$g(t) + h\mu(t)g^{\Delta}(t) = t^2 + 1 + ht(3t) = t^2 + 1 + 3t^2 h.$$

Moreover,

$$f'(t) = \cos t,$$

and thus

Example

$$f'(g(t) + h\mu(t)g^{\Delta}(t)) = \cos(t^2 + 1 + 3t^2 h).$$

We conclude that $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and

$$\begin{aligned}
(f \circ g)^{\Delta}(t) &= \int_0^1 \cos(t^2 + 1 + 3t^2 h)dh(3t) \\
&= \frac{3t}{3t^2} \int_0^1 \cos(t^2 + 1 + 3t^2 h)d(t^2 + 1 + 3t^2 h) \\
&= \frac{1}{t} \sin(t^2 + 1 + 3t^2 h)\Big|_{h=0}^{h=1} \\
&= \frac{1}{t}(\sin(4t^2 + 1) - \sin(t^2 + 1)) \\
&= \frac{2}{t} \sin \frac{3t^2}{2} \cos\left(\frac{5t^2}{2} + 1\right).
\end{aligned}$$

Example

Let

$$\mathbb{T} = 3^{\mathbb{N}_0}, \quad f(t) = \log(1+t^2), \quad g(t) = t^3 - 2t.$$

We have that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, $\sigma(t) = 3t$, $\mu(t) = 2t$, and

$$\begin{aligned} g^{\Delta}(t) &= (\sigma(t))^2 + t\sigma(t) + t^2 - 2 \\ &= 9t^2 + 3t^2 + t^2 - 2 \\ &= 13t^2 - 2, \end{aligned}$$

so that

Example

$$
\begin{aligned}
g(t) + h\mu(t)g^{\Delta}(t) &= t^3 - 2t + h(2t)(13t^2 - 2) \\
&= t^3 - 2t + (26t^3 - 4t)h.
\end{aligned}
$$

Moreover,

$$f'(t) = \frac{2t}{1+t^2},$$

and therefore

$$
\begin{aligned}
f'(g(t) + h\mu(t)g^{\Delta}(t)) &= \frac{2(t^3 - 2t) + 2(26t^3 - 4t)h}{1 + (t^3 - 2t + (26t^3 - 4t)h)^2} \\
&= 2\frac{t^3 - 2t + (26t^3 - 4t)h}{1 + (t^3 - 2t + (26t^3 - 4t)h)^2}.
\end{aligned}
$$

Example

We conclude that $f \circ g$ is delta differentiable and

$$
\begin{aligned}
(f \circ g)^{\Delta}(t) &= 2\int_0^1 \frac{t^3 - 2t + (26t^3 - 4t)h}{1 + (t^3 - 2t + (26t^3 - 4t)h)^2} dh(13t^2 - 2) \\
&= \frac{13t^2 - 2}{13t^3 - 2t} \\
&\quad \times \int_0^1 \frac{t^3 - 2t + (26t^3 - 4t)h}{1 + (t^3 - 2t + (26t^3 - 4t)h)^2} \mathrm{d}(t^3 - 2t + (26t^3 - 4t))h \\
&= \frac{1}{2t}\int_0^1 \frac{\mathrm{d}(t^3 - 2t + (26t^3 - 4t)h)^2}{1 + (t^3 - 2t + (26t^3 - 4t)h)^2}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{1}{2t}\log(t^3 - 2t + (26t^3 - 4t)h)\Big|_{h=0}^{h=1} \\
&= \frac{1}{2t}\left(\log(27t^3 - 6t) - \log(t^3 - 2t)\right) \\
&= \frac{1}{2t}\log\frac{27t^2 - 6}{t^2 - 2}.
\end{aligned}
$$

Theorem (Chain Rule)

Assume $v : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} = v(\mathbb{T})$ *is a time scale. Let* $w : \tilde{\mathbb{T}} \to \mathbb{R}$*. If* $v^{\Delta}(t)$ *and* $w^{\tilde{\Delta}}(v(t))$ *exist for* $t \in \mathbb{T}^{\kappa}$*, then*

$$(w \circ v)^{\Delta} = (w^{\tilde{\Delta}} \circ v)v^{\Delta}.$$

Proof.

