Time Scales Analysis
Lecture 14

Mean Value Theorems for Riemann Delta Integrals. Elementary Functions on Time Scales

Khaled Zennir

November 5, 2025

Theorem (Mean Value Theorem)

Let f and g be defined on $[a, b]$. Suppose f and g are Δ-integrable on $[a, b)$ and

$$m \leq f(t) \leq M, \quad g(t) \geq 0$$

for all $t \in [a, b]$. Then there exists $\lambda \in [m, M]$ such that

$$\int_a^b f(t)g(t)\Delta t = \lambda \int_a^b g(t)\Delta t.$$

Proof.

Since $g(t) \geq 0$ and $m \leq f(t) \leq M$ for all $t \in [a, b]$, we have

$$mg(t) \leq f(t)g(t) \leq Mg(t) \quad \text{for all} \quad t \in [a, b].$$

Hence, using Theorem **??**, we obtain

$$\int_a^b mg(t)\Delta t \leq \int_a^b f(t)g(t)\Delta t \leq \int_a^b Mg(t)\Delta t,$$

i.e.,

$$m\int_a^b g(t)\Delta t \leq \int_a^b f(t)g(t)\Delta t \leq M\int_a^b g(t)\Delta t. \tag{1}$$

$\square$

Proof.

If $\int_a^b g(t)\Delta t = 0$, then the assertion is valid.
If $\int_a^b g(t)\Delta t \neq 0$, then $\int_a^b g(t)\Delta t > 0$, and, using (1), we obtain

$$m \leq \frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g(t)\Delta t} \leq M.$$

Then there exists $\lambda \in [m, M]$ such that

$$\lambda = \frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g(t)\Delta t},$$

which completes the proof. $\square$

Lemma (Abel's Lemma)

Suppose that the numbers p_i, $1 \leq i \leq n$, satisfy the inequalities

$$p_1 \geq p_2 \geq \ldots \geq p_n \geq 0$$

and the numbers $S_k = \sum_{i=1}^{k} q_i$, $1 \leq k \leq n$, satisfy the inequalities

$$m \leq S_k \leq M \quad \text{for all} \quad k \in \{1, \ldots, n\},$$

where q_i, $i \in \{1, \ldots, n\}$, m and M are some numbers. Then

$$mp_1 \leq \sum_{i=1}^{n} p_i q_i \leq Mp_1.$$

Proof.

Let $l \in \{1, \ldots, n\}$ and j_r, $r \in \{1, \ldots, n\}$, be chosen so that

$$j_1, \ldots, j_l \in \{1, \ldots, n\} \quad \text{and} \quad q_{j_i} \geq 0 \quad \text{for} \quad i \in \{1, \ldots, l\}$$

and

$$j_{l+1}, \ldots, j_n \in \{1, \ldots, n\} \quad \text{and} \quad q_{j_i} \leq 0 \quad \text{for} \quad i \in \{l+1, \ldots, n\}.$$

Then

$$\sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{l} p_{j_i} q_{j_i} + \sum_{i=l+1}^{n} p_{j_i} q_{j_i}.$$

We have $\square$

Proof.

$$\sum_{i=1}^{l} p_{j_i} q_{j_i} \leq p_1 \sum_{i=1}^{l} q_{j_i} \quad \text{and} \quad \sum_{i=l+1}^{n} p_{j_i} q_{j_i} \leq p_n \sum_{i=l+1}^{n} q_{j_i}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{n} p_i q_i &\leq p_1 \sum_{i=1}^{l} q_{j_i} + p_n \sum_{i=l+1}^{n} q_{j_i} \\
&= (p_1 - p_n) \sum_{i=1}^{l} q_{j_i} + p_n \sum_{i=1}^{n} q_i \\
&\leq M(p_1 - p_n) + M p_n \\
&= M p_1.
\end{aligned}$$

$\square$

Proof.

Also,

$$\sum_{i=1}^{l} p_{j_i} q_{j_i} \geq p_n \sum_{i=1}^{l} q_{j_i} \quad \text{and} \quad \sum_{i=l+1}^{n} p_{j_i} q_{j_i} \geq p_1 \sum_{i=l+1}^{n} q_{j_i}.$$

Hence,

$$\begin{aligned} \sum_{i=1}^{n} p_i q_i &\geq p_n \sum_{i=1}^{l} q_{j_i} + p_1 \sum_{i=l+1}^{n} q_{j_i} \\ &= p_n \sum_{i=1}^{n} q_i + (p_1 - p_n) \sum_{i=l+1}^{n} q_{j_i} \\ &\geq m p_n + (p_1 - p_n) m \\ &= m p_1, \end{aligned}$$

which completes the proof. $\square$

Theorem (Second Mean Value Theorem)

Let f be a bounded function that is integrable on $[a, b]$. Let m_F and M_F be the infimum and supremum, respectively, of the function $F(t) = \int_a^t f(s)\Delta s$ on $[a, b]$. Then

1. *if a function g is nonincreasing with $g(t) \geq 0$ on $[a, b]$, then there exists a number Λ such that*

$$m_F \leq \Lambda \leq M_F \quad \text{and} \quad \int_a^b f(t)g(t)\Delta t = g(a)\Lambda,$$

2. *if g is any monotone function on $[a, b]$, then there exists some number Λ such that $m_F \leq \Lambda \leq M_F$ and*

$$\int_a^b f(t)g(t)\Delta t = (g(a) - g(b))\Lambda + g(b)\int_a^b f(t)\Delta t.$$

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Note that the functions f and fg are integrable on $[a, b]$. Then there exists a partition

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\} \in \mathcal{P}([a, b])$$

such that

$$\sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \varepsilon$$

and

$$\left| \sum_{i=1}^{n} f(t_{i-1}) g(t_{i-1})(t_i - t_{i-1}) - \int_a^b f(t) g(t) \Delta t \right| < \varepsilon, \tag{2}$$

where m_i and M_i are the infimum and supremum, respectively, of f on $[t_{i-1}, t_i)$. $\square$

Proof.

We have

$$\begin{aligned}\sum_{i=1}^{n} m_i g(t_{i-1})(t_i - t_{i-1}) &\leq \sum_{i=1}^{n} f(t_{i-1})g(t_{i-1})(t_i - t_{i-1}) \\ &\leq \sum_{i=1}^{n} M_i g(t_{i-1})(t_i - t_{i-1}).\end{aligned}$$

From Theorem 1, it follows that there exist numbers Λ_i, $1 \leq i \leq n$, such that

$$m_i \leq \Lambda_i \leq M_i \quad \text{and} \quad \int_{t_{i-1}}^{t_i} f(t)\Delta t = \Lambda_i(t_i - t_{i-1}).$$

□

Proof.

Consider the numbers

$$S_k = \sum_{i=1}^{k} \Lambda_i (t_i - t_{i-1}) = \int_a^{t_k} f(t)\Delta t, \quad 1 \le k \le n.$$

We have that

$$m_F \le S_k \le M_F, \quad 1 \le k \le n.$$

Let

$$p_i = g(t_{i-1}) \quad \text{and} \quad q_i = \Lambda_i (t_i - t_{i-1}), \quad 1 \le i \le n.$$

Since g is nonincreasing and $g(t) \ge 0$, we have that

$$p_1 \ge p_2 \ge \ldots \ge p_n \ge 0.$$

Therefore, using Lemma 2, we get

$$m_F g(a) \le \sum_{i=1}^{n} g(t_{i-1}) \Lambda_i (t_i - t_{i-1}) \le M_F g(a). \tag{3}$$

Proof.

