Time Scales Analysis
Lecture 12
Properties of the Riemann Delta Integral

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Definition

The *upper Darboux Δ-integral* $U(f)$ of f from a to b is defined by

$$U(f) = \inf\{U(f,P) : \ P \quad \text{is a partition of} \quad [a,b)\}$$

and the *lower Darboux Δ-integral* $L(f)$ is defined by

$$L(f) = \sup\{L(f,P) : \ P \quad \text{is a partition of} \quad [a,b)\}.$$

From (**??**), it follows that $U(f)$ and $L(f)$ are finite numbers and $L(f) \leq U(f)$.

Theorem

Let f be a bounded function on $[a, b)$. If P and Q are partitions of $[a, b)$ and Q is a refinement of P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof.

Let $P = \{t_0, t_1, t_2, \dots, t_n\}$. Without loss of generality, we suppose that

$$Q = \{t_0, t_1, \dots, t_k, t', t_{k+1}, \dots, t_n\},$$

i.e., $Q \setminus P = \{t'\}$. Define also

$$m_k^1 = \inf\{f(t) : t \in [t_k, t')\}, \quad m_k^2 = \inf\{f(t) : t \in [t', t_{k+1})\}$$

and

$$M_k^1 = \sup\{f(t) : t \in [t_k, t')\}, \quad M_k^2 = \sup\{f(t) : t \in [t', t_{k+1})\}.$$

We have □

Proof.

$$m_k^1 \geq m_{k+1}, \quad m_k^2 \geq m_{k+1}, \quad M_k^1 \leq M_{k+1}, \quad M_k^2 \leq M_{k+1}.$$

Then

$$\begin{aligned} L(f, Q) &= \sum_{i=1}^{k} m_i(t_i - t_{i-1}) + m_k^1(t' - t_k) + m_k^2(t_{k+1} - t') \\ &\quad + \sum_{i=k+2}^{n} m_i(t_i - t_{i-1}) \\ &\geq \sum_{i=1}^{k} m_i(t_i - t_{i-1}) + m_k(t' - t_k) + m_k(t_{k+1} - t') \\ &\quad + \sum_{i=k+2}^{n} m_i(t_i - t_{i-1}) \end{aligned}$$

□

Proof.

$$
\begin{aligned}
&= \sum_{i=1}^{k} m_i(t_i - t_{i-1}) + m_k(t_{k+1} - t_k) \\
&\quad + \sum_{i=k+2}^{n} m_i(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\
&= L(f, P)
\end{aligned}
$$

□

Proof.

and

$$
\begin{aligned}
U(f,Q) &= \sum_{i=1}^{k} M_i(t_i - t_{i-1}) + M_k^1(t' - t_k) + M_k^2(t_{k+1} - t') \\
&\quad + \sum_{i=k+2}^{n} M_i(t_i - t_{i-1}) \\
&\leq \sum_{i=1}^{k} M_i(t_i - t_{i-1}) + M_k(t' - t_k) + M_k(t_{k+1} - t') \\
&\quad + \sum_{i=k+2}^{n} M_i(t_i - t_{i-1})
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
&= \sum_{i=1}^{k} M_i(t_i - t_{i-1}) + M_k(t_{k+1} - t_k) \\
&\quad + \sum_{i=k+2}^{n} M_i(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \\
&= U(f, P),
\end{aligned}
$$

which completes the proof. □

Theorem

If f is a bounded function on $[a, b)$ and if P and Q are any two partitions of $[a, b)$, then $L(f, P) \leq U(f, Q)$.

Proof.

Note that $P \cup Q$ is also a partition of $[a, b)$. Since

$$P \subset P \cup Q \quad \text{and} \quad Q \subset P \cup Q,$$

we get

$$\begin{aligned} L(f, P) &\leq L(f, P \cup Q) \\ &\leq U(f, P \cup Q) \leq U(f, Q), \end{aligned}$$

which completes the proof. $\square$

Theorem

If f is a bounded function on $[a, b)$, then $L(f) \leq U(f)$.

Proof.