Let $\varepsilon \in (0,1)$ be arbitrarily chosen. We put

$$\varepsilon^* = \frac{\varepsilon}{1 + |v^{\Delta}(t)| + |w^{\tilde{\Delta}}(v(t))|}.$$

Note that $0 < \varepsilon^* < 1$. Since w is differentiable at t, there exists a neighbourhood U_1 of t such that

$$|v(\sigma(t)) - v(t) - (\sigma(t) - s)v^{\Delta}(t)| \leq \varepsilon^* |\sigma(t) - s| \quad \text{for all} \quad s \in U_1.$$

Since w is differentiable at $v(t)$, there exists a neighbourhood U_2 of $v(t)$ such that

$$|w(\tilde{\sigma}(v(t))) - w(r) - (\tilde{\sigma}(v(t)) - r)w^{\tilde{\Delta}}(v(t))| \leq \varepsilon^* |\tilde{\sigma}(v(t)) - r|$$

for all $r \in U_2$. Let $U = U_1 \cap v^{-1}(U_2)$ and let $s \in U$. $\square$

Proof.

Then $s \in U_1$, $v(s) \in U_2$, and

$$
\begin{aligned}
&|w(v(\sigma(t))) - w(v(s)) - (\sigma(t) - s)w^{\tilde{\Delta}}(v(t))v^{\Delta}(t)| \\
= \; & |w(v(\sigma(t))) - w(v(s)) - (\tilde{\sigma}(v(t)) - v(s))w^{\tilde{\Delta}}(v(t)) \\
& + [(\tilde{\sigma}(v(t)) - v(s)) - (\sigma(t) - s)v^{\Delta}(t)]w^{\tilde{\Delta}}(v(t))| \\
\leq \; & |w(v(\sigma(t))) - w(v(s)) - (\tilde{\sigma}(v(t)) - v(s))w^{\tilde{\Delta}}(v(t))| \\
& + |(\tilde{\sigma}(v(t)) - v(s)) - (\sigma(t) - s)v^{\Delta}(t)||w^{\tilde{\Delta}}(v(t))| \\
\leq \; & \varepsilon^* |\tilde{\sigma}(v(t)) - v(s)| + \varepsilon^* |\sigma(t) - s||w^{\tilde{\Delta}}(v(t))|
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&= \varepsilon^*|\tilde{\sigma}(v(t)) - v(s) - (\sigma(t) - s)v^{\Delta}(t) + (\sigma(t) - s)v^{\Delta}(t)| \\
&\quad + \varepsilon^*|\sigma(t) - s||w^{\tilde{\Delta}}(v(t))| \\
&\leq \varepsilon^*|\tilde{\sigma}(v(t)) - v(s) - (\sigma(t) - s)v^{\Delta}(t)| + \varepsilon^*|\sigma(t) - s||v^{\Delta}(t)| \\
&\quad + \varepsilon^*|\sigma(t) - s||w^{\tilde{\Delta}}(v(t))|
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
&\leq \quad \varepsilon^* \left(\varepsilon^* |\sigma(t) - s| + |\sigma(t) - s||v^{\Delta}(t)| + |\sigma(t) - s||w^{\tilde{\Delta}}(v(t))|\right) \\
&= \quad \varepsilon^*(\varepsilon^* + |v^{\Delta}(t)| + |w^{\tilde{\Delta}}(v(t))|)|\sigma(t) - s| \\
&\leq \quad \varepsilon^*(1 + |v^{\Delta}(t)| + |w^{\tilde{\Delta}}(v(t))|)|\sigma(t) - s| \\
&= \quad \varepsilon|\sigma(t) - s|,
\end{aligned}
$$

which completes the proof. □

Example

Let $\mathbb{T} = \left\{2^{2n} : n \in \mathbb{N}_0\right\}$, $v(t) = t^2$, $w(t) = t^2 + 1$. Then $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} = v(\mathbb{T}) = \left\{2^{4n} : n \in \mathbb{N}_0\right\}$ is a time scale. For $t \in \mathbb{T}$, $t = 2^{2n}$, $n \in \mathbb{N}_0$, we have