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{n} m_i g(t_{i-1})(t_i - t_{i-1}) &\leq \sum_{i=1}^{n} g(t_{i-1})\Lambda_i(t_i - t_{i-1}) \\ &\leq \sum_{i=1}^{n} M_i g(t_{i-1})(t_i - t_{i-1}).\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\sum_{i=1}^{n} g(t_{i-1})(f(t_{i-1}) - \Lambda_i)(t_i - t_{i-1})\right| &\leq \sum_{i=1}^{n}(M_i - m_i)g(t_{i-1})(t_i - t_{i-1}) \\ &\leq g(a)\sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1}) \\ &\leq g(a)\varepsilon.\end{aligned}$$

Proof.

From here and from (2), we get

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)\Delta t - \sum_{i=1}^{n} g(t_{i-1})\Lambda_i(t_i - t_{i-1})\right| \\
&= \Big|\int_a^b f(t)g(t)\Delta t - \sum_{i=1}^{n} f(t_{i-1})g(t_{i-1})(t_i - t_{i-1}) \\
&\quad + \sum_{i=1}^{n} g(t_{i-1})(f(t_{i-1}) - \Lambda_i)(t_i - t_{i-1})\Big| \\
&\leq \left|\int_a^b f(t)g(t)\Delta t - \sum_{i=1}^{n} f(t_{i-1})g(t_{i-1})(t_i - t_{i-1})\right| \\
&\quad + \left|\sum_{i=1}^{n} g(t_{i-1})(f(t_{i-1}) - \Lambda_i)(t_i - t_{i-1})\right| \\
&\leq \varepsilon + g(a)\varepsilon
\end{aligned}
$$

Proof.

whereupon

$$
\begin{aligned}
-\varepsilon - g(a)\varepsilon + \sum_{i=1}^{n} g(t_{i-1})\Lambda_i(t_i - t_{i-1}) &< \int_a^b f(t)g(t)\Delta t \\
&< \varepsilon + g(a)\varepsilon + \sum_{i=1}^{n} g(t_{i-1})\Lambda_i(t_i - t_i
\end{aligned}
$$

Thus, using (3), we get

$$
-\varepsilon - g(a)\varepsilon + m_F g(a) < \int_a^b f(t)g(t)\Delta t < \varepsilon + \varepsilon g(a) + M_F g(a).
$$

□

Proof.

Since $\varepsilon > 0$ was arbitrarily chosen, we obtain

$$m_F g(a) \leq \int_a^b f(t)g(t)\Delta t \leq M_F g(a).$$

If $g(a) = 0$, then we get that $\int_a^b f(t)g(t)\Delta t = 0$, which completes the proof.

Suppose $g(a) > 0$. Then

$$m_F \leq \frac{\int_a^b f(t)g(t)\Delta t}{g(a)} \leq M_F.$$

From here, there exists $\Lambda \in [m_F, M_F]$ such thatf

$$\Lambda = \frac{\int_a^b f(t)g(t)\Delta t}{g(a)},$$

which completes the proof. □

Proof.

Let g be an arbitrary nonincreasing function on $[a, b]$ and define $h(t) = g(t) - g(b)$, $t \in [a, b]$. We have that $h(t) \geq 0$ and h is nonincreasing on $[a, b]$. Then there exists $\Lambda \in [m_F, M_F]$ such that

$$\int_a^b f(t)h(t)\Delta t = h(a)\Lambda,$$

whereupon

$$\int_a^b f(t)(g(t) - g(b))\Delta t = (g(a) - g(b))\Lambda,$$

i.e.,

$$\int_a^b f(t)g(t)\Delta t = (g(a) - g(b))\Lambda + g(b)\int_a^b f(t)\Delta t.$$

The proof is complete. □

As above, one can prove the following theorem.

Theorem (Second Mean Value Theorem)

Let f be a bounded function that is integrable on $[a, b]$. Let m_F and M_F be the infimum and supremum, respectively, of the function $F(t) = \int_t^b f(s)\Delta s$ on $[a, b]$. Then

1. *if a function g is nondecreasing with $g(t) \geq 0$ on $[a, b]$, then there exists a number Λ such that*

$$m_F \leq \Lambda \leq M_F \quad \text{and} \quad \int_a^b f(t)g(t)\Delta t = g(b)\Lambda,$$

2. *if g is any monotone function on $[a, b]$, then there exists some number Λ such that $m_F \leq \Lambda \leq M_F$ and*

$$\int_a^b f(t)g(t)\Delta t = (g(b) - g(a))\Lambda + g(a)\int_a^b f(t)\Delta t.$$

Definition

Let $h > 0$.

1. The *Hilger complex numbers* are defined by

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : \; z \neq -\frac{1}{h} \right\}.$$

2. The *Hilger real axis* is defined as

$$\mathbb{R}_h = \left\{ z \in \mathbb{C} : \; z > -\frac{1}{h} \right\}.$$

3. The *Hilger alternative axis* is defined as

$$A_h = \left\{ z \in \mathbb{C} : \; z < -\frac{1}{h} \right\}.$$

Definition

1. The *Hilger imaginary circle* is defined by

$$\mathbb{I}_h = \left\{ z \in \mathbb{C} : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\}.$$

For $h = 0$, we set

$$\mathbb{C}_0 = \mathbb{C}, \quad \mathbb{R}_0 = \mathbb{R}, \quad \mathbb{A}_0 = \emptyset, \quad \mathbb{I}_0 = i\mathbb{R}.$$

Definition

Let $h > 0$ and $z \in \mathbb{C}_h$. We define the *Hilger real part* of z by

$$\mathrm{Re}_h(z) = \frac{|zh + 1| - 1}{h}$$

and the *Hilger imaginary part* by

$$\mathrm{Im}_h(z) = \frac{\mathrm{Arg}(zh + 1)}{h},$$

where $\mathrm{Arg}(z)$ denotes the principal argument of z, i.e.,

$$-\pi < \mathrm{Arg}(z) \leq \pi.$$

We note that

$$-\frac{1}{h} < \mathrm{Re}_h(z) < \infty \quad \text{and} \quad -\frac{\pi}{h} < \mathrm{Im}_h(z) < \frac{\pi}{h}.$$

In particular, $\mathrm{Re}_h(z) \in \mathbb{R}_h$.

Definition

Let $-\frac{\pi}{h} < w \leq \frac{\pi}{h}$. We define the *Hilger purely imaginary number* $\overset{\circ}{\iota}$ by

$$\overset{\circ}{\iota} w = \frac{e^{iwh} - 1}{h}.$$

Theorem

Let $z \in \mathbb{C}_h$. Then $\overset{\circ}{\iota}\, \mathrm{Im}_h(z) \in \mathbb{I}_h$.

Proof.