Let P and Q be arbitrarily chosen partitions of $[a, b)$. From Theorem 3, it follows that

$$L(f, P) \leq U(f, Q).$$

Hence,

$$L(f, P) \leq \inf_{Q} U(f, Q)$$

so that

$$L(f, P) \leq U(f),$$

and therefore □

Proof.

$$\sup_P L(f,P) \le U(f),$$

hence

$$L(f) \le U(f),$$

which completes the proof. $\square$

Definition

We say that f is Δ-*integrable* from a to b (or on $[a, b)$) provided $L(f) = U(f)$. We write $\int_a^b f(t)\Delta t$ for this common value. We call this integral the *Darboux* Δ-*integral*.

Theorem

If $L(f, P) = U(f, P)$ for some partition P of $[a, b)$, then the function f is Δ-integrable from a to b and

$$\int_a^b f(t)\Delta t = L(f, P) = U(f, P).$$

Proof.

From Theorem 4, it follows that

$$L(f, P) \leq L(f) \leq U(f) \leq U(f, P) = L(f, P),$$

which completes the proof. □

Theorem

A bounded function f on $[a, b)$ is Δ-integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b)$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof.

Let f be Δ-integrable. Then $L(f) = U(f)$. Let $\varepsilon > 0$ be arbitrarily chosen. Then there exist partitions P and Q of $[a, b)$ such that

$$L(f) - L(f, P) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f, Q) - U(f) < \frac{\varepsilon}{2}.$$

$\square$

Proof.

Assume that P is a refinement of Q. Then

$$U(f,P) \leq U(f,Q)$$

and

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,Q) - L(f,P) \\ &= U(f,Q) - U(f) + L(f) - L(f,P) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Assume that Q is a refinement of P. Then

$$L(f,Q) \geq L(f,P)$$

Proof.

$$\begin{aligned} U(f,Q) - L(f,Q) &\leq U(f,Q) - L(f,P) \\ &= U(f,Q) - U(f) + L(f) - L(f,P) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Suppose for each $\varepsilon > 0$, there exists a partition P so that

$$U(f,P) - L(f,P) < \varepsilon.$$

$\square$

Proof.

Hence, using that

$$U(f) \leq U(f,P) \quad \text{and} \quad -L(f) \leq -L(f,P),$$

we get

$$U(f) - L(f) < \varepsilon$$

for each $\varepsilon > 0$. Consequently, $U(f) = L(f)$, which completes the proof. $\square$

Theorem

For every $\delta > 0$, there exists at least one partition

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

of $[a, b)$ such that, for each $i \in \{1, \ldots, n\}$, either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$.

Proof.

Let $\delta > 0$ be arbitrarily chosen. If $b - a \leq \delta$, then, for every partition $P = \{t_0 = a < t_1 < \ldots < t_n = b\}$, we have $t_i - t_{i-1} \leq \delta$ for all $i \in \{1, \ldots, n\}$. $\square$

Proof.

Let $b - a > \delta$. We set $a = t_0$. Assume that t_0 is right-scattered. Let $t_1 = \sigma(t_0)$. Hence, $t_1 - t_0 \leq \delta$ or $t_1 - t_0 > \delta$ and $\rho(t_1) = t_0$. Assume that t_0 is right-dense. Then there exists $t_1 \in (a, b]$ such that $t_1 - t_0 \leq \delta$. If $t_1 = b$, then $P = \{t_0 = a < t_1 = b\}$ is the desired partition. Otherwise, we consider the interval $[t_1, b)$. $\square$

Definition

For given $\delta > 0$, we denote by $\mathcal{P}_\delta([a, b))$ the set of all partitions

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

that possess the property indicated in Theorem 8.

Theorem

A bounded function f on $[a, b)$ is Δ-integrable if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $P \in \mathcal{P}_\delta([a, b))$ implies

$$U(f, P) - L(f, P) < \varepsilon. \tag{1}$$

Proof.

