Time Scales Analysis
Lecture 11

Completely Delta Differentiable Functions. Introduction to Integral Calculus on Time Scales

Svetlin G. Georgiev

October 15, 2025

Definition

A function $f : \mathbb{T} \to \mathbb{R}$ is called *completely delta differentiable* at a point $t^0 \in \mathbb{T}^\kappa$ if there exist constants A_1 and A_2 such that

$$f(t^0) - f(t) = A_1(t^0 - t) + \alpha(t^0 - t) \quad \text{for all} \quad t \in U_\delta(t^0) \tag{1}$$

and

$$f(\sigma(t^0)) - f(t) = A_2(\sigma(t^0) - t) + \beta(\sigma(t^0) - t) \quad \text{for all} \quad t \in U_\delta(t^0), \tag{2}$$

where $U_\delta(t^0)$ is a δ-neighbourhood of t^0 and

$$\alpha = \alpha(t^0, t) \quad \text{and} \quad \beta = \beta(t^0, t)$$

are equal to zero for $t = t^0$ such that

$$\lim_{t \to t^0} \alpha(t^0, t) = \lim_{t \to t^0} \beta(t^0, t) = 0.$$

Theorem

Let a function $f : \mathbb{T} \to \mathbb{R}$ *be continuous and have the first-order delta derivative* f^{Δ} *in some* δ*-neighbourhood* $U_{\delta}(t^0)$ *of the point* $t^0 \in \mathbb{T}^{\kappa}$*. If* f^{Δ} *is continuous at the point* t^0*, then* f *is completely delta differentiable at* t^0*.*

Proof.

Using the definition of delta derivative, we have that

$$f(\sigma(t^0)) - f(t) = f^{\Delta}(t^0)(\sigma(t^0) - t) + \beta(\sigma(t^0) - t), \tag{3}$$

where $\beta = \beta(t^0, t)$ and $\beta \to 0$ as $t \to t^0$, i.e., (2) holds. Now, we will prove (1).

Let $t^0 \in \mathbb{T}$ be isolated. Then (1) is satisfied independently of A_1 and α because in this case, $U_\delta(t^0)$ consists of the single point t^0 for sufficiently small $\delta > 0$. □

Proof.

Let t^0 be right-dense. In this case, $\sigma(t^0) = t^0$, and (3) coincides with (1). Let t^0 be left-dense and right-scattered. Then for sufficiently small $\delta > 0$, any point $t \in U_\delta(t^0) \setminus \{t^0\}$ must satisfy $t < t^0$. By the mean value theorem, we obtain

$$f^\Delta(\xi_1)(t^0 - t) \leq f(t^0) - f(t) \leq f^\Delta(\xi_2)(t^0 - t),$$

where $\xi_1, \xi_2 \in [t, t^0)$. Since $\xi_1, \xi_2 \to t^0$ as $t \to t^0$ and f^Δ is continuous at t^0, we have

$$\lim_{t \to t^0} \frac{f(t^0) - f(t)}{t^0 - t} = f^\Delta(t^0).$$

Therefore,

$$\frac{f(t^0) - f(t)}{t^0 - t} = f^\Delta(t^0) + \alpha,$$

where $\alpha = \alpha(t^0, t)$ and $\alpha \to 0$ as $t \to t^0$. Consequently, (1) holds for $A_1 = f^\Delta(t^0)$. $\square$

Definition

Let $\overline{\mathbb{T}} = \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}$. If $\infty \in \overline{\mathbb{T}}$, then ∞ is called left-dense. If $-\infty \in \overline{\mathbb{T}}$, then $-\infty$ is called right-dense.

Remark

For any left-dense point $t_0 \in \overline{\mathbb{T}}$ *and for any* $\varepsilon > 0$*, we set*

$$L_\varepsilon(t_0) = \{t \in \mathbb{T} : \ 0 < t_0 - t < \varepsilon\}.$$

If $t_0 \in \overline{\mathbb{T}}$ *is left-dense, then* $L_\varepsilon(t_0)$ *is nonempty. If* $\infty \in \overline{\mathbb{T}}$*, then*

$$L_\varepsilon(\infty) = \left\{t \in \mathbb{T} : \ t > \frac{1}{\varepsilon}\right\}.$$

For a right-dense point $t_1 \in \overline{\mathbb{T}}$*, we define*

$$R_\varepsilon(t_1) = \{t \in \mathbb{T} : \ 0 < t - t_1 < \varepsilon\}.$$

For every right-dense point $t_1 \in \overline{\mathbb{T}}$*, the set* $R_\varepsilon(t_1)$ *is nonempty. If* $-\infty \in \overline{\mathbb{T}}$*, then*

$$R_\varepsilon(-\infty) = \left\{t \in \mathbb{T} : \ t < -\frac{1}{\varepsilon}\right\}.$$

Definition

Let $h : \mathbb{T} \to \mathbb{R}$.

