Time Scales Analysis
Lecture 13
Properties of the Riemann Delta Integral

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October 22, 2025

Theorem

If f is Δ-integrable on $[a, b)$, then $|f|$ is Δ-integrable on $[a, b)$ and

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b |f(t)|\Delta t.$$

Proof.

Let $\phi(x) = |x|$. Then

$$\phi : \mathbb{R} \to \mathbb{R}$$

satisfies the Lipschitz condition on every finite interval $[m, M]$. Hence, using Theorem **??**, we conclude that $|f|$ is Δ-integrable on $[a, b)$. Since □

Proof.

$$-|f(t)| \leq f(t) \leq |f(t)| \quad \text{for any} \quad t \in [a, b),$$

using Theorem **??**, we obtain

$$-\int_a^b |f(t)|\Delta t \leq \int_a^b f(t)\Delta t \leq \int_a^b |f(t)|\Delta t,$$

which completes the proof. □

Theorem

If f and g are Δ-integrable on $[a, b)$, then

$$\left|\int_a^b f(t)g(t)\Delta t\right| \le \int_a^b |f(t)g(t)|\Delta t \le \left(\sup_{t\in[a,b)} |f(t)|\right) \int_a^b |g(t)|\Delta t.$$

Proof.

Since f and g are Δ-integrable on $[a, b)$, using Theorem **??**, we have that $|fg|$ is Δ-integrable on $[a, b)$. $\square$

Proof.

By Theorem 1, we get

$$\left|\int_a^b f(t)g(t)\Delta t\right| \leq \int_a^b |f(t)g(t)|\Delta t.$$

Because

$$|f(t)g(t)| \leq \left(\sup_{t\in[a,b)} |f(t)|\right)|g(t)| \quad \text{for any} \quad t \in [a, b),$$

using Theorem **??**, we obtain

$$\begin{aligned}\int_a^b |f(t)g(t)|\Delta t &\leq \int_a^b \left(\sup_{t\in[a,b)} |f(t)|\right)|g(t)|\Delta t \\ &= \left(\sup_{t\in[a,b)} |f(t)|\right)\int_a^b |g(t)|\Delta t,\end{aligned}$$

which completes the proof.

Theorem

Let $\{f_n(t)\}_{n\in\mathbb{N}}$ be a sequence of Δ-integrable functions on $[a,b)$ and suppose that $f_n \to f$ uniformly on $[a,b)$ for a function f defined on $[a,b)$. Then f is Δ-integrable on $[a,b)$ and

$$\int_a^b f(t)\Delta t = \lim_{n\to\infty} \int_a^b f_n(t)\Delta t.$$

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since $f_n \to f$ uniformly on $[a,b)$, there exists $n_0 \in \mathbb{N}$ so that

$$|f_n(t) - f(t)| < \varepsilon \quad \text{for all} \quad t \in [a,b) \quad \text{and} \quad n \geq n_0.$$

$\square$

Proof.

Because f_n, $n \geq n_0$, are Δ-integrable on $[a, b)$, there exists a partition P of $[a, b)$ so that

$$U(f_n, P) - L(f_n, P) < \varepsilon, \quad n \geq n_0.$$

Note that

$$U(f_n - f, P) < \varepsilon(b - a), \quad L(f_n - f, P) > -\varepsilon(b - a), \quad n \geq n_0.$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= U(f - f_n + f_n, P) - L(f - f_n + f_n, P) \\ &\leq U(f - f_n, P) + U(f_n, P) - L(f - f_n, P) - L(f_n, P) \\ &= U(f - f_n, P) - L(f - f_n, P) + U(f_n, P) - L(f_n, P) \\ &< \varepsilon(b - a) + \varepsilon(b - a) + \varepsilon, \quad n \geq n_0. \end{aligned}$$

Proof.

Hence, using Theorem **??**, it follows that f is Δ-integrable on $[a,b)$. From here, $f_n - f$, $|f_n - f|$, $n \geq n_0$, are Δ-integrable on $[a,b)$. Then

$$\begin{aligned} \left| \int_a^b (f_n(t) - f(t))\Delta t \right| &= \left| \int_a^b f_n(t)\Delta t - \int_a^b f(t)\Delta t \right| \\ &\leq \int_a^b |f_n(t) - f(t)|\Delta t \\ &< \varepsilon \int_a^b \Delta t \\ &= \varepsilon(b-a), \end{aligned}$$

which completes the proof. $\square$

Theorem

Suppose that $\sum_{k=1}^{\infty} g_k$ is a series of Δ-integrable functions g_k on $[a,b)$ that converges uniformly to g on $[a,b)$. Then g is Δ-integrable and

$$\int_a^b g(t)\Delta t = \sum_{k=1}^{\infty} \int_a^b g_k(t)\Delta t.$$

Proof.