Let f be Δ-integrable on $[a, b)$. Let also $\varepsilon > 0$ be arbitrarily chosen. Then there exists a partition P_1 of $[a, b)$ such that

$$U(f, P_1) - L(f, P_1) < \varepsilon. \tag{2}$$

If $P_1 \in \mathcal{P}_\delta([a, b))$ for some $\delta > 0$, then the assertion follows. Let $P_1 \notin \mathcal{P}_\delta([a, b))$ for any $\delta > 0$. Then there exist $\delta > 0$ and $P \in \mathcal{P}_\delta([a, b))$ so that P is a refinement of P_1. Hence, □

Proof.

$$U(f, P) \leq U(f, P_1) \quad \text{and} \quad L(f, P_1) \geq L(f, P).$$

From here and from (2), we get (1). Suppose that for each $\varepsilon > 0$, there exists $\delta > 0$ so that $P \in \mathcal{P}_\delta([a, b))$ implies (1). Thus, using Theorem 7, it follows that f is Δ-integrable on $[a, b)$. $\square$

Definition

Let f be a bounded function on $[a, b)$. Assume

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

is a partition of $[a, b)$. In each interval $[t_{i-1}, t_i)$, $i \in \{1, \ldots, b\}$, choose an arbitrary point ξ_i and form the sum

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

We call S a *Riemann Δ-sum* of f corresponding to the partition P. We say that f is *Riemann Δ-integrable* from a to b (or on $[a, b)$) if there exists a number I with the following property.

Definition

For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann Δ-sum of f corresponding to a partition $P \in \mathcal{P}_\delta([a, b))$, independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)$, $i \in \{1, \ldots, n\}$. The number I is called the *Riemann Δ-integral* of f from a to b.

Theorem

A bounded function f on $[a, b)$ is Riemann Δ-integrable if and only if it is Darboux Δ-integrable, in which case the values of the integrals coincide.

Proof.

Suppose that f is Darboux Δ-integrable on $[a, b)$. Then

$$U(f) = L(f) = \int_a^b f(t)\Delta t.$$

Let $\varepsilon > 0$ be arbitrarily chosen. From Theorem 10, it follows that there exists $\delta > 0$ such that $P \in \mathcal{P}_\delta([a, b))$ satisfies

$$U(f, P) - L(f, P) < \varepsilon.$$

$\square$

Proof.

Note that

$$
\begin{aligned}
U(f,P) &< \varepsilon + L(f,P) \\
&\leq \varepsilon + L(f) \\
&= \varepsilon + \int_a^b f(t)\Delta t, \\
L(f,P) &\geq U(f,P) - \varepsilon \\
&\geq U(f) - \varepsilon \\
&= \int_a^b f(t)\Delta t - \varepsilon.
\end{aligned}
$$

Since

Proof.

$$\begin{aligned} 0 &\geq S - U(f,P) \\ &> S - \varepsilon - \int_a^b f(t)\Delta t, \end{aligned}$$

i.e.,

$$S - \int_a^b f(t)\Delta t \leq \varepsilon. \tag{3}$$

Also,

$$\begin{aligned} 0 &\leq S - L(f,P) \\ &\leq S - \int_a^b f(t)\Delta t + \varepsilon, \end{aligned}$$

i.e.,

Proof.

$$S - \int_a^b f(t)\Delta t \geq -\varepsilon.$$

From the last inequality and from (3), we get that

$$\left| S - \int_a^b f(t)\Delta t \right| \leq \varepsilon,$$

which proves that f is Riemann Δ-integrable and

$$I = \int_a^b f(t)\Delta t.$$

Assume that f is Riemann Δ-integrable in the sense of Definition 11 and let $\varepsilon > 0$. Let $\delta > 0$ and I be as given in Definition 11. We take a partition

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

□

Proof.

of $[a, b)$ such that $P \in \mathcal{P}_\delta([a, b))$. For each $i \in \{1, \dots, n\}$, we choose $\xi_i \in [t_{i-1}, t_i)$ so that $f(\xi_i) < m_i + \varepsilon$, where $m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)\}$. Hence,

$$
\begin{aligned}
S &= \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \\
&< \sum_{i=1}^{n} (m_i + \varepsilon)(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) + \varepsilon \sum_{i=1}^{n} (t_i - t_{i-1}) \\
&= L(f, P) + \varepsilon(b - a),
\end{aligned}
$$

i.e.,

$\square$

Proof.

$$L(f,P) > S - \varepsilon(b-a).$$

Also, we have

$$|S - I| < \varepsilon, \quad \text{i.e.,} \quad -\varepsilon + I < S < I + \varepsilon.$$

Hence,

$$\begin{aligned} L(f) &\geq L(f,P) \\ &> S - \varepsilon(b-a) \\ &> I - \varepsilon - \varepsilon(b-a). \end{aligned}$$

Because $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$L(f) \geq I. \tag{4}$$

Proof.