1. Let $t_0 \in \overline{\mathbb{T}}$ be left-dense. We define
$$\liminf_{t \to t_0-} h(t) = \lim_{\varepsilon \to 0+} \inf_{t \in L_\varepsilon(t_0)} h(t), \quad \limsup_{t \to t_0-} h(t) = \lim_{\varepsilon \to 0+} \sup_{t \in L_\varepsilon(t_0)} h(t).$$
2. Let $t_1 \in \overline{\mathbb{T}}$ be right-dense. We define
$$\liminf_{t \to t_1+} h(t) = \lim_{\varepsilon \to 0+} \inf_{t \in R_\varepsilon(t_1)} h(t), \quad \limsup_{t \to t_1+} h(t) = \lim_{\varepsilon \to 0+} \sup_{t \in R_\varepsilon(t_1)} h(t).$$

Theorem (L'Hôpital's Rule)

Let f and g be differentiable on $\mathbb{T}$ and

$$\lim_{t \to t_0-} f(t) = \lim_{t \to t_0-} g(t) = 0 \quad \textit{for some left-dense} \quad t_0 \in \overline{\mathbb{T}}. \tag{4}$$

Suppose there exists $\varepsilon > 0$ such that

$$g(t) > 0, \quad g^{\Delta}(t) < 0 \quad \textit{for all} \quad t \in L_{\varepsilon}(t_0). \tag{5}$$

Then

$$\liminf_{t \to t_0-} \frac{f^{\Delta}(t)}{g^{\Delta}(t)} \leq \liminf_{t \to t_0-} \frac{f(t)}{g(t)} \leq \limsup_{t \to t_0-} \frac{f(t)}{g(t)} \leq \limsup_{t \to t_0-} \frac{f^{\Delta}(t)}{g^{\Delta}(t)}.$$

Proof.

Let $\delta \in (0, \varepsilon]$. We set

$$\alpha = \inf_{\tau \in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}, \quad \beta = \sup_{\tau \in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}.$$

Then, using (5),

$$\alpha g^\Delta(\tau) \geq f^\Delta(\tau) \geq \beta g^\Delta(\tau) \quad \text{for any} \quad \tau \in L_\delta(t_0).$$

Hence,

$$\int_s^t \alpha g^\Delta(\tau)\Delta\tau \geq \int_s^t f^\Delta(\tau)\Delta\tau \geq \int_s^t \beta g^\Delta(\tau)\Delta\tau$$

for all $s, t \in L_\delta(t_0)$, $s < t$. Therefore, $\square$

Proof.

$$\alpha(g(t)-g(s)) \geq f(t)-f(s) \geq \beta(g(t)-g(s))$$

for all $s, t \in L_\delta(t_0)$, $s < t$. From here,

$$-\alpha g(s) \geq -f(s) \geq -\beta g(s) \quad \text{for all} \quad s \in L_\delta(t_0)$$

as $t \to t_0-$, whereupon

$$\alpha \leq \frac{f(s)}{g(s)} \leq \beta \quad \text{for all} \quad s \in L_\delta(t_0).$$

Letting $\delta \to 0+$, we get the desired result. $\square$

Example

Let $\mathbb{T} = [-1,1) \cup \{1,2,4,\ldots\}$, where $[-1,1)$ is the real-valued interval. We compute

$$I = \lim_{t \to 1-} \frac{t^3 - 2t^2 - t + 2}{t^3 - 7t + 6}.$$

Note that $t = 1$ is left-dense and $f^{\Delta}(t) = f'(t)$ for all $t \in [-1,1)$. Then

$$\begin{aligned} I &= \lim_{t \to 1-} \frac{t^2(t-2) - (t-2)}{t^3 - t - 6t + 6} \\ &= \lim_{t \to 1-} \frac{(t^2-1)(t-2)}{t(t-1)(t+1) - 6(t-1)} \end{aligned}$$