$$\sigma(t) = \inf\left\{2^{2l} :\ 2^{2l} > 2^{2n},\ l \in \mathbb{N}_0\right\} = 2^{2n+2} = 4t$$

and

$$v^{\Delta}(t) = \sigma(t) + t = 5t.$$

For $t \in \tilde{\mathbb{T}}$, $t = 2^{4n}$, $n \in \mathbb{N}_0$, we have

$$\tilde{\sigma}(t) = \inf\left\{2^{4l} :\ 2^{4l} > 2^{4n},\ l \in \mathbb{N}_0\right\} = 2^{4n+4} = 16t.$$

Example

Also, for $t \in \mathbb{T}$, we have

$$(w \circ v)(t) = w(v(t)) = v^2(t) + 1 = t^4 + 1$$

and

$$\begin{aligned}(w \circ v)^{\Delta}(t) &= (\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3 \\ &= 64t^3 + 16t^3 + 4t^3 + t^3 \\ &= 85t^3.\end{aligned}$$

Thus,

Example

$$
\begin{aligned}
\left(w^{\tilde{\Delta}} \circ v\right)(t) &= \tilde{\sigma}(v(t)) + v(t) \\
&= 16v(t) + v(t) \\
&= 17v(t) \\
&= 17t^2
\end{aligned}
$$

and

$$\left(w^{\tilde{\Delta}} \circ v\right)(t)v^{\Delta}(t) = 17t^2(5t) = 85t^3.$$

Consequently,

$$(w \circ v)^{\Delta}(t) = (w^{\tilde{\Delta}} \circ v(t))v^{\Delta}(t), \quad t \in \mathbb{T}^{\kappa}.$$

Example

Let $\mathbb{T} = \{n+1 : n \in \mathbb{N}_0\}$, $v(t) = t^2$, $w(t) = t$. Then $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} = \{(n+1)^2 : n \in \mathbb{N}_0\}$ is a time scale. For $t \in \mathbb{T}$, $t = n+1$, $n \in \mathbb{N}_0$, we have

$$\sigma(t) = \inf\{l+1 : \ l+1 > n+1, \ l \in \mathbb{N}_0\} = n+2 = t+1$$

and

$$v^{\Delta}(t) = \sigma(t) + t = t+1+t = 2t+1.$$

For $t \in \tilde{\mathbb{T}}$, $t = (n+1)^2$, $n \in \mathbb{N}_0$, we have

$$\begin{aligned}\tilde{\sigma}(t) &= \{(l+1)^2 : \ (l+1)^2 > (n+1)^2, \ l \in \mathbb{N}_0\} = (n+2)^2 \\ &= (n+1)^2 + 2(n+1) + 1 = t + 2\sqrt{t} + 1.\end{aligned}$$

Example

Hence, for $t \in \mathbb{T}$, we get

$$(w^{\tilde{\Delta}} \circ v)(t) = 1, \quad (w^{\tilde{\Delta}} \circ v)(t)v^{\Delta}(t) = 1(2t+1) = 2t+1,$$

and thus

$$w \circ v(t) = v(t) = t^2, \quad (w \circ v)^{\Delta}(t) = \sigma(t) + t = 2t+1.$$

Consequently,

$$(w \circ v)^{\Delta}(t) = (w^{\tilde{\Delta}} \circ v(t))v^{\Delta}(t), \quad t \in \mathbb{T}^{\kappa}.$$

Example

Let $\mathbb{T} = \{2^n : n \in \mathbb{N}_0\}$, $v(t) = t$, $w(t) = t^2$. Then $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $v(\mathbb{T}) = \mathbb{T}$. For $t \in \mathbb{T}$, $t = 2^n$, $n \in \mathbb{N}_0$, we have

$$\sigma(t) = \inf\{2^l : \; 2^l > 2^n, \; l \in \mathbb{N}_0\} = 2^{n+1} = 2t$$

and

$$v^{\Delta}(t) = 1.$$

Moreover,

$$(w \circ v)(t) = w(v(t)) = v^2(t) = t^2,$$

and thus

$$(w \circ v)^{\Delta}(t) = \sigma(t) + t = 2t + t = 3t.$$

Example

Therefore,

$$(w^{\Delta} \circ v)(t) = \sigma(v(t)) + v(t) = 2v(t) + v(t) = 3v(t) = 3t,$$

so that

$$(w^{\Delta} \circ v)(t)v^{\Delta}(t) = 3t.$$

Consequently,

$$(w \circ v)^{\Delta}(t) = (w^{\tilde{\Delta}} \circ v)(t)v^{\Delta}(t), \quad t \in \mathbb{T}^{\kappa}.$$