Since $\sum_{k=1}^{\infty} g_k$ is uniformly convergent to g on $[a,b)$, the sequence

$$\left\{ S_n = \sum_{k=1}^{n} g_k \right\}_{k\in\mathbb{N}}$$

is uniformly convergent to g on $[a,b)$. $\square$

Proof.

Hence, using Theorem 3, it follows that g is Δ-integrable on $[a, b)$ and

$$\lim_{n\to\infty}\int_a^b S_n(t)\Delta t = \int_a^b g(t)\Delta t,$$

whereupon

$$\lim_{n\to\infty}\int_a^b \sum_{k=1}^{n} g_k(t)\Delta t = \int_a^b g(t)\Delta t$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_a^b g_k(t)\Delta t = \int_a^b g(t)\Delta t,$$

which completes the proof. $\square$

Theorem

Let f be Δ-integrable on $[a, b)$. If f has a Δ-antiderivative

$$F : [a, b] \to \mathbb{R},$$

then

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Proof.

Since f is a Δ-antiderivative of f on $[a, b]$, we have that

$$F^{\Delta}(t) = f(t) \quad \text{for all} \quad t \in [a, b].$$

Let $\varepsilon > 0$ be arbitrarily chosen. □

Proof.

Because f is Δ-integrable on $[a, b)$, there exists a partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a, b)$ so that

$$U(f, P) - L(f, P) < \varepsilon. \tag{1}$$

For every $i \in \{1, \ldots, n\}$, there exist $\xi_i, \eta_i \in (t_{i-1}, t_i)$ so that

$$F^{\Delta}(\eta_i)(t_i - t_{i-1}) \leq F(t_i) - F(t_{i-1}) \leq F^{\Delta}(\xi_i)(t_i - t_{i-1}),$$

whereupon

$$f(\eta_i)(t_i - t_{i-1}) \leq F(t_i) - F(t_{i-1}) \leq f(\xi_i)(t_i - t_{i-1}).$$

Hence, $\square$

Proof.

$$m_i(t_i - t_{i-1}) \leq F(t_i) - F(t_{i-1}) \leq M_i(t_i - t_{i-1}),$$

where

$$m_i = \inf_{t \in [t_{i-1}, t_i)} f(t), \quad M_i = \sup_{t \in [t_{i-1}, t_i)} f(t),$$

and

$$\sum_{i=1}^{n} m_i(t_i - t_{i-1}) \leq \sum_{i=1}^{n} (F(t_i) - F(t_{i-1})) \leq \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Therefore, □

Proof.

$$L(f,P) \leq F(b) - F(a) \leq U(f,P).$$

Since

$$L(f,P_1) \leq \int_a^b f(t)\Delta t \leq U(f,P_1)$$

for all partitions P_1 of $[a,b)$, (1) yields

$$\left| \int_a^b f(t)\Delta t - (F(b) - F(a)) \right| < \varepsilon,$$

which completes the proof. $\square$

Example

Let $\mathbb{T} = \mathbb{Z}$. Then $\sigma(t) = t + 1$, $\mu(t) = 1$, and

$$\int_t^{t+1} (\tau^3 + \tau^2 + \tau + 1)\Delta\tau = t^3 + t^2 + t + 1.$$

Example

Let $\mathbb{T} = \mathbb{N}_0^4$. Then $\sigma(t) = (\sqrt[4]{t} + 1)^4$, $\mu(t) = (\sqrt[4]{t} + 1)^4 - t$, and

$$\int_t^{(\sqrt[4]{t}+1)^4} \tau^2 \Delta\tau = \left((\sqrt[4]{t} + 1)^4 - t\right) t^2.$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Then $\sigma(t) = 3t$, $\mu(t) = 2t$, and

$$\int_t^{3t} \sin \tau \Delta \tau = 2t \sin t.$$

Theorem

If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, *and* $f, g \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T})$, *then*

1. $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$
2. $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t,$
3. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$
4. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$
5. $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t,$

Theorem

1. $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$,
2. $\int_a^a f(t)\Delta t = 0$,
3. *if* $|f(t)| \leq g(t)$ *on* $[a, b]$, *then*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t,$$

4. *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$.