Example

$$
\begin{aligned}
&= \lim_{t\to 1-} \frac{(t-1)(t+1)(t-2)}{(t-1)(t^2+t-6)} \\
&= \lim_{t\to 1-} \frac{(t+1)(t-2)}{t^2+t-6} \\
&= \frac{-2}{-4} \\
&= \frac{1}{2}.
\end{aligned}
$$

Example

Let $\mathbb{T} = [-3, 0) \cup \mathbb{N}_0$, where $[-3, 0)$ is the real-valued interval. We find

$$I = \lim_{t \to 0-} \frac{t \cos t - \sin t}{t}.$$

Note that $t = 0$ is left-dense and $f^{\Delta}(t) = f'(t)$ for $t \in [-3, 0)$. Hence,

$$\begin{aligned} I &= \lim_{t \to 0-} \frac{(t \cos t - \sin t)'}{t'} \\ &= \lim_{t \to 0-} (\cos t - t \sin t - \cos t) \\ &= \lim_{t \to 0-} (-t \sin t) \\ &= 0. \end{aligned}$$

Example

Let $\mathbb{T} = [-3, 1) \cup 2^{\mathbb{N}_0}$, where $[-3, 1)$ is the real-valued interval. We compute

$$l = \lim_{t \to 0} \frac{\tan t - \sin t}{t - \sin t}.$$

Here, $t = 0$ is left-dense and $f^{\Delta}(t) = f'(t)$ for all $t \in [-3, 1)$. Then

$$\begin{aligned} l &= \lim_{t \to 0} \frac{(\tan t - \sin t)'}{(t - \sin t)'} \\ &= \lim_{t \to 0} \frac{\frac{1}{\cos^2 t} - \cos t}{1 - \cos t} \\ &= \lim_{t \to 0} \frac{1 - \cos^3 t}{\cos^2 t(1 - \cos t)} \end{aligned}$$

Example

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{(1-\cos t)(1+\cos t+\cos^2 t)}{\cos^2 t(1-\cos t)} \\
&= \lim_{t\to 0} \frac{1+\cos t+\cos^2 t}{\cos^2 t} \\
&= 3.
\end{aligned}
$$

Theorem (L'Hôpital's Rule)

Let f and g be differentiable on $\mathbb{T}$ and

$$\lim_{t\to t_0-} g(t) = \infty \quad \text{for some left-dense point} \quad t_0 \in \overline{\mathbb{T}}. \tag{6}$$

Assume there exists $\varepsilon > 0$ such that

$$g(t) > 0, \quad g^{\Delta}(t) > 0 \quad \text{for all} \quad t \in L_{\varepsilon}(t_0). \tag{7}$$

Then

$$\lim_{t\to t_0-} \frac{f^{\Delta}(t)}{g^{\Delta}(t)} = r \in \overline{\mathbb{R}}$$

implies

$$\lim_{t\to t_0-} \frac{f(t)}{g(t)} = r.$$

Proof.

Let $r \in \mathbb{R}$ and $c > 0$. Then there exists $\delta \in (0, \varepsilon]$ such that

$$\left| \frac{f^{\Delta}(\tau)}{g^{\Delta}(\tau)} - r \right| \leq c \quad \text{for all} \quad \tau \in L_{\delta}(t_0).$$

Hence, using (7), we obtain

$$-cg^{\Delta}(\tau) \leq f^{\Delta}(\tau) - rg^{\Delta}(\tau) \leq cg^{\Delta}(\tau) \quad \text{for all} \quad \tau \in L_{\delta}(t_0).$$

From here,

$$-\int_s^t cg^{\Delta}(\tau)\Delta\tau \leq \int_s^t (f^{\Delta}(\tau) - rg^{\Delta}(\tau))\Delta\tau \leq \int_s^t cg^{\Delta}(\tau)\Delta\tau$$

for all $s, t \in L_{\delta}(t_0)$, $s < t$. Therefore,

$$-c(g(t) - g(s)) \leq f(t) - f(s) - r(g(t) - g(s)) \leq c(g(t) - g(s)),$$

i.e., □

Proof.

$$(r-c)(g(t)-g(s)) \le f(t)-f(s) \le (c+r)(g(t)-g(s))$$

for all $s, t \in L_\delta(t_0)$, $s < t$. Hence,

$$(r-c)\left(1-\frac{g(s)}{g(t)}\right) \le \frac{f(t)}{g(t)} - \frac{f(s)}{g(t)} \le (c+r)\left(1-\frac{g(s)}{g(t)}\right)$$

for all $s, t \in L_\delta(t_0)$, $s < t$. Letting $t \to t_0-$ and using (6), we find

$$r-c \le \liminf_{t\to t_0-} \frac{f(t)}{g(t)} \le \limsup_{t\to t_0-} \frac{f(t)}{g(t)} \le c+r.$$

Now, we let $c \to 0+$, and then $\lim_{t\to t_0-} \frac{f(t)}{g(t)}$ exists and equals r. □

Proof.

