Time Scales Analysis
Lecture 10

Extreme Values. Convex and Concave Functions. Completely Delta Differentiable Functions

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Assume that f is defined on $D_f \subset \mathbb{T}$. Let $t_0 \in \mathbb{T}$.

Definition

We say that $f(t_0)$ is a *local maximum value* of f if there exists $\delta > 0$ so that

$$f(t) \leq f(t_0) \quad \text{for all} \quad t \in (t_0 - \delta, t_0 + \delta) \cap D_f$$

and $f(\rho(t_0)), f(\sigma(t_0)) \leq f(t_0)$, or a *local minimum value* of f if there exists $\delta > 0$ such that

$$f(t) \geq f(t_0) \quad \text{for all} \quad t \in (t_0 - \delta, t_0 + \delta) \cap D_f$$

and $f(\rho(t_0)), f(\sigma(t_0)) \geq f(t_0)$. The point t_0 is called a *local extreme point* of f, more specifically, a *local maximum* or *local minimum* point of f.

Theorem

Let f be delta and nabla differentiable in a neighbourhood $(t_0 - \delta, t_0 + \delta)$ of t_0. If $f^{\Delta}(t) \leq 0$ in $[t_0, t_0 + \delta)$ and $f^{\nabla}(t) \geq 0$ in $(t_0 - \delta, t_0]$, then t_0 is a local maximum point of f.

Proof.

Let t_0 be an isolated point. Then

$$\rho(t_0) < t_0 < \sigma(t_0)$$

and

$$f^{\Delta}(t_0) = \frac{f(\sigma(t_0)) - f(t_0)}{\sigma(t_0) - t_0} \leq 0, \quad f^{\nabla}(t_0) = \frac{f(t_0) - f(\rho(t_0))}{t_0 - \rho(t_0)} \geq 0.$$

Therefore, $f(t_0) \geq f(\sigma(t_0))$ and $f(t_0) \geq f(\rho(t_0))$. Also, there exists $\delta_1 > 0$ such that $f(t) \leq f(t_0)$ for all $t \in (t_0 - \delta_1, t_0 + \delta_1)$. Consequently, t_0 is a local maximum point.

Let t_0 be left-dense and right-scattered. As above, we have that $f(\sigma(t_0)) \leq f(t_0)$. Since t_0 is left-dense, we have that $f^{\nabla}(t_0) = f'(t_0)$ and there exists $\delta_1 > 0$ so that $f^{\nabla}(t) = f'(t)$ for every $t \in (t_0 - \delta_1, t_0]$. □

Proof.

For every $t_1 \in (t_0 - \delta_1, t_0]$, there exists $\xi_1 \in (t_1, t_0)$ such that

$$f'(\xi_1) = \frac{f(t_1) - f(t_0)}{t_1 - t_0}.$$

Because $f'(\xi_1) \geq 0$, we obtain that $f(t_1) \leq f(t_0)$. Consequently, there exists $\delta_2 > 0$, $\delta_2 \leq \delta_1$, such that for every $t \in (t_0 - \delta_2, t_0 + \delta_2)$, we have that $f(t) \leq f(t_0)$. Therefore, t_0 is a local maximum point. The cases when t_0 is left-scattered and right-dense and when t_0 is dense are left to the reader for exercise. $\square$

As in the proof of Theorem 2, one can prove the following theorem.

Theorem

Let f be delta and nabla differentiable in a neighbourhood $(t_0 - \delta, t_0 + \delta)$ of t_0. If $f^{\Delta}(t) \geq 0$ in $[t_0, t_0 + \delta)$ and $f^{\nabla}(t) \leq 0$ in $(t_0 - \delta, t_0]$, then t_0 is a local minimum point of f.