We have

$$\overset{\circ}{\iota}\,\mathrm{Im}_h(z) = \frac{e^{iw\,\mathrm{Im}_h(z)} - 1}{h}$$

and

$$\begin{aligned}
\left|\overset{\circ}{\iota}\,\mathrm{Im}_h(z) + \frac{1}{h}\right| &= \left|\frac{e^{iw\,\mathrm{Im}_h(z)} - 1}{h} + \frac{1}{h}\right| \\
&= \frac{\left|e^{iw\,\mathrm{Im}_h(z)}\right|}{h} \\
&= \frac{1}{h},
\end{aligned}$$

completing the proof. □

Theorem

We have

$$\lim_{h\to 0}[\mathrm{Re}_h(z) + \mathring{\iota}\,\mathrm{Im}_h(z)] = \mathrm{Re}(z) + i\,\mathrm{Im}(z).$$

Proof.

We have

$$\begin{aligned}
z &= \mathrm{Re}(z) + \mathrm{Im}(z),\\
zh+1 &= (\mathrm{Re}(z) + i\,\mathrm{Im}(z))h + 1\\
&= h\,\mathrm{Re}(z) + 1 + ih\,\mathrm{Im}(z),\\
\mathrm{Arg}(zh+1) &= \arcsin\frac{h\,\mathrm{Im}(z)}{\sqrt{(h\,\mathrm{Re}(z)+1)^2 + h^2\,\mathrm{Im}^2(z)}},
\end{aligned}$$

□

Proof.

$$\begin{aligned}
\mathrm{Im}_h(z) &= \frac{\mathrm{Arg}(zh+1)}{h} \\
&= \frac{1}{h}\arcsin\frac{h\,\mathrm{Im}(z)}{\sqrt{(h\,\mathrm{Re}(z)+1)^2+h^2\,\mathrm{Im}^2(z)}}, \\
|zh+1| &= \sqrt{(h\,\mathrm{Re}(z)+1)^2+h^2\,\mathrm{Im}^2(z)}, \\
\mathrm{Re}_h(z) &= \frac{|zh+1|-1}{h} \\
&= \frac{\sqrt{(h\,\mathrm{Re}(z)+1)^2+h^2\,\mathrm{Im}^2(z)}-1}{h}.
\end{aligned}$$

Hence, □

Proof.

$$
\begin{aligned}
\lim_{h\to 0} \mathrm{Re}_h(z) &= \lim_{h\to 0} \frac{\sqrt{(h\,\mathrm{Re}(z)+1)^2 + h^2\,\mathrm{Im}^2(z)} - 1}{h} \\
&= \lim_{h\to 0} \frac{(h\,\mathrm{Re}(z)+1)\,\mathrm{Re}(z) + h\,\mathrm{Im}^2(z)}{\sqrt{(h\,\mathrm{Re}(z)+1)^2 + h^2\,\mathrm{Im}^2(z)}} \\
&= \mathrm{Re}(z)
\end{aligned}
$$

and

$$
\lim_{h\to 0} \mathrm{Im}_h(z) = \lim_{h\to 0} \frac{1}{h} \arcsin \frac{h\,\mathrm{Im}(z)}{\sqrt{(h\,\mathrm{Re}(z)+1)^2 + h^2\,\mathrm{Im}^2(z)}}
$$

$\square$

Proof.

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{1}{\sqrt{1-\frac{h^2\operatorname{Im}^2(z)}{(h\operatorname{Re}(z)+1)^2+h^2\operatorname{Im}^2(z)}}} \\
&\quad\times \frac{\operatorname{Im}(z)\sqrt{(h\operatorname{Re}(z)+1)^2+h^2\operatorname{Im}^2(z)} - h\operatorname{Im}(z)\frac{(h\operatorname{Re}(z)+1)\operatorname{Re}(z)+h\operatorname{Im}^2(z)}{\sqrt{(h\operatorname{Re}(z)+1)^2+h^2\operatorname{Im}^2(z)}}}{(h\operatorname{Re}(z)+1)^2+h^2\operatorname{Im}^2(z)} \\
&= \operatorname{Im}(z),
\end{aligned}
$$

which completes the proof. □

Theorem

Let $-\frac{\pi}{h} < w \leq \frac{\pi}{h}$. *Then*

$$|\mathring{\iota} w|^2 = \frac{4}{h^2} \sin^2 \frac{wh}{2}.$$

Proof.

We have

$$\mathring{\iota} w = \frac{e^{iwh} - 1}{h} = \frac{\cos(wh) - 1 + i \sin(wh)}{h}.$$

Hence,

$$\begin{aligned} |\mathring{\iota} w|^2 &= \frac{(\cos(wh) - 1)^2}{h^2} + \frac{\sin^2(wh)}{h^2} \\ &= \frac{\cos^2(wh) - 2\cos(wh) + 1 + \sin^2(wh)}{h^2} \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= \frac{2(1-\cos(wh))}{h^2} \\
&= \frac{4}{h^2}\sin^2\frac{wh}{2},
\end{aligned}
$$

cmpleting the proof. □

Definition

The *circle plus addition* $\oplus$ on $\mathbb{C}_h$ is defined by

$$z \oplus w = z + w + zwh.$$

Theorem

$(\mathbb{C}_h, \oplus)$ is an Abelian group.

Proof.

Let $z, w \in \mathbb{C}_h$. Then $z, w \in \mathbb{C}$ and $z, w \neq -\frac{1}{h}$. Therefore, $z \oplus w \in \mathbb{C}$. Since

$$\begin{aligned} h(z \oplus w) + 1 &= h(z + w + zwh) + 1 = 1 + hz + hw + zwh^2 \\ &= 1 + hz + hw(1 + hz) = (1 + hw)(1 + hz) \neq 0, \end{aligned}$$

we conclude that $z \oplus w \in \mathbb{C}_h$. Also,

$$0 \oplus z = z \oplus 0 = z,$$

i.e., 0 is the additive identity for $\oplus$. □

Proof.

For $z \in \mathbb{C}_h$, we have

$$\begin{aligned} z \oplus \left(-\frac{z}{1+zh}\right) &= z - \frac{z}{1+zh} - z\frac{z}{1+zh}h \\ &= \frac{z^2 h}{1+zh} - \frac{z^2 h}{1+zh} \\ &= 0, \end{aligned}$$

i.e., the additive inverse of z under the addition $\oplus$ is $-\frac{z}{1+zh}$. We note that $\square$

Proof.

$$-\frac{z}{1+zh} \in \mathbb{C}$$

and

$$1 - \frac{zh}{1+zh} = \frac{1}{1+zh} \neq 0,$$

i.e., $-\frac{z}{1+zh} \neq -\frac{1}{h}$. Therefore, $-\frac{z}{1+zh} \in \mathbb{C}_h$. For $z, w, v \in \mathbb{C}_h$, we have

$$\begin{aligned}
(z \oplus w) \oplus v &= (z + w + zwh) \oplus v \\
&= z + w + zwh + v + (z + w + zwh)vh \\
&= z + w + zwh + v + zvh + wvh + zwvh^2
\end{aligned}$$

$\square$

Proof.

and

$$\begin{aligned} z \oplus (w \oplus v) &= z + (w \oplus v) + z(w \oplus v)h \\ &= z + w + v + wvh + z(w + v + wvh)h \\ &= z + w + v + wvh + zwh + zvh + zwvh^2. \end{aligned}$$

Consequently,

$$z \oplus (w \oplus v) = (z \oplus w) \oplus v,$$

i.e., the associative law holds in $(\mathbb{C}_h, \oplus)$. □

Proof.