Example

Let $U = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$,

$$\mathbb{T} = U \cup (1-U) \cup (1+U) \cup (2-U) \cup (2+U) \cup \{0, 1, 2\},$$

and

$$\begin{aligned} f(t) &= \cos t, \\ g(t) &= (2+t^2)(1-t^2), \quad t \in \mathbb{T}. \end{aligned}$$

We will find $(f \circ g)^{\Delta}(t)$, $t \in \mathbb{T}^{\kappa}$. We have

$$f'(t) = -\sin t, \quad t \in \mathbb{T}.$$

We have the following cases.

Example

Let $t = \frac{1}{2}$. We have

$$g^{\Delta}\left(\frac{1}{2}\right) = -\frac{151}{64}.$$

We get

$$\begin{aligned} \mu\left(\frac{1}{2}\right) &= \frac{3}{4} - \frac{1}{2} \\ &= \frac{1}{4}. \end{aligned}$$

Next,

$$\begin{aligned} g\left(\frac{1}{2}\right) &= \left(2 + \frac{1}{4}\right)\left(1 - \frac{1}{4}\right) \\ &= \frac{9}{4} \cdot \frac{3}{4} \\ &= \frac{27}{16}. \end{aligned}$$

Example

Then

$$
\begin{aligned}
(f \circ g)^{\Delta}(t) &= -g^{\Delta}\left(\frac{1}{2}\right)\int_{0}^{1} \sin\left(g\left(\frac{1}{2}\right) + h\mu\left(\frac{1}{2}\right)g^{\Delta}\left(\frac{1}{2}\right)\right) dh \\
&= \frac{151}{64}\int_{0}^{1} \sin\left(\frac{27}{16} + h\left(\frac{1}{4}\right)\left(-\frac{151}{64}\right)\right) dh \\
&= \frac{151}{64}\int_{0}^{1} \cos\left(\frac{27}{16} - \frac{151}{256}h\right) dh
\end{aligned}
$$

Example

$$
\begin{aligned}
&= -4\int_0^1 \cos\left(\frac{27}{16}-\frac{151}{256}h\right) d\left(\frac{27}{16}-\frac{151}{256}h\right) \\
&= 4\sin\left(\frac{27}{16}-\frac{151}{256}h\right)\Big|_{h=0}^{h=1} \\
&= -4\left(\sin\left(\frac{151}{256}\right)+\sin\left(\frac{27}{16}\right)\right).
\end{aligned}
$$

Example

Let $t = \frac{3}{2}$. We have

$$g^{\Delta}\left(\frac{3}{2}\right) = -\frac{1195}{64}.$$

We find

$$\begin{aligned} \mu\left(\frac{3}{2}\right) &= \sigma\left(\frac{3}{2}\right) - \frac{3}{2} \\ &= \frac{7}{4} - \frac{3}{2} \\ &= \frac{1}{4}. \end{aligned}$$

Next, we get

$$\begin{aligned} g\left(\frac{3}{2}\right) &= \left(2 + \frac{9}{4}\right)\left(1 - \frac{9}{4}\right) \\ &= \frac{17}{4}\left(-\frac{5}{4}\right) = -\frac{85}{16}. \end{aligned}$$