Example

Let $\mathbb{T} = \mathbb{Z}$. Consider the function

$$f(t) = t^2 - 5t + 4.$$

Then

$$f^{\Delta}(t) = \sigma(t) + t - 5 = t + 1 + t - 5 = 2t - 4$$

and

$$f^{\nabla}(t) = \rho(t) + t - 5 = t - 1 + t - 5 = 2t - 6.$$

Hence,

$$f^{\Delta}(t) \leq 0 \quad \text{and} \quad f^{\nabla}(t) \geq 0$$

iff

$$2t - 4 \leq 0 \quad \text{and} \quad 2t - 6 \geq 0$$

iff

$$t \leq 2 \quad \text{and} \quad t \geq 3.$$

Example

Therefore, f has no local maximum points. Also,

$$f^{\Delta}(t) \geq 0 \quad \text{and} \quad f^{\nabla}(t) \leq 0$$

iff

$$2t - 4 \geq 0 \quad \text{and} \quad 2t - 6 \leq 0$$

iff

$$t \geq 2 \quad \text{and} \quad t \leq 3.$$

Consequently, $t = 2$ and $t = 3$ are local minimum points. We have

$$f(2) = f(3) = -2.$$

Example

Let $\mathbb{T} = \mathbb{Z}$. We will find the local extreme values of the function

$$f(t) = \frac{t+1}{t^2+1}.$$

Here, $\sigma(t) = t+1$, $\rho(t) = t-1$. Also,

$$\begin{aligned} f^{\Delta}(t) &= \frac{t^2+1-(t+1)(\sigma(t)+t)}{(t^2+1)((t+1)^2+1)} \\ &= \frac{t^2+1-(t+1)(t+1+t)}{(t^2+1)(t^2+2t+2)} \\ &= \frac{t^2+1-(t+1)(2t+1)}{(t^2+1)(t^2+2t+2)} \\ &= \frac{t^2+1-(2t^2+t+2t+1)}{(t^2+1)(t^2+2t+2)} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{t^2+1-2t^2-3t-1}{(t^2+1)(t^2+2t+2)} \\
&= \frac{-t^2-3t}{(t^2+1)(t^2+2t+2)} \\
&= -\frac{t(t+3)}{(t^2+1)(t^2+2t+2)}
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\nabla}(t) &= \frac{t^2+1-(t+1)(\rho(t)+t)}{(t^2+1)((t-1)^2+1)} \\
&= \frac{t^2+1-(t+1)(2t-1)}{(t^2+1)(t^2-2t+2)}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{t^2+1-(2t^2-t+2t-1)}{(t^2+1)(t^2-2t+2)} \\
&= \frac{t^2+1-2t^2-t+1}{(t^2+1)(t^2-2t+2)} \\
&= \frac{-t^2-t+2}{(t^2+1)(t^2-2t+2)} \\
&= -\frac{t^2+t-2}{(t^2+1)(t^2-2t+2)} \\
&= -\frac{(t+2)(t-1)}{(t^2+1)(t^2-2t+2)}.
\end{aligned}
$$

Hence,

Example

$$f^{\Delta}(t) \leq 0 \quad \text{and} \quad f^{\nabla}(t) \geq 0$$

iff

$$-\frac{t(t+3)}{(t^2+1)(t^2+2t+2)} \leq 0 \quad \text{and} \quad -\frac{(t+2)(t-1)}{(t^2+1)(t^2-2t+2)} \geq 0$$

iff

$$t(t+3) \geq 0 \quad \text{and} \quad (t-1)(t+2) \leq 0$$

so that

$$t = 0 \quad \text{and} \quad t = 1.$$

Therefore,

$$f_{\max} = f(0) = f(1) = 1.$$

Also,

$$f^{\Delta}(t) \geq 0 \quad \text{and} \quad f^{\nabla}(t) \leq 0$$

iff

Example

$$-\frac{t(t+3)}{(t^2+1)(t^2+2t+2)} \geq 0 \quad \text{and} \quad -\frac{(t+2)(t-1)}{(t^2+1)(t^2-2t+2)} \leq 0$$

iff

$$t(t+3) \leq 0 \quad \text{and} \quad (t-1)(t+2) \geq 0$$

so that

$$t = -2 \quad \text{and} \quad t = -1.$$

Consequently,

$$f_{\min} = f(-2) = \frac{-2+1}{4+1} = -\frac{1}{5}$$

and

$$f_{\min} = f(-1) = 0.$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We will find the extreme values of the function