For $z, w \in \mathbb{C}_h$, we have

$$\begin{aligned} z \oplus w &= z + w + zwh \\ &= w + z + wzh \\ &= w \oplus z, \end{aligned}$$

which completes the proof. $\square$

Example

Let $z \in \mathbb{C}_h$ and $w \in \mathbb{C}$ be such that $z + w \in \mathbb{C}_h$. We will simplify the expression

$$A = z \oplus \frac{w}{1 + hz}.$$

We have

$$\begin{aligned} A &= z + \frac{w}{1 + hz} + \frac{zw}{a + hz} h \\ &= z + \frac{(1 + hz)w}{1 + hz} \\ &= z + w. \end{aligned}$$

Theorem

For $z \in \mathbb{C}_h$, we have

$$z = \mathrm{Re}_h(z) \oplus \mathring{\iota}\,\mathrm{Im}_h(z).$$

Proof.

We have

$$\begin{aligned} \mathrm{Re}_h(z) \oplus \mathring{\iota}\,\mathrm{Im}_h(z) &= \frac{|zh+1|-1}{h} \oplus \mathring{\iota}\frac{\mathrm{Arg}(zh+1)}{h} \\ &= \frac{|zh+1|-1}{h} \oplus \frac{e^{i\,\mathrm{Arg}(zh+1)}-1}{h} \\ &= \frac{|zh+1|-1}{h} + \frac{e^{i\,\mathrm{Arg}(zh+1)}-1}{h} \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
& +\frac{|zh+1|-1}{h}\frac{e^{i\,\mathrm{Arg}(zh+1)}-1}{h}h \\
= & \ \frac{1}{h}\Big(|zh+1|-1+e^{i\,\mathrm{Arg}(zh+1)}-1 \\
& +|zh+1|e^{i\,\mathrm{Arg}(zh+1)} \\
& -|zh+1|-e^{i\,\mathrm{Arg}(zh+1)}+1\Big) \\
= & \ \frac{1}{h}\left(|zh+1|e^{i\,\mathrm{Arg}(zh+1)}-1\right) \\
= & \ \frac{1}{h}(zh+1-1)=z,
\end{aligned}
$$

completing the proof. □

Definition

Let $n \in \mathbb{N}$ and $z \in \mathbb{C}_h$. We define the *circle dot multiplication* $\odot$ by

$$n \odot z = z \oplus z \oplus \cdots \oplus z.$$

Theorem

Let $n \in \mathbb{N}$ and $z \in \mathbb{C}_h$. Then

$$n \odot z = \frac{(zh+1)^n - 1}{h}. \tag{4}$$

Proof.

Let $n = 2$. Then

$$\begin{aligned}
2 \odot z &= z \oplus z \\
&= z + z + z^2 h \\
&= 2z + zh \\
&= \frac{1}{h}(z^2 h^2 + 2zh) \\
&= \frac{1}{h}(z^2 h^2 + 2zh + 1 - 1) \\
&= \frac{(zh+1)^2 - 1}{h}.
\end{aligned}$$

$\square$

Proof.

Assume

$$n \odot z = \frac{(zh+1)^n - 1}{h}$$

for some $n \in \mathbb{N}$. We will prove that

$$(n+1) \odot z = \frac{(zh+1)^{n+1} - 1}{h}.$$

Indeed,

$$\begin{aligned}
(n+1) \odot z &= (n \odot z) \oplus z \\
&= \frac{(zh+1)^n - 1}{h} \oplus z
\end{aligned}$$

□

Proof.

$$
\begin{aligned}
&= \frac{(zh+1)^n - 1}{h} + z + \frac{(zh+1)^n - 1}{h} zh \\
&= \frac{(zh+1)^n - 1 + zh + zh(zh+1)^n - zh}{h} \\
&= \frac{(zh+1)^{n+1} - 1}{h}.
\end{aligned}
$$

Hence, we conclude that (4) holds for all $n \in \mathbb{N}$. $\square$

Definition

Let $z \in \mathbb{C}_h$. We define the *circle minus* of z as

$$\ominus z = \frac{-z}{1 + zh}.$$

Theorem

Let $z \in \mathbb{C}_h$. Then $\ominus z$ is the additive inverse of z under the operation $\oplus$, i.e.,

$$\ominus(\ominus z) = z.$$

Proof.

We have

$$\begin{aligned} \ominus(\ominus z) &= -\frac{\ominus z}{1+(\ominus z)h} \\ &= -\frac{\frac{-z}{1+zh}}{1+\frac{-z}{1+zh}h} \\ &= \frac{\frac{z}{1+zh}}{\frac{1+zh-zh}{1+zh}} = z, \end{aligned}$$

cmpleting the proof. □

Definition

Let $z, w \in \mathbb{C}_h$. We define the *circle minus subtraction* by

$$z \ominus w = z \oplus (\ominus w).$$

Remark

For $z, w \in \mathbb{C}_h$, we have

$$\begin{aligned} z \ominus w &= z \oplus (\ominus w) \\ &= z + (\ominus w) + z(\ominus w)h \\ &= z - \frac{w}{1+wh} - \frac{zwh}{1+wh} \\ &= \frac{z + zwh - w - zwh}{1+wh} = \frac{z-w}{1+wh}, \end{aligned}$$

Remark

i.e.,

$$z \ominus w = \frac{z - w}{1 + wh}. \tag{5}$$

Theorem

Let $z \in \mathbb{C}_h$. Then $\overline{z} = \ominus z$ iff $z \in \mathbb{I}_h$.

Proof.

By (5), we have

$$\overline{z} = \ominus z,$$

i.e.,

$$\overline{z} = -\frac{z}{1+zh},$$

i.e.,

$$\overline{z} + \overline{z}zh = -z,$$

i.e.,

$$2\,\mathrm{Re}(z) + |z|^2 h = 0.$$

Also,

$$\left| z + \frac{1}{h} \right| = \frac{1}{h},$$

i.e.,

$\square$

Proof.

$$\left|z+\frac{1}{h}\right|^2=\frac{1}{h^2},$$

i.e.,

$$\left(\mathrm{Re}(z)+\frac{1}{h}\right)^2+\mathrm{Im}^2(z)=\frac{1}{h^2},$$

i.e.,

$$\mathrm{Re}^2(z)+\frac{2}{h}\mathrm{Re}(z)+\frac{1}{h^2}+\mathrm{Im}^2(z)=\frac{1}{h^2},$$

i.e.,

$$|z|^2+\frac{2}{h}\mathrm{Re}(z)=0,$$

, i.e.,

$$2\,\mathrm{Re}(z)+|z|^2h=0,$$

which completes the proof. □

Theorem

Let $-\frac{\pi}{h} < w \leq \frac{\pi}{h}$. *Then*

$$\ominus(\overset{\circ}{\iota} w) = \overline{\overset{\circ}{\iota} w}$$

Proof.

We have

$$\begin{aligned}
\ominus(\overset{\circ}{\iota} w) &= -\frac{\overset{\circ}{\iota} w}{1 + \overset{\circ}{\iota} wh} \\
&= -\frac{\frac{e^{iwh}-1}{h}}{1 + \frac{e^{iwh}-1}{h} h} \\
&= -\frac{e^{iwh} - 1}{he^{iwh}} = \frac{e^{-iwh} - 1}{h} = \overline{\overset{\circ}{\iota} w},
\end{aligned}$$

completing the proof. □

Definition

Let $z \in \mathbb{C}_h$. The *circle square* of z is defined by

$$z^{②} = (-z)(\ominus z).$$

Remark

We have

$$z^{②} = -z\frac{-z}{1+zh} = \frac{z^2}{1+zh}.$$

Theorem

For $z \in \mathbb{C}_h$, we have

$$(\ominus z)^{②} = z^{②}.$$

Proof.