Proof.

Since $f, g \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T})$, they possess antiderivatives F and G. We have

$$F^{\Delta}(t) = f(t) \quad \text{and} \quad G^{\Delta}(t) = g(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}.$$

For all $t \in \mathbb{T}^{\kappa}$, we have

$$(F+G)^{\Delta}(t) = F^{\Delta}(t) + G^{\Delta}(t).$$

Hence,

$$\begin{aligned}
\int_a^b (f(t)+g(t))\Delta t &= (F+G)(b) - (F+G)(a) \\
&= F(b) - F(a) + G(b) - G(a) \\
&= \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.
\end{aligned}$$

$\square$

Proof.

Since

$$\alpha F^{\Delta}(t) = (\alpha F)^{\Delta}(t) = \alpha f(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa},$$

we get

$$\begin{aligned} \int_a^b \alpha f(t)\Delta t &= (\alpha F)(b) - (\alpha F)(a) \\ &= \alpha(F(b) - F(a)) \\ &= \alpha \int_a^b f(t)\Delta t. \end{aligned}$$

$\square$

Proof.

We have

$$\begin{aligned}\int_a^b f(t)\Delta t &= F(b) - F(a)\\&= -(F(a) - F(b))\\&= -\int_b^a f(t)\Delta t.\end{aligned}$$

We have

$$\begin{aligned}\int_a^b f(t)\Delta t &= F(b) - F(a)\\&= F(c) - F(a) + F(b) - F(c)\\&= \int^c f(t)\Delta t + \int^b f(t)\Delta t.\end{aligned}$$

Proof.

For all $t \in \mathbb{T}^\kappa$, we have

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t),$$

i.e.,

$$f(\sigma(t))g^\Delta(t) = (fg)^\Delta(t) - f^\Delta(t)g(t).$$

Hence, by using items 1 and 2, we get

$$\begin{aligned}
\int_a^b f(\sigma(t))g^\Delta(t)\Delta t &= \int_a^b \Big((fg)^\Delta(t) - f^\Delta(t)g(t)\Big)\Delta t \\
&= \int_a^b (fg)^\Delta(t)\Delta t - \int_a^b f^\Delta(t)g(t)\Delta t \\
&= (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t.
\end{aligned}$$

$\square$

Proof.

For all $t \in \mathbb{T}^{\kappa}$, we have

$$(fg)^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$

i.e.,

$$f(t)g^{\Delta}(t) = (fg)^{\Delta}(t) - f^{\Delta}(t)g(\sigma(t)).$$

Hence, by using items 1 and 2, we find

$$\begin{aligned} \int_a^b f(t)g^{\Delta}(t)\Delta t &= \int_a^b (fg)^{\Delta}(t)\Delta t - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t \\ &= (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t. \end{aligned}$$

$\square$

Proof.

We have

$$\int_a^a f(t)\Delta t = F(a) - F(a) = 0.$$

We note that

$$|F^\Delta(t)| \leq G^\Delta(t) \quad \text{on} \quad [a, b].$$

Thus, employing Theorem **??**, we get

$$|F(b) - F(a)| \leq G(b) - G(a),$$

i.e.,

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b g(t)\Delta t.$$

□

Example

Let $\mathbb{T} = \mathbb{Z}$. We will compute

$$I = \int_{-2}^{3} (t^2 + t + 1)\Delta t.$$

First way. We have

$$\begin{aligned} t^2 &= \frac{1}{3}(t^3)^\Delta - \frac{1}{2}(t^2)^\Delta + \frac{1}{6} \\ &= \left(\frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{t}{6}\right)^\Delta, \\ t &= \frac{1}{2}(t^2)^\Delta - \frac{1}{2} \\ &= \left(\frac{1}{2}t^2 - \frac{t}{2}\right)^\Delta, \end{aligned}$$

Example

Hence,

$$\begin{aligned} t^2+t+1 &= \left(\frac{1}{3}t^3-\frac{1}{2}t^2+\frac{t}{6}\right)^{\Delta}+\left(\frac{1}{2}t^2-\frac{1}{2}t\right)^{\Delta}+t^{\Delta} \\ &= \left(\frac{1}{3}t^3-\frac{1}{2}t^2+\frac{t}{6}+\frac{1}{2}t^2-\frac{t}{2}+t\right)^{\Delta} \\ &= \left(\frac{1}{3}t^3+\frac{2}{3}t\right)^{\Delta}. \end{aligned}$$