Example

Then

$$
\begin{aligned}
(f\circ g)^{\Delta}\left(\frac{3}{2}\right) &= -g^{\Delta}\left(\frac{3}{2}\right)\int_{0}^{1}\sin\left(g\left(\frac{3}{2}\right)+h\mu\left(\frac{3}{2}\right)g^{\Delta}\left(\frac{3}{2}\right)\right)dh \\
&= \frac{1195}{64}\int_{0}^{1}\sin\left(-\frac{85}{16}+\frac{1}{4}\left(-\frac{1195}{64}\right)h\right)dh \\
&= \frac{1195}{64}\int_{0}^{1}\sin\left(-\frac{85}{16}-\frac{1195}{256}h\right)dh
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{1195}{64}\int_0^1 \sin\left(-\frac{85}{16}-\frac{1195}{256}h\right) d\left(-\frac{85}{16}-\frac{1195}{256}h\right)\\
&= 4\cos\left(-\frac{85}{16}-\frac{1195}{256}h\right)\Bigg|_{h=0}^{h=1}\\
&= 4\left(\cos\left(\frac{85}{16}\right)-\cos\left(\frac{2555}{256}\right)\right).
\end{aligned}
$$

Let $t = 0$. We have

$$g^{\Delta}(0) = 0.$$

Hence,

$$
\begin{aligned}
(f\circ g)^{\Delta}(0) &= g^{\Delta}(0)\int_0^1 \sin\left(g(0)+h\mu(0)g^{\Delta}(0)\right) dh\\
&= 0.
\end{aligned}
$$

Example

Next, we have

$$g(1) = 0.$$

Then

$$\begin{aligned}(f \circ g)^{\Delta}(1) &= -g^{\Delta}(1) \int_{0}^{1} \sin\left(g(1) + h\mu(1)g^{\Delta}(1)\right) dh \\ &= 6 \int_{0}^{1} \sin 0 dh \\ &= 0.\end{aligned}$$

Let $t = 2$. We have

$$g^{\Delta}(2) = -36.$$

Example

We get

$$\begin{aligned}\mu(2) &= \sigma(2) - 2\\ &= 2 - 2\\ &= 0.\end{aligned}$$

Next, we find

$$\begin{aligned}g(2) &= (2+4)(1-4)\\ &= -18.\end{aligned}$$

Hence,

Example

$$
\begin{aligned}
(f \circ g)^{\Delta}(2) &= -g^{\Delta}(2) \int_0^2 \sin\left(g(2) + h\mu(2) g^{\Delta}(2)\right) dh \\
&= 36 \int_0^1 \sin(-18) dh \\
&= -36 \sin(18).
\end{aligned}
$$

Let $t \in U \setminus \left\{\frac{1}{2}\right\}$. We have

$$g^{\Delta}(t) = -15t^3 - 3t.$$

We find

$$
\begin{aligned}
\mu(t) &= \sigma(t) - t \\
&= 2t - t = t.
\end{aligned}
$$

Example

$$(f\circ g)^{\Delta}(t) = -g^{\Delta}(t)\int\limits_0^1 \sin\left(g(t)+h\mu(t)g^{\Delta}(t)\right)dh$$

$$= -\frac{1}{\mu(t)}\int\limits_0^1 \sin\left(g(t)+h\mu(t)g^{\Delta}(t)\right)d\left(g(t)+h\mu(t)g^{\Delta}(t)\right)$$

$$= \frac{1}{\mu(t)}\cos\left(g(t)+h\mu(t)g^{\Delta}(t)\right)\Big|_{h=0}^{h=1}$$

$$= \frac{1}{\mu(t)}\left(\cos\left(g(t)+\mu(t)g^{\Delta}(t)\right)-\cos(g(t))\right)$$

$$= \frac{1}{t}\left(\cos\left((2+t^2)(1-t^2)-t(15t^3+3t)\right)-\cos((2+t^2)(1-t^2))\right).$$

Example

Let $t \in (1-U)\setminus\left\{\frac{1}{2}\right\}$. We have

$$g^{\Delta}(t) = -\frac{15t^3 + 11t^2 + 17t + 5}{8}.$$

We find

$$\begin{aligned}\mu(t) &= \sigma(t) - t \\ &= \frac{1+t}{2} - t \\ &= \frac{1-t}{2}.\end{aligned}$$