Let $\eta_i \in [t_{i-1}, t_i)$ be so that

$$f(\eta_i) > M_i - \varepsilon,$$

where

$$M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)\}, \quad i \in \{1, \ldots, n\}.$$

Hence,

$$\begin{aligned} S &= \sum_{i=1}^{n} f(\eta_i)(t_i - t_{i-1}) \\ &> \sum_{i=1}^{n} (M_i - \varepsilon)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) - \varepsilon \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= U(f, P) - \varepsilon(b - a). \end{aligned}$$

Proof.

$$U(f,P) < S + \varepsilon(b-a).$$

From here,

$$\begin{aligned} U(f) &\leq U(f,P) \\ &< S + \varepsilon(b-a) \\ &< I + \varepsilon + \varepsilon(b-a). \end{aligned}$$

□

Proof.

Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$U(f) \leq I.$$

From the last inequality and from (4), we obtain

$$I \leq L(f) \leq U(f) \leq I,$$

i.e.,

$$L(f) = U(f) = I.$$

This shows that f is Darboux Δ-integrable and $\int_a^b f(t)\Delta t = I$. $\square$

Remark

In our definition of $\int_a^b f(t)\Delta t$*, we assumed that* $a < b$*. We remove this restriction with the following definitions.*

$$\int_a^a f(t)\Delta t = 0, \quad \int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t \quad \text{if} \quad a > b.$$

Theorem

Let $a, b \in \mathbb{T}$. *Then every constant function*

$$f(t) = c, \quad t \in \mathbb{T},$$

is Δ*-integrable from* a *to* b *and*

$$\int_a^b c\Delta t = c(b-a).$$

Proof.

Without loss of generality, we assume that $a < b$. For any partition

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\},$$

we have

$$U(f, P) = L(f, P) = c(b - a).$$

Therefore,

$$U(f) = L(f) = c(b - a),$$

which completes the proof. □

Theorem

Let t be an arbitrary point in $\mathbb{T}$. Every function f defined on $\mathbb{T}$ is Δ-integrable from t to $\sigma(t)$ and

$$\int_t^{\sigma(t)} f(s)\Delta s = (\sigma(t) - t)f(t).$$

Proof.

Let $\sigma(t) = t$. Then the assertion is valid. Let $\sigma(t) > t$. Then a single partition of $[t, \sigma(t))$ is

$$P = \{t_0 = t < t_1 = \sigma(t)\}.$$

Since $[t, \sigma(t)) = \{t\}$, we have that

$$U(f, P) = L(f, P) = (\sigma(t) - t)f(t).$$

Therefore, $U(f) = L(f) = (\sigma(t) - t)f(t)$, which completes the proof. $\square$

Theorem

Let f be Δ-integrable on $[a, b)$ and let M and m be its supremum and infimum on $[a, b)$, respectively. Assume $\phi : \mathbb{R} \to \mathbb{R}$ is a function defined on $[m, M]$ such that

$$|\phi(x) - \phi(y)| \leq B|x - y|$$

for some positive constant B and for all $x, y \in [m, M]$. Then the composite function $h = \phi \circ f)$ is Δ-integrable on $[a, b)$.

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is Δ-integrable on $[a, b)$, using Theorem 7, there exists a partition

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

of $[a, b)$ such that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{B}.$$

Define

$$m_i = \inf_{t \in [t_{i-1}, t_i)} f(t), \quad M_i = \sup_{t \in [t_{i-1}, t_i)} f(t)$$

and

$$m_i^* = \inf_{t \in [t_{i-1}, t_i)} h(t), \quad M_i^* = \sup_{t \in [t_{i-1}, t_i)} h(t).$$

□

Proof.