$$f(t) = \frac{t^2+2}{t+2} \quad \text{for} \quad t \geq 4.$$

Here, $\sigma(t) = 2t$, $\rho(t) = \frac{1}{2}t$ for all $t \in \mathbb{T}$ and $t \geq 4$. Then, for $t \geq 4$, we have

$$\begin{aligned} f^{\Delta}(t) &= \frac{(\sigma(t)+t)(t+2)-(t^2+2)}{(t+2)(2t+2)} \\ &= \frac{3t(t+2)-(t^2+2)}{2(t+1)(t+2)} \\ &= \frac{3t^2+6t-t^2-2}{2(t+1)(t+2)} \\ &= \frac{t^2+3t-1}{(t+1)(t+2)} \end{aligned}$$

Example

and

$$\begin{aligned} f^{\nabla}(t) &= \frac{(\rho(t)+t)(t+2)-(t^2+2)}{(t+2)\left(\frac{1}{2}t+2\right)} \\ &= \frac{\frac{3}{2}t(t+2)-t^2-2}{(t+2)\left(\frac{1}{2}t+2\right)} \\ &= \frac{\frac{3}{2}t^2+3t-t^2-2}{(t+2)\left(\frac{1}{2}t+2\right)} \\ &= \frac{\frac{1}{2}t^2+3t-2}{(t+2)\left(\frac{1}{2}t+2\right)} \\ &= \frac{t^2+6t-4}{(t+2)(t+4)}. \end{aligned}$$

Note that $f^{\Delta}(t) \geq 0$ and $f^{\nabla}(t) \geq 0$ for all $t \geq 4$. Therefore, the function f has no local extreme values.

Definition

Suppose that f is Δ-differentiable and ∇-differentiable at t_0. We say that t_0 is a *critical point* of f if

$$f^{\Delta}(t_0) \leq 0 \quad \text{and} \quad f^{\nabla}(t_0) \geq 0$$

or

$$f^{\Delta}(t_0) \geq 0 \quad \text{and} \quad f^{\nabla}(t_0) \leq 0.$$

The least (greatest) value of a continuous function f on a given interval $[a, b]$ is attained at the critical points of f or at the endpoints of the interval $[a, b]$.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We will prove that

$$\frac{t^2+2}{t+3} \geq \frac{3}{4} \quad \text{for all} \quad t \in \mathbb{T}.$$

We have $\sigma(t) = 2t$ for all $t \in \mathbb{T}$ and

$$\begin{aligned} f^{\Delta}(t) &= \frac{(t+\sigma(t))(t+3) - (t^2+2)}{(t+3)(2t+3)} \\ &= \frac{3t(t+3) - t^2 - 2}{(t+3)(2t+3)} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{3t^2 + 9t - t^2 - 2}{(t+3)(2t+3)} \\
&= \frac{2t^2 + 9t - 2}{(t+3)(2t+3)} \geq 0
\end{aligned}
$$

for all $t \in \mathbb{T}$. Consequently, f is increasing in $\mathbb{T}$. Hence

$$
f(t) \geq f(1) = \frac{3}{4} \quad \text{for all} \quad t \in \mathbb{T}.
$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. We will find a positive constant a such that

$$1 + a \log t \leq t^2 \quad \text{for all} \quad t \in \mathbb{T}.$$

Let

$$f(t) = t^2 - a \log t - 1, \quad t \in \mathbb{T}.$$

Here, $\sigma(t) = 3t$ for all $t \in \mathbb{T}$ and

$$\begin{aligned} f^{\Delta}(t) &= \sigma(t) + t - a\frac{\log \sigma(t) - \log t}{\sigma(t) - t} \\ &= 3t + t - a\frac{\log(3t) - \log t}{3t - t} \\ &= 4t - a\frac{\log 3}{2t} \quad \text{for all} \quad t \in \mathbb{T}. \end{aligned}$$