Example

Hence,

$$
\begin{aligned}
(f\circ g)^{\Delta}(t) &= -g^{\Delta}(t)\int_{0}^{1}\sin\left(g(t)+h\mu(t)g^{\Delta}(t)\right)dh\\
&= -\frac{1}{\mu(t)}\int_{0}^{1}\sin\left(g(t)+h\mu(t)g^{\Delta}(t)\right)d\left(g(t)+h\mu(t)g^{\Delta}(t)\right)\\
&= \frac{1}{\mu(t)}\cos\left(g(t)+h\mu(t)g^{\Delta}(t)\right)\Big|_{h=0}^{h=1}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{1}{\mu(t)}\left(\cos\left(g(t)+\mu(t)g^{\Delta}(t)\right)-\cos(g(t))\right)\\
&= \frac{2}{1-t}\left(\cos\left((2+t^2)(1-t^2)-\frac{(1-t)(15t^3+11t^2+17t+5)}{16}\right)-\text{co}\right.
\end{aligned}
$$

Let $t \in (1+U)\setminus\left\{\frac{3}{2}\right\}$. We have

$$g^{\Delta}(t) = -\frac{15t^3+11t^2+17t+5}{8}.$$

We find

$$
\begin{aligned}
\mu(t) &= \sigma(t)-t\\
&= 2t-1-t\\
&= t-1.
\end{aligned}
$$

Example

Hence,

$$
\begin{aligned}
(f \circ g)^{\Delta}(t) &= -g^{\Delta}(t) \int_0^1 \sin\left(g(t) + h\mu(t)g^{\Delta}(t)\right) dh \\
&= -\frac{1}{\mu(t)} \int_0^1 \sin\left(g(t) + h\mu(t)g^{\Delta}(t)\right) d\left(g(t) + h\mu(t)g^{\Delta}(t)\right) \\
&= \frac{1}{\mu(t)} \cos\left(g(t) + h\mu(t)g^{\Delta}(t)\right) \Big|_{h=0}^{h=1} \\
&= \frac{1}{\mu(t)} \left(\cos\left(g(t) + \mu(t)g^{\Delta}(t)\right) - \cos(g(t))\right) \\
&= \frac{1}{t} \left(\cos\left((2+t^2)(1-t^2) - \frac{(t-1)(15t^3+11t^2+17t+5)}{8}\right.\right.
\end{aligned}
$$

Let $t \in (2-U) \setminus \left\{\frac{3}{2}\right\}$. We have

Example

$$g^{\Delta}(t) = -\frac{15t^3 + 22t^2 + 32t + 8}{8}.$$

We find

$$\begin{aligned}
\mu(t) &= \sigma(t) - t \\
&= \frac{t+2}{2} - t \\
&= \frac{2-t}{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(f \circ g)^{\Delta}(t) &= -g^{\Delta}(t) \int_0^1 \sin\left(g(t) + h\mu(t)g^{\Delta}(t)\right) dh \\
&= -\frac{1}{\mu(t)} \int_0^1 \sin\left(g(t) + h\mu(t)g^{\Delta}(t)\right) d\left(g(t) + h\mu(t)g^{\Delta}(t)\right)
\end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{1}{\mu(t)} \cos\left(g(t) + h\mu(t)g^{\Delta}(t)\right)\Big|_{h=0}^{h=1} \\
&= \frac{1}{\mu(t)} \left(\cos\left(g(t) + \mu(t)g^{\Delta}(t)\right) - \cos(g(t))\right) \\
&= \frac{2}{2-t}\Bigg(\cos\left((2+t^2)(1-t^2) - \frac{(2-t)(15t^3+22t^2+32t+8)}{16}\right) \\
&\quad - \cos((2+t^2)(1-t^2))\Bigg).
\end{aligned}
$$