We have

$$\begin{aligned} (\ominus z)^{②} &= -(\ominus z)(\ominus(\ominus z)) \\ &= \frac{z}{1+zh} z \\ &= \frac{z^2}{1+zh} \\ &= z^{②}, \end{aligned}$$

completing the proof. □

Theorem

For $z \in \mathbb{C}_h$, we have

$$1 + zh = \frac{z^2}{z^{②}}.$$

Proof.

We have

$$\begin{aligned} \frac{z^2}{z^{②}} &= \frac{z^2}{\frac{z^2}{1+zh}} \\ &= 1 + zh, \end{aligned}$$

completing the proof. $\square$

Theorem

For $z \in \mathbb{C}_h$, we have

$$z + (\ominus z) = z^{②} h.$$

Proof.

We have

$$z^{②} h = \frac{z^2}{1 + zh} h$$

and

$$\begin{aligned} z + (\ominus z) &= z - \frac{z}{1+zh} \\ &= \frac{z^2 h}{1+zh}, \end{aligned}$$

which completes the proof. $\square$

Theorem

For $z \in \mathbb{C}_h$, we have

$$z \oplus z^{②} = z + z^2.$$

Proof.

We have

$$\begin{aligned} z \oplus z^{②} &= z + z^{②} + zz^{②}h \\ &= z + \frac{z^2}{1+zh} + \frac{z^3h}{1+zh} \\ &= z + \frac{z^2(1+zh)}{1+zh} \\ &= z + z^2, \end{aligned}$$

completing the proof. □

Theorem

Let $-\frac{\pi}{h} < w \leq \frac{\pi}{h}$. *Then*

$$-(\overset{\circ}{\iota} w)^{②} = \frac{4}{h^2} \sin^2\left(\frac{wh}{2}\right).$$

Proof.

We have

$$\begin{aligned} -(\overset{\circ}{\iota} w)^{②} &= -(\overset{\circ}{\iota} w)(\ominus \overset{\circ}{\iota} w) \\ &= (\overset{\circ}{\iota} w)\overline{\overset{\circ}{\iota} w} \\ &= |\overset{\circ}{\iota} w|^2 \\ &= \frac{4}{h^2} \sin^2\left(\frac{wh}{2}\right), \end{aligned}$$

For $h > 0$, we define the strip

$$\mathbb{Z}_h = \{z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h}\}.$$

For $h = 0$, we set $\mathbb{Z}_0 = \mathbb{C}$.

Definition

For $h > 0$, we define the *cylinde transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h} \mathrm{Log}(1 + zh),$$

where Log is the principal logarithm function. Moreover, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Remark

We note that

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h}$$

for $z \in \mathbb{Z}_h$.

Definition

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}(\mathbb{T})$ or $\mathcal{R}$.

Definition

In $\mathcal{R}$, we define the *circle plus addition* by

$$f \oplus g = f + g + \mu fg, \quad f, g \in \mathcal{R}.$$

Definition

The group $(\mathcal{R}, \oplus)$ is called the *regressive group*.

Definition

For $f \in \mathcal{R}$, we define the *circle minus* by

$$\ominus f = -\frac{f}{1+\mu f}.$$

Definition

We define the *circle minus subtraction* $\ominus$ on $\mathcal{R}$ by

$$f \ominus g = f \oplus (\ominus g), \quad f, g \in \mathcal{R}.$$

Remark

For $f, g \in \mathcal{R}$, we have

$$\begin{aligned} f \ominus g &= f \oplus (\ominus g) \\ &= f \oplus \left(-\frac{g}{1+\mu g}\right) \\ &= f - \frac{g}{1+\mu g} - \frac{\mu f g}{1+\mu g} \\ &= \frac{f-g}{1+\mu g}. \end{aligned}$$

Theorem

Let $f, g \in \mathcal{R}$. Then

1. $f \ominus f = 0$,
2. $\ominus(\ominus f) = f$,
3. $f \ominus g \in \mathcal{R}$,
4. $\ominus(f \ominus g) = g \ominus f$,
5. $\ominus(f \oplus g) = (\ominus f) \oplus (\ominus g)$,
6. $f \oplus \frac{g}{1+\mu f} = f + g$.

Proof.

We have

$$\begin{aligned} f \ominus f &= f \oplus (\ominus f) \\ &= f \oplus \left(-\frac{f}{1+\mu f}\right) \\ &= f - \frac{f}{1+\mu f} - \frac{f^2\mu}{1+\mu f} \\ &= \frac{f+\mu f^2 - f - \mu f^2}{1+\mu f} \\ &= 0. \end{aligned}$$

$\square$

Proof.

We have

$$\begin{aligned}
\ominus(\ominus f) &= \ominus\left(-\frac{f}{1+\mu f}\right) \\
&= \frac{\frac{f}{1+\mu f}}{1-\frac{\mu f}{1+\mu f}} \\
&= f.
\end{aligned}$$

$\square$

Proof.

We have

$$\begin{aligned} 1+\mu f\ominus g &= 1+\frac{\mu f-\mu g}{1+\mu g} \\ &= \frac{1+\mu f}{1+\mu g}\neq 0. \end{aligned}$$

We note that $\frac{f-g}{1+\mu g}$ is rd-continuous. Therefore, $f\ominus g\in\mathcal{R}$. □

Proof.

We have

$$
\begin{aligned}
\ominus(f \ominus g) &= \ominus\left(\frac{f-g}{1+\mu g}\right) \\
&= -\frac{\frac{f-g}{1+\mu g}}{1+\mu\frac{f-g}{1+\mu g}} \\
&= -\frac{f-g}{1+\mu f} \\
&= \frac{g-f}{1+\mu f} \\
&= g \ominus f.
\end{aligned}
$$

$\square$

Proof.

We have

$$\begin{aligned}
\ominus(f \oplus g) &= \ominus(f + g + \mu fg) \\
&= -\frac{f + g + \mu fg}{1 + \mu f + \mu g + \mu^2 fg} \\
&= -\frac{f + g + \mu fg}{(1 + \mu f)(1 + \mu g)}.
\end{aligned}$$

Since

$$\ominus f = -\frac{f}{1 + \mu f} \quad \text{and} \quad \ominus g = -\frac{g}{1 + \mu g},$$

we also have

Proof.

$$
\begin{aligned}
(\ominus f) \oplus (\ominus g) &= \ominus f + (\ominus g) + \mu(\ominus f)(\ominus g) \\
&= -\frac{f}{1+\mu f} - \frac{g}{1+\mu g} + \frac{\mu f g}{(1+\mu f)(1+\mu g)} \\
&= \frac{-f(1+\mu g) - g(1+\mu f) + \mu f g}{(1+\mu f)(1+\mu g)} \\
&= -\frac{f+g+\mu f g}{(1+\mu f)(1+\mu g)}.
\end{aligned}
$$

$\square$

Proof.