Then, for every $s, \tau \in [t_{i-1}, t_i)$, we have

$$\begin{aligned} h(s) - h(\tau) &\leq |h(s) - h(\tau)| \\ &= |\phi(f(s)) - \phi(f(\tau))| \\ &\leq B|f(s) - f(\tau)| \\ &\leq B(M_i - m_i). \end{aligned} \tag{5}$$

There exist sequences $\{s_k\}_{k\in\mathbb{N}}$ and $\{\tau_k\}_{k\in\mathbb{N}}$ □

Proof.

of points of $[t_{i-1}, t_i)$ such that

$$h(s_k) \to M_i^* \quad \text{and} \quad h(\tau_k) \to m_i^*$$

as $k \to \infty$. Thus, using (5), we obtain

$$h(s_k) - h(\tau_k) \le B(M_i - m_i),$$

whereupon

$$M_i^* - m_i^* \le B(M_i - m_i).$$

From here,

$$\begin{aligned} U(h, P) - L(h, P) &= \sum_{i=1}^{n} (M_i^* - m_i^*)(t_i - t_{i-1}) \\ &\le \sum_{i=1}^{n} B(M_i - m_i)(t_i - t_{i-1}) \end{aligned}$$

Proof.

$$
\begin{aligned}
&= B\left(\sum_{i=1}^{n} M_i(t_i - t_{i-1}) - \sum_{i=1}^{n} m_i(t_i - t_{i-1})\right) \\
&= B(U(f,P) - L(f,P)) \\
&< B\frac{\varepsilon}{B} \\
&= \varepsilon.
\end{aligned}
$$

Thus, using Theorem 7, it follows that h is Δ-integrable on $[a,b)$. □

Theorem

Let f be Δ-integrable on $[a,b)$ and let M and m be its supremum and infimum on $[a,b)$, respectively. Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function on $[m,M]$. Then the composite function $h = \phi \circ f$ is Δ-integrable on $[a,b)$.

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. We take a partition

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

of $[a,b)$ so that

$$\sup_{t \in [t_{i-1}, t_i)} h(t) - \inf_{t \in [t_{i-1}, t_i)} h(t) < \frac{\varepsilon}{b-a}, \quad i \in \{1, \ldots, n\}.$$

Then □

Proof.

$$\begin{aligned} U(h,P) - L(h,P) &= \sum_{i=1}^{n} \left(\sup_{t\in[t_{i-1},t_i)} h(t) - \inf_{t\in[t_{i-1},t_i)} h(t) \right)(t_i - t_{i-1}) \\ &< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= \varepsilon. \end{aligned}$$

From here and from Theorem 7, it follows that h is Δ-integrable on $[a,b]$. □

Corollary

If f is Δ-integrable on $[a, b)$, then, for an arbitrary positive number α, the function $|f|^{\alpha}$ is Δ-integrable on $[a, b)$.

Proof.

We take the function $\phi(x) = |x|^{\alpha}$ and apply Theorem 17. □

Theorem

Let f be a bounded function that is Δ-integrable on $[a, b)$. Then f is Δ-integrable on every subinterval $[c, d)$ of the interval $[a, b)$.

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is Δ-integrable on $[a, b)$, using Theorem 7, there exists a partition P of $[a, b)$ such that $\square$

Proof.

$$U(f, P) - L(f, P) < \varepsilon.$$

We set $P' = P \cup \{c\} \cup \{d\}$. Then P''is a refinement of P and

$$U(f, P') - L(f, P') < \varepsilon.$$

Let $P'' = P' \cap [c, d]$. Then P'' is a partition of $[c, d)$ and $P'' \subset P'$. Hence,

$$U(f, P'') - L(f, P'') \leq U(f, P') - L(f, P') < \varepsilon.$$

From here and from Theorem 7, it follows that f is Δ-integrable on $[c, d)$. $\square$

Theorem

Let f and g be Δ-integrable functions on $[a, b)$ and $c \in \mathbb{R}$. Then

1. *cf is Δ-integrable on $[a, b)$ and*

$$\int_a^b cf(t)\Delta t = c \int_a^b f(t)\Delta t,$$

2. *$f + g$ is Δ-integrable on $[a, b)$ and*

$$\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.$$

Proof.