Therefore,

Example

$$
\begin{aligned}
I &= \int_{-2}^{3} \left(\frac{1}{3}t^3 + \frac{2}{3}t\right)^{\Delta} \Delta t \\
&= \left(\frac{1}{3}t^3 + \frac{2}{3}t\right)\Big|_{t=-2}^{t=3} \\
&= \left(\frac{27}{3} + 2\right) - \left(-\frac{8}{3} - \frac{4}{3}\right) \\
&= 11 + 4 \\
&= 15.
\end{aligned}
$$

Second way. Since all points of $\mathbb{T}$ are isolated and $\mu(t) = 1$, we have that

Example

$$
\begin{aligned}
I &= \sum_{t=-2}^{2}(t^2+t+1) \\
&= (4-2+1)+(1-1+1)+1+(1+1+1)+(4+2+1) \\
&= 3+1+1+3+7 \\
&= 15.
\end{aligned}
$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We will compute

$$I = \int_1^4 \left(\frac{\sin\frac{t}{2}\sin\frac{3t}{2}}{t} + t^2 \right) \Delta t.$$

First way. Note that $\sigma(t) = 2t$, $\mu(t) = t$, and

$$\begin{aligned} (\cos t)^\Delta &= -2\frac{\sin\frac{t}{2}\sin\frac{3t}{2}}{t}, \\ t^2 &= \frac{1}{7}(t^3)^\Delta. \end{aligned}$$

Hence,

Example

$$
\begin{aligned}
\frac{\sin\frac{t}{2}\sin\frac{3t}{2}}{t} + t^2 &= -\frac{1}{2}(\cos t)^{\Delta} + \frac{1}{7}(t^3)^{\Delta} \\
&= \left(-\frac{1}{2}\cos t + \frac{1}{7}t^3\right)^{\Delta}.
\end{aligned}
$$

Therefore,

Example

$$
\begin{aligned}
I &= \int_1^4 \left(-\frac{1}{2}\cos t + \frac{1}{7}t^3\right)^{\Delta} \Delta t \\
&= \left(-\frac{1}{2}\cos t + \frac{1}{7}t^3\right)\Big|_{t=1}^{t=4} \\
&= -\frac{1}{2}\cos 4 + \frac{64}{7} + \frac{1}{2}\cos 1 - \frac{1}{7} \\
&= -\frac{1}{2}(\cos 4 - \cos 1) + 9 \\
&= \sin\frac{3}{2}\sin\frac{5}{2} + 9.
\end{aligned}
$$

Example

Second way. Since all points of $\mathbb{T}$ are isolated, we obtain

$$
\begin{aligned}
I &= \sum_{t=1,2} \mu(t) \left(\frac{\sin \frac{t}{2} \sin \frac{3t}{2}}{t} + t^2 \right) \\
&= \sin \frac{1}{2} \sin \frac{3}{2} + 1 + 2 \left(\frac{\sin 1 \sin 3}{2} + 4 \right) \\
&= \sin \frac{1}{2} \sin \frac{3}{2} + \sin 1 \sin 3 + 9 \\
&= \frac{1}{2} (\cos 1 - \cos 2 + \cos 2 - \cos 4) + 9 \\
&= \frac{1}{2}(\cos 1 - \cos 4) + 9 \\
&= -\sin \frac{-3}{2} \sin \frac{5}{2} + 9
\end{aligned}
$$

Example

Let $\mathbb{T} = [-1, 0] \cup 3^{\mathbb{N}_0}$, where $[-1, 0]$ is the real-valued interval. Define

$$f(t) = \begin{cases} \frac{1}{(t+2)^3} - \frac{1}{8} & \text{for} \quad t \in [-1, 0) \\ 0 & \text{for} \quad t = 0 \\ t^2 - t & \text{for} \quad t \in 3^{\mathbb{N}_0}. \end{cases}$$