We have

$$\begin{aligned} f \oplus \frac{g}{1+\mu f} &= f + \frac{g}{1+\mu f} + \frac{\mu f g}{1+\mu f} \\ &= f + g. \end{aligned}$$

The proof is complete. $\square$

Definition

If $f \in \mathcal{R}$, then we define the *generalized exponential function* by

$$e_f(t,s) = e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \quad \text{for} \quad s,t \in \mathbb{T}.$$

Remark

In fact, using the definition for the cylinder transformation, we have

$$e_f(t,s) = e^{\int_s^t \frac{1}{\mu(\tau)} \operatorname{Log}(1+\mu(\tau)f(\tau))\Delta\tau} \quad \textit{for} \quad s,t \in \mathbb{T}.$$

Theorem (Semigroup Property)

If $f \in \mathcal{R}$, *then*

$$e_f(t,r)e_f(r,s) = e_f(t,s) \quad \textit{for all} \quad t,r,s \in \mathbb{T}.$$

Proof.

We have

$$\begin{aligned} e_f(t,r)e_f(r,s) &= e^{\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} e^{\int_s^r \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau + \int_s^r \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} = e_f(t,s), \end{aligned}$$

completing the proof. □

Theorem

Let $f \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then

$$e_f^{\Delta}(t, t_0) = f(t) e_f(t, t_0).$$

Proof.

If $\sigma(t) > t$, then

$$\begin{aligned} e_f^{\Delta}(t, t_0) &= \frac{e_f(\sigma(t), t_0) - e_f(t, t_0)}{\mu(t)} \\ &= \frac{e^{\int_{t_0}^{\sigma(t)} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} - e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau}}{\mu(t)} \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= \frac{e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau + \int_{t}^{\sigma(t)} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} - e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau}}{\mu(t)} \\
&= \frac{e^{\int_{t_0}^{\sigma(t)} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} - 1}{\mu(t)} e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\
&= \frac{e^{\mu(t)\xi_{\mu(t)}(f(t))} - 1}{\mu(t)} e_f(t, t_0) \\
&= f(t)e_f(t, t_0).
\end{aligned}
$$

□

Proof.

If $\sigma(t) = t$, then

$$
\begin{aligned}
&|e_f(t,t_0) - e_f(s,t_0) - f(t)e_f(t,t_0)(t-s)| \\
&= |e_f(t,t_0) - e_f(t,t_0)e_f(s,t) - f(t)e_f(t,t_0)(t-s)| \\
&= |e_f(t,t_0)||1 - e_f(s,t) - f(t)(t-s)| \\
&= |e_f(t,t_0)|\Big|1 - \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)
\end{aligned}
\tag{6}
$$

$\square$

Proof.

$$+\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - f(t)(t-s)\Big|$$

$$\leq |e_f(t,t_0)|\Big(\Big|1-\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)\Big|$$

$$+\Big|\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - f(t)(t-s)\Big|\Big)$$

$$\leq |e_f(t,t_0)|\Big(\Big|1-\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)\Big|$$

$$+\Big|\int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau\Big|\Big).$$

$\square$

Proof.

Since $\sigma(t) = t$ and $f \in \mathrm{C_{rd}}$, we get

$$\lim_{r \to t} \xi_{\mu(r)}(f(r)) = \xi_0(f(t)).$$

Therefore, there exists a neighbourhood U_1 of t such that

$$|\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t))| < \frac{\varepsilon}{3|e_f(t, t_0)|} \quad \text{for all} \quad \tau \in U_1.$$

Let $s \in U_1$. Then

$$|e_f(t, t_0)| \left| \int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau \right| \leq \frac{\varepsilon}{3}|t - s|. \tag{7}$$

$\square$

Proof.

Also, using that

$$\lim_{z \to 0} \frac{1 - z - e^{-z}}{z} = 0,$$

we conclude that there exists a neighbourhood U_2 of t so that if $s \in U_2$ and $s \leq t$, then we have

$$\left| \frac{1 - \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)}{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \right| < \varepsilon^*,$$

where

$$\varepsilon^* = \min\left\{1, \frac{\varepsilon}{1 + 3|f(t)||e_f(t,t_0)|}\right\}.$$

Let $s \in U = U_1 \cap U_2$. Then

□

Proof.

$$
\begin{aligned}
& |e_f(t,t_0)| \left| 1 - \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t) \right| \\
= \quad & |e_f(t,t_0)| \frac{\left| 1 - \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t) \right|}{\left| \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau \right|} \left| \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau \right| \\
\leq \quad & |e_f(t,t_0)| \varepsilon^* \left| \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau \right|
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
&\leq |e_f(t,t_0)|\varepsilon^* \left\{ \left| \int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau \right| + |f(t)||t-s| \right\} \\
&\leq |e_f(t,t_0)| \left| \int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau \right| + |e_f(t,t_0)|\varepsilon^* |f(t)||t-s| \\
&\leq \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t-s| \\
&= \frac{2\varepsilon}{3}|t-s|.
\end{aligned}
$$

$\square$

Proof.

From the last inequality and from (6) and (7), we conclude that

$$\begin{aligned} |e_f(t,t_0) - e_f(s,t_0) - f(t)e_f(t,t_0)(t-s)| &< \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t- \\ &= \varepsilon|t-s|, \end{aligned}$$

which completes the proof. □

Corollary

Let $f \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then $e_f(\cdot, t_0)$ is a solution to the Cauchy problem

$$y^{\Delta}(t) = f(t)y(t), \quad y(t_0) = 1. \tag{8}$$

Corollary

Let $f \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then $e_f(\cdot, t_0)$ is the unique solution of the problem (8).

Proof.

Let y be any solution of the problem (8). Then

$$\begin{aligned}\left(\frac{y}{e_f(\cdot, t_0)}\right)^{\Delta}(t) &= \frac{y^{\Delta}(t)e_f(t, t_0) - y(t)e_f^{\Delta}(t, t_0)}{e_f(\sigma(t), t_0)e_f(t, t_0)} \\ &= \frac{f(t)y(t)e_f(t, t_0) - y(t)f(t)e_f(t, t_0)}{e_f(\sigma(t), t_0)e_f(t, t_0)} \\ &= 0.\end{aligned}$$

Consequently, $y = ce_f(\cdot, t_0)$, where c is a constant. Thus,

$$1 = y(t_0) = ce_f(t_0, t_0) = c.$$

Therefore, $y = e_f(\cdot, t_0)$. $\square$

Theorem

Let $f \in \mathcal{R}$. Then

$$e_f(\sigma(t), s) = (1 + \mu(t) f(t)) e_f(t, s).$$

Proof.

We have

$$\begin{aligned} e_f(\sigma(t), s) - e_f(t, s) &= \mu(t) e_f^{\Delta}(t, s) \\ &= \mu(t) f(t) e_f(t, s), \end{aligned}$$

which completes the proof. $\square$

Theorem

Let $f \in \mathcal{R}$. Then

$$e_f(t,s) = \frac{1}{e_f(s,t)} = e_{\ominus f}(s,t).$$

Proof.

We have

$$\begin{aligned} e_f(t,s) &= e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{-\int_t^s \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= \frac{1}{e^{\int_t^s \xi_{\mu(\tau)}(f(\tau))\Delta\tau}} \\ &= \frac{1}{e_f(s,t)}. \end{aligned}$$

□

Proof.