Let $c > 0$. Suppose $\varepsilon > 0$ is arbitrarily chosen. Since f is Δ-integrable on $[a, b)$, there exists a partition P of $[a, b)$ such that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{c}.$$

Note that

$$U(cf, P) = cU(f, P), \quad L(cf, P) = cL(f, P)$$

and thus

$$U(cf) = cU(f), \quad L(cf) = cL(f).$$

Hence, $\square$

Proof.

$$
\begin{aligned}
U(cf, P) - L(cf, P) &= cU(f, P) - cL(f, P) \\
&< c\frac{\varepsilon}{c} \\
&= \varepsilon.
\end{aligned}
$$

Therefore, cf is Δ-integrable on $[a, b)$. Also,

$$U(cf) = cU(f) = cL(f) = L(cf),$$

i.e., $\square$

Proof.

$$\int_a^b cf(t)\Delta t = c\int_a^b f(t)\Delta t.$$

Let $c = -1$. Then

$$U(-f, P) = -L(f, P), \quad L(-f, P) = -U(f, P)$$

and thus

$$U(-f) = -L(f), \quad L(-f) = -U(f).$$

Therefore,

$$U(-f) = -L(f) = -U(f) = L(-f).$$

Consequently, $-f$ is Δ-integrable on $[a, b)$ and

$$\int_a^b (-f(t))\Delta t = -\int_a^b f(t)\Delta t.$$

$\square$

Proof.

Let $c < 0$. Then

$$\begin{aligned}\int_a^b cf(t)\Delta t &= \int_a^b -(-c)f(t)\Delta t \\ &= -\int_a^b (-c)f(t)\Delta t \\ &= -(-c)\int_a^b f(t)\Delta t \\ &= c\int_a^b f(t)\Delta t,\end{aligned}$$

which completes the proof.
Let $\varepsilon > 0$ be arbitrarily chosen. □

Proof.

Since f and g are Δ-integrable on $[a,b)$, there exist partitions P_1 and P_2 of $[a,b)$ so that

$$U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2} \quad \text{and} \quad U(g,P_2)-L(g,P_2)<\frac{\varepsilon}{2}.$$

We set $P=P_1\cup P_2$. Then P is a refinement of P_1 and P_2. Hence,

$$U(f,P)-L(f,P)\leq U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2},$$

$$U(g,P)-L(g,P)\leq U(g,P_2)-L(g,P_2)<\frac{\varepsilon}{2},$$

and

Proof.

$$
\begin{aligned}
U(f+g,P) - L(f+g,P) &\leq U(f,P) + U(g,P) - L(f,P) - L(g,P) \\
&= U(f,P) - L(f,P) + U(g,P) - L(g,P) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Thus, using Theorem 7, we conclude that $f+g$ is Δ-integrable on $[a,b)$. Also, $\square$

Proof.

$$
\begin{aligned}
\int_a^b (f(t)+g(t))\Delta t &= U(f+g) \\
&\leq U(f+g,P) \\
&\leq U(f,P)+U(g,P) \\
&< \frac{\varepsilon}{2}+L(f,P)+\frac{\varepsilon}{2}+L(g,P) \\
&= \varepsilon+L(f,P)+L(g,P) \\
&\leq \varepsilon+L(f)+L(g) \\
&= \varepsilon+\int^b f(t)\Delta t+\int^b g(t)\Delta t
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\int_a^b (f(t)+g(t))\Delta t &= L(f+g) \\
&\geq L(f+g,P) \\
&\geq L(f,P)+L(g,P) \\
&> U(f,P)-\frac{\varepsilon}{2}+U(g,P)-\frac{\varepsilon}{2} \\
&= U(f,P)+U(g,P)-\varepsilon \\
&\geq U(f)+U(g)-\varepsilon \\
&= \int_a^b f(t)\Delta t+\int_a^b g(t)\Delta t-\varepsilon,
\end{aligned}
$$

Proof.

$$
\begin{aligned}
-\varepsilon + \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t &\leq \int_a^b (f(t)+g(t))\Delta t \\
&\leq \varepsilon + \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.
\end{aligned}
$$

Because $\varepsilon > 0$ was arbitrarily chosen, we obtain that

$$
\int_a^b (f(t)+g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.
$$

$\square$

Theorem

Let f and g be Δ-integrable on $[a, b)$. Then fg is Δ-integrable on $[a, b)$.