Example

Since

$$\frac{\log 3}{2t} \leq \frac{\log 3}{2} \quad \text{for all} \quad t \in \mathbb{T},$$

we conclude that

$$4t - a\frac{\log 3}{2t} \geq 4 - a\frac{\log 3}{2} \quad \text{for all} \quad t \in \mathbb{T}.$$

Hence, if $0 < a < \frac{8}{\log 3}$, then f is increasing in $\mathbb{T}$. From here,

$$f(t) \geq f(1) = 0 \quad \text{for all} \quad t \in \mathbb{T} \quad \text{and for} \quad 0 < a < \frac{8}{\log 3}.$$

Suppose that $f : \mathbb{T} \to \mathbb{R}$.

Definition

The function f is called *convex* if for all $t_1, t_2 \in \mathbb{T}$ and for all $\lambda \in [0, 1]$, the inequality

$$f(\lambda t_1 + (1-\lambda)t_2) \leq \lambda f(t_1) + (1-\lambda) f(t_2)$$

holds.

Definition

The function f is called *strictly convex* if for all $t_1, t_2 \in \mathbb{T}$ with $t_1 \neq t_2$ and for all $\lambda \in (0, 1)$, the inequality

$$f(\lambda t_1 + (1-\lambda)t_2) < \lambda f(t_1) + (1-\lambda) f(t_2)$$

holds.

Definition

The function f is said to be (*strictly*) *concave* if $-f$ is (strictly) convex.

Theorem

Let f be twice delta differentiable on (a, b) and $f^{\Delta\Delta}(t) \geq 0$ for all $t \in (a, b)$. Then f is convex.

Proof.

Let $t_1, t_2 \in \mathbb{T}$, $t_1 < t_2$, and $\lambda \in (0,1)$. Then

$$\lambda f(t_1) + (1-\lambda) f(t_2) - f(\lambda t_1 + (1-\lambda) t_2)$$

$$= \lambda f(t_1) + (1-\lambda) f(t_2) - (1-\lambda+\lambda) f(\lambda t_1 + (1-\lambda) t_2)$$

$$= \lambda (f(t_1) - f(\lambda t_1 + (1-\lambda) t_2)) + (1-\lambda)(f(t_2) - f(\lambda t_1 + (1-\lambda) t_2). \tag{1}$$

By the mean value theorem, it follows that there exist □

Proof.

$$\xi_1 \in (t_1, \lambda t_1 + (1-\lambda)t_2) \quad \text{and} \quad \xi_2 \in (\lambda t_1 + (1-\lambda)t_2, t_2)$$

so that

$$\begin{aligned} f(t_1) - f(\lambda t_1 + (1-\lambda)t_2) &\geq f^{\Delta}(\xi_1)(t_1 - \lambda t_1 - (1-\lambda)t_2) \\ &= (1-\lambda) f^{\Delta}(\xi_1)(t_1 - t_2) \end{aligned}$$

and

$$\begin{aligned} f(t_2) - f(\lambda t_1 + (1-\lambda)t_2) &\geq f^{\Delta}(\xi_2)(t_2 - \lambda t_1 - (1-\lambda)t_2) \\ &= \lambda f^{\Delta}(\xi_2)(t_2 - t_1). \end{aligned}$$

□

Proof.

By (1), we obtain

$$\begin{aligned} &\lambda f(t_1) + (1-\lambda) f(t_2) - f(\lambda t_1 + (1-\lambda) t_2) \\ &\geq \lambda(1-\lambda) f^{\Delta}(\xi_1)(t_1 - t_2) + \lambda(1-\lambda) f^{\Delta}(\xi_2)(t_2 - t_1) \qquad (2) \\ &= \lambda(1-\lambda)(f^{\Delta}(\xi_1) - f^{\Delta}(\xi_2))(t_1 - t_2). \end{aligned}$$

By the mean value theorem, it follows that there exists $\xi_3 \in (\xi_1, \xi_2)$ so that □

Proof.

$$f^{\Delta}(\xi_1) - f^{\Delta}(\xi_2) \leq f^{\Delta\Delta}(\xi_3)(\xi_1 - \xi_2).$$

From the last inequality and from (2), we obtain

$$\begin{aligned} \lambda f(t_1) + (1-\lambda)f(t_2) - f(\lambda t_1 + (1-\lambda)t_2) &\geq \lambda(1-\lambda)f^{\Delta\Delta}(\xi_3) \\ &\quad \times(\xi_1 - \xi_2)(t_1 - t_2) \\ &\geq 0, \end{aligned}$$

which completes the proof. □

As in Theorem 21, one can prove the following theorem.