Let $r = \infty$ and $c > 0$. Then there exists $\delta \in (0, \varepsilon]$ such that

$$\frac{f^{\Delta}(\tau)}{g^{\Delta}(\tau)} \geq \frac{1}{c} \quad \text{for all} \quad \tau \in L_{\delta}(t_0),$$

and hence

$$f^{\Delta}(\tau) \geq \frac{1}{c} g^{\Delta}(\tau) \quad \text{for all} \quad \tau \in L_{\delta}(t_0).$$

From here,

$$\int_s^t f^{\Delta}(\tau)\Delta\tau \geq \int_s^t \frac{1}{c} g^{\Delta}(\tau)\Delta\tau \quad \text{for all} \quad s, t \in L_{\delta}(t_0), \quad s < t,$$

whereupon

$$f(t) - f(s) \geq \frac{1}{c}(g(t) - g(s)) \quad \text{for all} \quad s, t \in L_{\delta}(t_0), \quad s < t.$$

Therefore, $\square$

Proof.

$$\frac{f(t)}{g(t)} - \frac{f(s)}{g(t)} \geq \frac{1}{c}\left(1 - \frac{g(s)}{g(t)}\right) \quad \text{for all} \quad s, t \in L_\delta(t_0), \quad s < t.$$

Then letting $t \to t_0-$, we find

$$\lim_{t \to t_0-} \frac{f(t)}{g(t)} \geq \frac{1}{c}.$$

Now letting $c \to 0+$, we obtain

$$\lim_{t \to t_0-} \frac{f(t)}{g(t)} = \infty = r.$$

The case $r = -\infty$ is left for an exercise. This completes the proof. □

Definition

Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function $F : \mathbb{T} \to \mathbb{R}$ for which

$$F^{\Delta}(t) = f(t) \quad \text{holds for all} \quad t \in \mathbb{T}^{\kappa}$$

is called the *indefinite integral* of a regulated function f and denoted by

$$\int f(t)\Delta t = F(t) + c,$$

where c is an arbitrary constant. We define the *Cauchy integral* by

$$\int_{\tau}^{s} f(t)\Delta t = F(s) - F(\tau) \quad \text{for all} \quad \tau, s \in \mathbb{T}.$$

Example

Let $\mathbb{T} = \mathbb{Z}$. Then $\sigma(t) = t + 1$, $t \in \mathbb{T}$. Assume $f(t) = 3t^2 + 5t + 2$, $t \in \mathbb{T}$. Since

$$\begin{aligned}
(t^3 + t^2)^\Delta &= (\sigma(t))^2 + t\sigma(t) + t^2 + \sigma(t) + t \\
&= (t+1)^2 + t(t+1) + t^2 + t + 1 + t \\
&= t^2 + 2t + 1 + t^2 + t + t^2 + 2t + 1 \\
&= 3t^2 + 5t + 2,
\end{aligned}$$

we have

$$\int (3t^2 + 5t + 2)\Delta t = t^3 + t^2 + c.$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}}$ and define $f : \mathbb{T} \to \mathbb{R}$ by $f(t) = 2\sin\frac{t}{2}\cos\frac{3t}{2}$, $t \in \mathbb{T}$. In this case, we have that $\sigma(t) = 2t$. Since

$$\begin{aligned}(\sin t)^{\Delta} &= \frac{\sin\sigma(t) - \sin t}{\sigma(t) - t} \\ &= \frac{\sin(2t) - \sin t}{t} \\ &= \frac{2}{t}\sin\frac{t}{2}\cos\frac{3t}{2},\end{aligned}$$

we get

$$\int \frac{2}{t}\sin\frac{t}{2}\cos\frac{3t}{2}\Delta t = \sin t + c.$$

Example

Let $\mathbb{T} = \mathbb{N}_0^2$ and define $f : \mathbb{T} \to \mathbb{R}$ by $f(t) = \frac{1}{1+2\sqrt{t}} \log \frac{(\sqrt{t}+1)^2}{t}$, $t \in \mathbb{T}$. Since $\sigma(t) = (\sqrt{t}+1)^2$ and

$$
\begin{aligned}
(\log t)^\Delta &= \frac{\log \sigma(t) - \log t}{\sigma(t) - t} \\
&= \frac{\log(\sqrt{t}+1)^2 - \log t}{(\sqrt{t}+1)^2 - t} \\
&= \frac{1}{1+2\sqrt{t}} \log \frac{(\sqrt{t}+1)^2}{t},
\end{aligned}
$$

we get

$$
\int \frac{1}{1+2\sqrt{t}} \log \frac{(1+\sqrt{t})^2}{t} \Delta t = \log t + c.
$$