Proof.

Let $\phi(x) = x^2$. Then $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition on any finite interval $[m, M]$. We observe that
$f^2(t) = \phi(f(t))$. $\square$

Proof.

Thus, using Theorem 16, it follows that f^2 is Δ-integrable on $[a, b)$. Because f and g are Δ-integrable on $[a, b)$, using Theorem 20, we get that $f + g$, $-g$, $f - g$ are Δ-integrable functions on $[a, b)$. From here, $(f + g)^2$ and $(f - g)^2$ are Δ-integrable functions on $[a, b)$. Thus, using Theorem 20, we conclude that

$$\frac{1}{4}(f+g)^2, \quad -\frac{1}{4}(f-g)^2, \quad \text{and} \quad \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$$

are Δ-integrable on $[a, b)$, which completes the proof. $\square$

Theorem

Let f be a function defined on $[a, b)$ and let $c \in \mathbb{T}$ with $a < c < b$. If f is Δ-integrable from a to c and from c to b, then f is Δ-integrable on $[a, b)$ and

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t. \tag{6}$$

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is Δ-integrable on $[a, c)$, there exists a partition P_1 of $[a, c)$ so that □

Proof.

$$U_a^c(f, P_1) - L_a^c(f, P_1) < \frac{\varepsilon}{2}.$$

Because f is Δ-integrable on $[c, b)$, there exists a partition P_2 of $[c, b)$ such that

$$U_c^b(f, P_2) - L_c^b(f, P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then

$$\begin{aligned} U_a^b(f, P) - L_a^b(f, P) &= U_a^c(f, P_1) + U_c^b(f, P_2) - L_a^c(f, P_1) - L_c^b(f, P_2) \\ &= U_a^c(f, P_1) - L_a^c(f, P_1) + U_c^b(f, P_2) - L_c^b(f, P_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

Proof.

Consequently, f is Δ-integrable on $[a, b)$. Also,

$$\begin{aligned}
\int_a^b f(t)\Delta t &\leq U_a^b(f, P) \\
&= U_a^c(f, P_1) + U_c^b(f, P_2) \\
&< L_a^c(f, P_1) + L_c^b(f, P_2) + \varepsilon \\
&\leq \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t + \varepsilon
\end{aligned}$$

and

$\square$

Proof.

$$\begin{aligned}
\int_a^b f(t)\Delta t &\geq L_a^b(f,P) \\
&= L_a^c(f,P_1) + L_c^b(f,P_2) \\
&> U_a^c(f,P_1) + U_c^b(f,P_2) - \varepsilon \\
&\geq \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t - \varepsilon,
\end{aligned}$$

i.e.,

$$\begin{aligned}
\int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t - \varepsilon &\leq \int_a^b f(t)\Delta t \\
&\leq \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t + \varepsilon.
\end{aligned}$$

Theorem

If f and g are Δ-integrable on $[a, b)$ and $f(t) \leq g(t)$ for all $t \in [a, b)$, then

$$\int_a^b f(t)\Delta t \leq \int_a^b g(t)\Delta t.$$

Proof.

Since f and g are Δ-integrable on $[a, b)$, we have that $h = g - f$ is Δ-integrable on $[a, b)$. Because $h(t) \geq 0$ for all $t \in [a, b)$, $\square$

Proof.

we conclude that $L(h, P) \geq 0$ for all partitions P of $[a, b)$. Therefore

$$L(f) = \int_a^b h(t)\Delta t \geq 0,$$

i.e.,

$$\int_a^b (g(t) - h(t))\Delta t \geq 0.$$

Now applying Theorem 20, we obtain

$$\int_a^b g(t)\Delta t - \int_a^b f(t)\Delta t \geq 0,$$

which completes the proof. □