Now, we fix $t_0 \in \mathbb{T}$ and consider the problem

$$y^{\Delta}(t) = (\ominus f)(t)y(t), \quad y(t_0) = 1.$$

Its solution is $e_{\ominus f}(t,s)$. Also,

$$\begin{aligned} \left(\frac{1}{e_f(\cdot,s)}\right)^{\Delta}(t) &= -\frac{e_f^{\Delta}(t,s)}{e_f(\sigma(t),s)e_f(t,s)} \\ &= -\frac{f(t)e_f(t,s)}{(1+\mu(t)f(t))e_f(t,s)e_f(t,s)} \end{aligned}$$

□

Proof.

$$
\begin{aligned}
&= -\frac{f(t)}{(1+\mu(t)f(t))e_f(t,s)} \\
&= (\ominus f)(t)\frac{1}{e_f(t,s)}.
\end{aligned}
$$

Therefore,

$$\frac{1}{e_f(t,s)} = e_{\ominus f}(t,s),$$

completing the proof. □

Theorem

Let $f, g \in \mathcal{R}$. Then

$$e_f(t,s)e_g(t,s) = e_{f\oplus g}(t,s).$$

Proof.

We have

$$\begin{aligned} e_f(t,s)e_g(t,s) &= e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} e^{\int_s^t \xi_{\mu(\tau)}(g(\tau))\Delta\tau} \\ &= e^{\int_s^t (\xi_{\mu(\tau)}(f(\tau))+\xi_{\mu(\tau)}(g(\tau)))\Delta\tau} \end{aligned}$$

□

Proof.

$$
\begin{aligned}
&= e^{\int_s^t \frac{1}{\mu(\tau)}(\mathrm{Log}(1+\mu(\tau)f(\tau))+\mathrm{Log}(1+\mu(\tau)g(\tau)))\Delta\tau} \\
&= e^{\int_s^t \frac{1}{\mu(\tau)}\mathrm{Log}((1+\mu(\tau)f(\tau))(1+\mu(\tau)g(\tau)))\Delta\tau} \\
&= e^{\int_s^t \frac{1}{\mu(\tau)}\mathrm{Log}(1+\mu(\tau)(f(\tau)+g(\tau)+\mu(\tau)f(\tau)g(\tau)))\Delta\tau} \\
&= e^{\int_s^t \xi_{\mu(\tau)}((f\oplus g)(\tau))\Delta\tau} \\
&= e_{f\oplus g}(t,s),
\end{aligned}
$$

completing the proof. □

Theorem

Let $f, g \in \mathcal{R}$. Then

$$\frac{e_f(t,s)}{e_g(t,s)} = e_{f \ominus g}(t,s).$$

Proof.

We have

$$\begin{aligned} \frac{e_f(t,s)}{e_g(t,s)} &= e_f(t,s)e_{\ominus g}(t,s) \\ &= e_{f \oplus (\ominus g)}(t,s) \\ &= e_{f \ominus g}(t,s), \end{aligned}$$

completing the proof. $\square$

Theorem

Let $f \in \mathcal{R}$. Then

$$e_f(t,\sigma(s))e_f(s,r) = \frac{1}{1+\mu(s)f(s)}e_f(t,r).$$

Proof.

We have

$$\begin{aligned} e_f(t,\sigma(s))e_f(s,r) &= e^{\int_{\sigma(s)}^{t}\xi_{\mu(\tau)}(f(\tau))\Delta\tau}e^{\int_{r}^{s}\xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{\int_{\sigma(s)}^{s}\xi_{\mu(\tau)}(f(\tau))\Delta\tau+\int_{s}^{t}\xi_{\mu(\tau)}(f(\tau))\Delta\tau+\int_{r}^{s}\xi_{\mu(\tau)}(f(\tau))\Delta\tau} \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= e^{-\xi_{\mu(s)}(f(s))\mu(s)+\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\
&= e^{-\operatorname{Log}(1+f(s)\mu(s))} e^{\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\
&= \frac{1}{1+\mu(s)f(s)} e_f(t,r),
\end{aligned}
$$

completing the proof. □

Theorem

Let $f, g \in \mathcal{R}$. Then

$$e^{\Delta}_{f \ominus g}(t, t_0) = \frac{(f(t) - g(t))e_f(t, t_0)}{e_g(\sigma(t), t_0)}.$$

Proof.

We have

$$e^{\Delta}_{f \ominus g}(t, t_0) = \left(\frac{e_f(t, t_0)}{e_g(t, t_0)}\right)^{\Delta}$$

□

Proof.

$$
\begin{aligned}
&= \frac{e_f^{\Delta}(t,t_0)e_g(t,t_0) - e_f(t,t_0)e_g^{\Delta}(t,t_0)}{e_g(t,t_0)e_g(\sigma(t),t_0)} \\
&= \frac{f(t)e_f(t,t_0)e_g(t,t_0) - g(t)e_f(t,t_0)e_g(t,t_0)}{e_g(t,t_0)e_g(\sigma(t),t_0)} \\
&= \frac{(f(t)-g(t))e_f(t,t_0)}{e_g(\sigma(t),t_0)},
\end{aligned}
$$

completing the proof. □

Theorem

Let $f \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$. Then

$$(e_f(c, \cdot))^{\Delta}(t) = -f(t)e_f(c, \sigma(t))$$

and

$$\int_a^b f(t)e_f(c, \sigma(t))\Delta t = e_f(c, a) - e_f(c, b).$$

Proof.

We have

$$f(t)e_f(c, \sigma(t)) \quad = \quad f(t)e_{\ominus f}(\sigma(t), c)$$

$\square$

Proof.

$$
\begin{aligned}
&= f(t)(1+\mu(t)\ominus f(t))e_{\ominus f}(t,c) \\
&= f(t)\left(1-\frac{\mu(t)f(t)}{1+\mu(t)f(t)}\right)e_{\ominus f}(t,c) \\
&= \frac{f(t)}{1+\mu(t)f(t)}e_{\ominus f}(t,c) \\
&= \frac{f(t)}{1+\mu(t)f(t)}e_{\ominus f}(t,c) \\
&= -(\ominus f)(t)e_{\ominus f}(t,c) \\
&= -e^{\Delta}_{\ominus f}(t,c) \\
&= -e^{\Delta}_{f}(c,t).
\end{aligned}
$$

Proof.

Hence,

$$\begin{aligned}
\int_a^b f(t)e_f(c,\sigma(t))\Delta t &= -\int_a^b e_f^{\Delta}(c,t)\Delta t \\
&= -e_f(c,t)\Big|_{t=a}^{t=b} \\
&= e_f(c,a) - e_f(c,b),
\end{aligned}$$

which completes the proof. $\square$

Example

Assume $1+\mu(t)\frac{2}{t} \neq 0$ and $1+\mu(t)\frac{5}{t} \neq 0$ for all $t \in \mathbb{T} \cap (0,\infty)$. Let $t_0 \in \mathbb{T} \cap (0,\infty)$. We will evaluate the integral

$$I = \int_{t_0}^{t} \frac{e_{\frac{5}{s}}(s,t_0)}{s e^{\sigma}_{\frac{2}{s}}(s,t_0)} \Delta s.$$

We have

$$\begin{aligned} \left(\frac{5}{s} - \frac{2}{s}\right) \frac{e_{\frac{5}{s}}(s,t_0)}{e^{\sigma}_{\frac{2}{s}}(s,t_0)} &= \frac{3}{s} \frac{e_{\frac{5}{s}}(s,t_0)}{e^{\sigma}_{\frac{2}{s}}(s,t_0)} \\ &= e^{\Delta}_{\frac{5}{s} \ominus \frac{2}{s}}(s,t_0) \end{aligned}$$