Let $t \in (2+U)\setminus\left\{\frac{3}{2}\right\}$. We have

$$g^{\Delta}(t) = -15t^3 + 34t^2 - 31t + 10.$$

We find

$$\mu(t) \;=\; \sigma(t) - t = 2t - 2 - t = t - 2.$$

Example

Hence,

$$
\begin{aligned}
(f \circ g)^{\Delta}(t) &= -g^{\Delta}(t) \int_{0}^{1} \sin\left(g(t) + h\mu(t)g^{\Delta}(t)\right) dh \\
&= -\frac{1}{\mu(t)} \int_{0}^{1} \sin\left(g(t) + h\mu(t)g^{\Delta}(t)\right) d\left(g(t) + h\mu(t)g^{\Delta}(t)\right) \\
&= \frac{1}{\mu(t)} \cos\left(g(t) + h\mu(t)g^{\Delta}(t)\right) \Big|_{h=0}^{h=1}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{1}{\mu(t)}\left(\cos\left(g(t)+\mu(t)g^{\Delta}(t)\right)-\cos(g(t))\right)\\
&= \frac{1}{t}\Bigg(\cos\left((2+t^2)(1-t^2)+(t-2)(-15t_3^34t^2-31t+10)\right)\\
&\quad -\cos((2+t^2)(1-t^2))\Bigg).
\end{aligned}
$$

Theorem (Derivative of the Inverse)

Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} = v(\mathbb{T})$ is a time scale. Then

$$\left((v^{-1})^{\tilde{\Delta}} \circ v\right)(t) = \frac{1}{v^{\Delta}(t)}$$

for any $t \in \mathbb{T}^{\kappa}$ such that $v^{\Delta}(t) \neq 0$.

Proof.

Let $w = v^{-1} : \tilde{\mathbb{T}} \to \mathbb{T}$ in Theorem 19. □

Example

Let $\mathbb{T} = \mathbb{N}$ and $v(t) = t^2 + 1$. Then $\sigma(t) = t + 1$, $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing, and

$$v^{\Delta}(t) = \sigma(t) + t = 2t + 1.$$

Hence,

$$\left(\left(v^{-1}\right)^{\tilde{\Delta}} \circ v \right)(t) = \frac{1}{v^{\Delta}(t)} = \frac{1}{2t+1}.$$

Example

Let $\mathbb{T} = \{n+3 : n \in \mathbb{N}_0\}$, $v(t) = t^2$. Then $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing, $\sigma(t) = t+1$, and

$$v^{\Delta}(t) = \sigma(t) + t = 2t+1.$$

Hence,

$$\left(\left(v^{-1}\right)^{\tilde{\Delta}} \circ v\right)(t) = \frac{1}{v^{\Delta}(t)} = \frac{1}{2t+1}.$$

Example

Let $\mathbb{T} = \left\{2^{n^2} : n \in \mathbb{N}_0\right\}$, $v(t) = t^3$. Then $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing, and for $t \in \mathbb{T}$, $t = 2^{n^2}$, $n \in \mathbb{N}_0$, $n = (\log_2 t)^{\frac{1}{2}}$, we have

$$\sigma(t) = \inf\left\{2^{l^2} :\ 2^{l^2} > 2^{n^2},\ l \in \mathbb{N}_0\right\} = 2^{(n+1)^2} = 2^{n^2} 2^{2n+1} = t2^{2(\log_2 t)^{\frac{1}{2}}+1}.$$

Then

Example

$$v^{\Delta}(t) = (\sigma(t))^2 + t\sigma(t) + t^2 = t^2 2^{4(\log_2 t)^{\frac{1}{2}}+2} + t^2 2^{2(\log_2 t)^{\frac{1}{2}}+1} + t^2.$$

Hence,

$$\left(\left(v^{-1}\right)^{\tilde{\Delta}} \circ v\right)(t) = \frac{1}{t^2 2^{4(\log_2 t)^{\frac{1}{2}}+2} + t^2 2^{2(\log_2 t)^{\frac{1}{2}}+1} + t^2}.$$

Definition

If f is defined on $[t_0, b) \subset \mathbb{T}$, then the *right-hand derivative* of f at t_0 is defined to be

$$f_+^\Delta(t_0) = \lim_{t \to t_0+} \frac{f(\sigma(t_0)) - f(t)}{\sigma(t_0) - t},$$

while if f is defined on $(a, t_0] \subset \mathbb{T}$, then the left-hand derivative of f at t_0 is defined by

$$f_-^\Delta(t_0) = \lim_{t \to t_0-} \frac{f(\sigma(t_0)) - f(t)}{\sigma(t_0) - t}.$$