We will compute

$$I = \int_{-1}^{3} f(t)\Delta t.$$

We have

Example

$$
\begin{aligned}
I &= \int_{-1}^{0} f(t)\Delta t + \int_{1}^{3} f(t)\Delta t \\
&= \int_{-1}^{0} \left(\frac{1}{(t+2)^3} - \frac{1}{8} \right) \mathrm{d}t + \int_{1}^{3} (t^2 - t)\Delta t \\
&= -\frac{1}{2(t+2)^2}\Big|_{t=-1}^{t=0} - \frac{1}{8} + 2t(t^2 - t)\Big|_{t=1} \\
&= -\frac{1}{4} + \frac{1}{2} \\
&= \frac{1}{4}.
\end{aligned}
$$

Theorem

Let f be a function which is Δ-integrable from a to b. Let

$$F(t) = \int_a^t f(s)\Delta s, \quad t \in [a, b].$$

Then f is continuous on $[a, b]$. Further, let $t_0 \in [a, b)$ and suppose that f is continuous at t_0 if t_0 is right-dense. Then f is Δ-differentiable at t_0 and $F^{\Delta}(t_0) = f(t_0)$.

Proof.

Let $B > 0$ be such that

$$|f(t)| \leq B \quad \text{for all} \quad t \in [a,b].$$

Let $\varepsilon > 0$ be arbitrarily chosen. Take $t_1, t_2 \in [a,b]$ such that $|t_1 - t_2| < \frac{\varepsilon}{B}$. Then

$$\begin{aligned} |F(t_1) - F(t_2)| &= \left| \int_a^{t_1} f(s)\Delta s - \int_a^{t_2} f(s)\Delta s \right| \\ &= \left| \int_a^{t_1} f(s)\Delta s + \int_{t_1}^{a} f(s)\Delta s - \int_{t_1}^{a} f(s)\Delta s - \int_a^{t_2} f(s)\Delta s \right| \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= \left|\int_{t_1}^{t_2} f(s)\Delta s\right| \\
&\le \left|\int_{t_1}^{t_2} |f(s)|\Delta s\right| \\
&\le B|t_2 - t_1| \\
&< \varepsilon.
\end{aligned}
$$

□

Proof.

Therefore, f is uniformly continuous on $[a, b]$. Let t_0 be right-scattered. Since f is continuous on $[a, b]$, it is Δ-differentiable at t_0. Hence,

$$\begin{aligned} F^{\Delta}(t_0) &= \frac{F(\sigma(t_0)) - F(t_0)}{\sigma(t_0) - t_0} \\ &= \frac{1}{\sigma(t_0) - t_0}\left[\int_a^{\sigma(t_0)} f(s)\Delta s - \int_a^{t_0} f(s)\Delta s\right] \\ &= \frac{1}{\sigma(t_0) - t_0}\int_{t_0}^{\sigma(t_0)} f(s)\Delta s \\ &= \frac{1}{\sigma(t_0) - t_0}(\sigma(t_0) - t_0)f(t_0) \\ &= f(t_0). \end{aligned}$$

□

Proof.

Let t_0 be right-dense and assume that f is continuous at t_0. Then

$$\begin{aligned} F^{\Delta}(t_0) &= \lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0} \\ &= \lim_{t \to t_0} \frac{1}{t - t_0} \left[\int_a^t f(s)\Delta s - \int_a^{t_0} f(s)\Delta s \right] \qquad (2) \\ &= \lim_{t \to t_0} \frac{1}{t - t_0} \int_{t_0}^t f(s)\Delta s. \end{aligned}$$

□

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is continuous at t_0, there exists $\delta > 0$ such that $s \in [a, b)$ and $|s - t_0| < \delta$ imply $|f(s) - f(t_0)| < \varepsilon$. Then

$$\begin{aligned}
\left| \frac{1}{t - t_0} \int_{t_0}^{t} f(s)\Delta s - f(t_0) \right| &= \left| \frac{1}{t - t_0} \int_{t_0}^{t} (f(s) - f(t_0))\Delta s \right| \\
&\leq \frac{1}{|t - t_0|} \left| \int_{t_0}^{t} |f(s) - f(t_0)\Delta s \right| \\
&< \frac{\varepsilon}{|t - t_0|} \left| \int_{t_0}^{t} \Delta s \right| \\
&= \varepsilon
\end{aligned}$$

□

Proof.

for all $t \in [a, b]$ such that

$$|t - t_0| < \delta$$

and

$$t \neq t_0.$$

Thus, using (2), we get

$$F^{\Delta}(t_0) = f(t_0),$$

which completes the proof. □