Example

$$
\begin{aligned}
&= e^{\Delta}_{\frac{\frac{3}{s}}{1+\mu(s)\frac{2}{s}}}(s,t_0)\\
&= e^{\Delta}_{\frac{3}{s+2\mu(s)}}(s,t_0).
\end{aligned}
$$

Consequently,

$$\frac{1}{s}\frac{e_{\frac{5}{s}}(s,t_0)}{e^{\sigma}_{\frac{2}{s}}(s,t_0)} = \frac{1}{3}e^{\Delta}_{\frac{3}{s+2\mu(s)}}(s,t_0).$$

Hence,

$$
\begin{aligned}
I &= \frac{1}{3}\int_{t_0}^{t} e^{\Delta}_{\frac{3}{s+2\mu(s)}}(s,t_0)\Delta s\\
&= \frac{1}{3}e_{\frac{3}{s+2\mu(s)}}(s,t_0)\Big|_{s=t_0}^{s=t} = \frac{1}{3}e_{\frac{3}{t+2\mu(t)}}(t,t_0) - \frac{1}{3}.
\end{aligned}
$$

Example

Let $\alpha \in \mathbb{R}$. Suppose that the exponentials

$$e_{\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}}(t, t_0) \quad \text{and} \quad e_{\frac{\alpha}{t}}(t, t_0)$$

exist for all $t \in \mathbb{T} \cap (0, \infty)$. We will prove that

1. $$\frac{e_{\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}}(t, t_0)}{e_{\frac{\alpha}{t}}(t, t_0)} = e_{\frac{\alpha-1}{\sigma(t)}}(t, t_0),$$

2. $$\frac{e_{\frac{\alpha-1}{\sigma(t)}}(t, t_0)}{e_{\frac{\alpha}{t}}(t, t_0)} = e_{-\frac{1}{\sigma(t)}}(t, t_0) = \frac{t_0}{t}$$

for all $t, t_0 \in \mathbb{T} \cap (0, \infty)$.

Example

1. We have

$$\left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right) \ominus \frac{\alpha}{t} = \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right) \oplus \left(\ominus\frac{\alpha}{t}\right)$$

$$= \frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)} + \left(\ominus\frac{\alpha}{t}\right) + \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right)\left(\ominus\frac{\alpha}{t}\right)\mu(t)$$

$$= \frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)} - \frac{\frac{\alpha}{t}}{1+\frac{\alpha}{t}\mu(t)}$$

Example

$$
\begin{aligned}
& + \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)} \right) \left(- \frac{\frac{\alpha}{t}}{1 + \frac{\alpha}{t}\mu(t)} \right) \mu(t) \\
= \quad & \frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)} - \frac{\alpha}{t + \alpha\mu(t)} \\
& - \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)} \right) \frac{\alpha}{t + \alpha\mu(t)} \mu(t)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right)\left(1 - \frac{\alpha\mu(t)}{t+\alpha\mu(t)}\right) - \frac{\alpha}{t+\alpha\mu(t)} \\
&= \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right)\frac{t}{t+\alpha\mu(t)} - \frac{\alpha}{1+\alpha\mu(t)} \\
&= \frac{\alpha^2\sigma(t) - t(\alpha-1)^2}{t\sigma(t)}\frac{t}{t+\alpha\mu(t)} - \frac{\alpha}{t+\alpha\mu(t)}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{\alpha^2(t+\mu(t)) - t\alpha^2 + 2\alpha t - t}{\sigma(t)} \frac{1}{t+\alpha\mu(t)} - \frac{\alpha}{t+\alpha\mu(t)} \\
&= \frac{\alpha^2\mu(t) + 2\alpha t - t}{\sigma(t)(t+\alpha\mu(t))} - \frac{\alpha}{t+\alpha\mu(t)} \\
&= \frac{\alpha^2\mu(t) + 2\alpha t - t - \alpha\mu(t) - \alpha t}{\sigma(t)(t+\alpha\mu(t))}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{\alpha\mu(t)(\alpha-1)+(\alpha-1)t}{\sigma(t)(t+\alpha\mu(t))} \\
&= \frac{(\alpha-1)(t+\alpha\mu(t))}{\sigma(t)(t+\alpha\mu(t))} \\
&= \frac{\alpha-1}{\sigma(t)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0)}{e_{\frac{\alpha}{t}}(t,t_0)} &= e_{\left(\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}\right)\ominus\frac{\alpha}{t}}(t,t_0) \\
&= e_{\frac{\alpha-1}{t}}(t,t_0).
\end{aligned}
$$

Example

We have

$$
\begin{aligned}
\frac{\alpha-1}{\sigma(t)} \ominus \frac{\alpha}{t} &= \frac{\alpha-1}{\sigma(t)} \oplus \left(\ominus \frac{\alpha}{t}\right) \\
&= \frac{\alpha-1}{\sigma(t)} + \left(\ominus \frac{\alpha}{t}\right) + \frac{\alpha-1}{\sigma(t)} \left(\ominus \frac{\alpha}{t}\right) \mu(t) \\
&= \frac{\alpha-1}{\sigma(t)} - \frac{\frac{\alpha}{t}}{1+\mu(t)\frac{\alpha}{t}} - \frac{\alpha-1}{\sigma(t)} \frac{\frac{\alpha}{t}}{1+\mu(t)\frac{\alpha}{t}} \mu(t)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{\alpha - 1}{\sigma(t)} - \frac{\alpha}{t + \alpha\mu(t)} - \frac{\alpha - 1}{\sigma(t)} \frac{\alpha}{t + \alpha\mu(t)} \mu(t) \\
&= \frac{\alpha - 1}{\sigma(t)} \left(1 - \frac{\alpha\mu(t)}{t + \alpha\mu(t)} \right) - \frac{\alpha}{t + \alpha\mu(t)} \\
&= \frac{t(\alpha - 1)}{\sigma(t)(t + \alpha\mu(t))} - \frac{\alpha}{t + \alpha\mu(t)}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{\alpha t - t - \alpha\mu(t) - \alpha t}{\sigma(t)(t + \alpha\mu(t))} \\
&= -\frac{t + \alpha\mu(t)}{\sigma(t)(t + \alpha\mu(t))} \\
&= -\frac{1}{\sigma(t)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{e_{\frac{\alpha-1}{\sigma(t)}}(t, t_0)}{e_{\frac{\alpha}{t}}(t, t_0)} &= e_{\frac{\alpha-1}{\sigma(t)} \ominus \frac{\alpha}{t}}(t, t_0) \\
&= e_{-\frac{1}{\sigma(t)}}(t, t_0)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= e^{\int_{t_0}^{t} \xi_{\mu(\tau)}\left(-\frac{1}{\sigma(\tau)}\right)\Delta\tau} \\
&= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} \operatorname{Log}\left(1-\frac{\mu(\tau)}{\sigma(\tau)}\right)\Delta\tau} \\
&= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} \operatorname{Log} \frac{\tau}{\sigma(\tau)}\Delta\tau} \\
&= e^{-\operatorname{Log}\tau \Big|_{\tau=t_0}^{\tau=t}} \\
&= e^{-\operatorname{Log}\frac{t}{t_0}} \\
&= \frac{t_0}{t}.
\end{aligned}
$$