Theorem

Let f be twice delta differentiable on (a, b) and $f^{\Delta\Delta}(t) \leq 0$ for all $t \in (a, b)$. Then f is concave.

Example

Let $\mathbb{T} = \mathbb{Z}$. Consider

$$f(t) = t^3 - 7t^2 + t - 10.$$

Here, $\sigma(t) = t + 1$ and

$$\begin{aligned}
f^{\Delta}(t) &= (\sigma(t))^2 + t\sigma(t) + t^2 - 7(\sigma(t) + t) + 1 \\
&= (t+1)^2 + t(t+1) + t^2 - 7(t + 1 + t) + 1 \\
&= t^2 + 2t + 1 + t^2 + t + t^2 - 14t - 7 + 1 \\
&= 3t^2 - 11t - 5, \\
f^{\Delta\Delta}(t) &= 3(\sigma(t) + t) - 11
\end{aligned}$$

Example

Hence,

$$f^{\Delta\Delta}(t) \geq 0 \quad \text{for} \quad t \geq 2 \quad \text{and} \quad f^{\Delta\Delta}(t) \leq 0 \quad \text{for} \quad t \leq 1.$$

Therefore, f is convex for $t \geq 2$ and concave for $t \leq 1$.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider

$$f(t) = t^4 - t^3 - t^2 - t.$$

Here, $\sigma(t) = 2t$ and

$$\begin{aligned}
f^{\Delta}(t) &= (\sigma(t))^3 + t(\sigma(t))^2 + t^2\sigma(t) + t^3 \\
&\quad -((\sigma(t))^2 + t\sigma(t) + t^2) - (\sigma(t) + t) - 1 \\
&= 8t^3 + 4t^3 + 2t^3 + t^3 - (4t^2 + 2t^2 + t^2) - (2t + t) - 1 \\
&= 15t^3 - 7t^2 - 3t - 1,
\end{aligned}$$

Example

$$
\begin{aligned}
f^{\Delta\Delta}(t) &= 15((\sigma(t))^2 + t\sigma(t) + t^2) - 7(\sigma(t) + t) - 3 \\
&= 15(4t^2 + 2t^2 + t^2) - 7(2t + t) - 3 \\
&= 105t^2 - 21t - 3.
\end{aligned}
$$

Hence, $f^{\Delta\Delta}(t) > 0$ for all $t \in \mathbb{T}$. Therefore, the function f is strictly convex in $\mathbb{T}$.

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Consider the function

$$f(t) = \frac{t-3}{t+2}.$$

We have $\sigma(t) = 3t$ and

$$\begin{aligned} f^{\Delta}(t) &= \frac{t+2-(t-3)}{(t+2)(3t+2)} \\ &= \frac{5}{3t^2+2t+6t+4} \\ &= \frac{5}{3t^2+8t+4}, \end{aligned}$$

Example

$$
\begin{aligned}
f^{\Delta\Delta}(t) &= -5\frac{3(\sigma(t)+t)+8}{(3t^2+8t+4)(3(\sigma(t))^2+8\sigma(t)+4)} \\
&= -5\frac{12t+8}{(3t^2+8t+4)(27t^2+24t+4)} \\
&= -20\frac{3t+2}{(3t^2+8t+4)(27t^2+24t+4)} \\
&< 0 \quad \text{for all} \quad t \in \mathbb{T}.
\end{aligned}
$$

Therefore, f is a strictly concave function in $\mathbb{T}$.