Remark

f is differentiable at t_0 if and only if $f_+^\Delta(t_0)$ and $f_-^\Delta(t_0)$ exist and

$$f^\Delta(t_0) = f_-^\Delta(t_0) = f_+^\Delta(t_0).$$

Example

Consider

$$f(t) = \begin{cases} t^2 + t & \text{for} \quad t \in \{1, 2, 3\}, \\ \\ t + 1 & \text{for} \quad t \in [-1, 1), \end{cases}$$

where $[-1, 1)$ is the real-valued interval. Note that f is continuous on

$$[-1, 1) \cup \{1, 2, 3\}.$$

Also,

$$\begin{aligned} f_-^\Delta(1) &= \lim_{t \to 1-} \frac{f(\sigma(1)) - f(t)}{\sigma(1) - t} \\ &= \lim_{t \to 1-} \frac{f(2) - f(t)}{2 - t} \\ &= \lim_{t \to 1-} \frac{6 - (t + 1)}{2 - t} = \lim_{t \to 1-} \frac{5 - t}{2 - t} = 4 \end{aligned}$$

Example

and

$$
\begin{aligned}
f_+^{\Delta}(1) &= \lim_{t\to 1+} \frac{f(\sigma(1)) - f(t)}{\sigma(1) - t} \\
&= \lim_{t\to 1+} \frac{f(2) - f(t)}{2 - t} \\
&= \lim_{t\to 1+} \frac{6 - t^2 - t}{2 - t} \\
&= \lim_{t\to 1+} \frac{(2 - t)(t + 3)}{2 - t} \\
&= \lim_{t\to 1+} (t + 3) \\
&= 4.
\end{aligned}
$$

Therefore,

Example

Consider

$$f(t) = \begin{cases} t+3 & \text{for} \quad t \in [-2, 2), \\ \\ t^2 + t & \text{for} \quad t \in \{2, 4, 8\}. \end{cases}$$

We have

$$\begin{aligned} f_-^{\Delta}(2) &= \lim_{t \to 2-} \frac{f(\sigma(2)) - f(t)}{\sigma(2) - t} \\ &= \lim_{t \to 2-} \frac{f(4) - f(t)}{4 - t} \\ &= \lim_{t \to 2-} \frac{20 - (t+3)}{4 - t} = \lim_{t \to 2-} \frac{17 - t}{4 - t} = \frac{15}{2} \end{aligned}$$

Example

and

$$
\begin{aligned}
f_+^{\Delta}(t) &= \lim_{t\to 2+} \frac{f(\sigma(2)) - f(t)}{\sigma(2) - t} \\
&= \lim_{t\to 2+} \frac{f(4) - f(t)}{4 - t} \\
&= \lim_{t\to 2+} \frac{20 - t^2 - t}{4 - t} \\
&= 7.
\end{aligned}
$$

Therefore,

$$f_-^{\Delta}(2) \neq f_+^{\Delta}(2).$$

Hence, f is not differentiable at $t = 2$.

Example

Consider

$$f(t) = \begin{cases} 1 & \text{for} \quad t \in \{3,5,7,9\}, \\ \\ 3 & \text{for} \quad t \in [0,3), \end{cases}$$

where $[0,3)$ is the real-valued interval. We have

$$\begin{aligned} f_{-}^{\Delta}(3) &= \lim_{t \to 3-} \frac{f(\sigma(3)) - f(t)}{\sigma(3) - t} \\ &= \lim_{t \to 3-} \frac{f(5) - f(t)}{5 - t} \\ &= \lim_{t \to 3-} \frac{1-3}{5-t} = \lim_{t \to 3-} \frac{-2}{5-t} = -1 \end{aligned}$$

Example

and

$$\begin{aligned} f_{+}^{\Delta}(3) &= \lim_{t \to 3+} \frac{f(\sigma(3)) - f(t)}{\sigma(3) - t} \\ &= \lim_{t \to 3+} \frac{f(5) - f(t)}{5 - t} \\ &= \lim_{t \to 3+} \frac{1 - 1}{5 - t} \\ &= 0. \end{aligned}$$

Consequently,

$$f_{-}^{\Delta}(3) \neq f_{+}^{\Delta}(3).$$

Hence, f is not differentiable at 3.