Theorem

Every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau \quad \text{for} \quad t \in \mathbb{T},$$

is an antiderivative of f.

Proof.

Since f is rd-continuous, it is regulated. Let F be such that

$$F^{\Delta}(t) = f(t) \quad \text{for} \quad t \in D.$$

We have that F is pre-differentiable with D. Let $t \in \mathbb{T}^{\kappa} \setminus D$. Then t is right-dense. Since f is rd-continuous, f is continuous at t. Let $\varepsilon > 0$ be arbitrarily chosen. Then there exists a neighbourhood U of t such that

$$|f(s) - f(t)| \leq \varepsilon \quad \text{for all} \quad s \in U.$$

We define

$$h(\tau) = F(\tau) - f(t)(\tau - t_0) \quad \text{for} \quad \tau \in \mathbb{T}.$$

Then h is pre-differentiable with D and □

Proof.

$$h^{\Delta}(\tau) = F^{\Delta}(\tau) - f(t)$$

$$= f(\tau) - f(t) \quad \text{for all} \quad \tau \in D.$$

Hence,

$$|h^{\Delta}(s)| = |f(s) - f(t)|$$

$$\leq \varepsilon \quad \text{for all} \quad s \in D \cap U.$$

Therefore, □

Proof.

$$\sup_{s \in D \cap U} |h^{\Delta}(s)| \leq \varepsilon,$$

whereupon

$$\begin{aligned} |F(t) - F(r) - f(t)(t-r)| &= |h(t) + f(t)(t - t_0) - (h(r) \\ &\quad + f(t)(r - t_0)) - f(t)(t-r)| \\ &= |h(t) - h(r)| \\ &\leq \left\{ \sup_{s \in D \cap U} |h^{\Delta}(s)| \right\} |t - r| \\ &\leq \varepsilon |t - r|, \end{aligned}$$

which shows that f is differentiable at t and $F^{\Delta}(t) = f(t)$. □

Let $a < b$ be points in $\mathbb{T}$ and $[a, b)$ be the half-closed bounded interval in $\mathbb{T}$.

Definition

A *partition* of $[a, b)$ is any finite ordered subset

$$P = \{t_0, t_1, t_2, \ldots, t_n\} \subset [a, b],$$

where

$$a = t_0 < t_1 < \ldots < t_n = b.$$

The number n depends on the particular partition. The intervals

$$[t_{i-1}, t_i), \quad i \in \{1, \ldots, n\}$$

are called *subintervals* of the partition P.

Let f be a real-valued bounded function on $[a, b)$. We set

$$M = \sup\{f(t) : t \in [a, b)\}, \quad m = \inf\{f(t) : t \in [a, b)\}$$

and

$$M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)\}, \quad m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)\}, \quad i \in \{1, \ldots,$$

Definition

The *upper Darboux* Δ*-sum* $U(f, P)$ and the *lower Darboux* Δ*-sum* $L(f, P)$ of f with respect to P are defined by

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}), \quad L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

respectively.

Theorem

We have

$$m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a). \tag{8}$$

Proof.

We have

$$m_i \geq m \quad \text{and} \quad M_i \leq M \quad \text{for all} \quad i \in \{1, \ldots, n\}.$$

Then □

Proof.

$$
\begin{aligned}
L(f, P) &= \sum_{i=1}^{n} m_i (t_i - t_{i-1}) \\
&\geq m \sum_{i=1}^{n} (t_i - t_{i-1}) \\
&= m(b - a),
\end{aligned}
$$

$$
\begin{aligned}
U(f, P) &= \sum_{i=1}^{n} M_i (t_i - t_{i-1}) \\
&\leq M \sum_{i=1}^{n} (t_i - t_{i-1}) = M(b - a),
\end{aligned}
$$

which completes the proof. $\square$