Time Scales Analysis
Lecture 15
Trigonometric and Hyperbolic Functions. The Taylor Formula

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Definition

If $f \in \mathcal{R}$, then we define the *generalized exponential function* by

$$e_f(t,s) = e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \quad \text{for} \quad s,t \in \mathbb{T}.$$

Remark

In fact, using the definition for the cylinder transformation, we have

$$e_f(t,s) = e^{\int_s^t \frac{1}{\mu(\tau)} \mathrm{Log}(1+\mu(\tau)f(\tau))\Delta\tau} \quad \textit{for} \quad s,t \in \mathbb{T}.$$

Theorem (Semigroup Property)

If $f \in \mathcal{R}$, *then*

$$e_f(t,r)e_f(r,s) = e_f(t,s) \quad \text{for all} \quad t,r,s \in \mathbb{T}.$$

Proof.

We have

$$\begin{aligned} e_f(t,r)e_f(r,s) &= e^{\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} e^{\int_s^r \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau + \int_s^r \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} = e_f(t,s), \end{aligned}$$

completing the proof. □

Theorem

Let $f \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then

$$e_f^{\Delta}(t, t_0) = f(t)e_f(t, t_0).$$

Proof.

If $\sigma(t) > t$, then

$$\begin{aligned} e_f^{\Delta}(t, t_0) &= \frac{e_f(\sigma(t), t_0) - e_f(t, t_0)}{\mu(t)} \\ &= \frac{e^{\int_{t_0}^{\sigma(t)} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} - e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau}}{\mu(t)} \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= \frac{e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau + \int_{t}^{\sigma(t)} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} - e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau}}{\mu(t)} \\
&= \frac{e^{\int_{t_0}^{\sigma(t)} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} - 1}{\mu(t)} e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\
&= \frac{e^{\mu(t)\xi_{\mu(t)}(f(t))} - 1}{\mu(t)} e_f(t, t_0) \\
&= f(t)e_f(t, t_0).
\end{aligned}
$$

□

Proof.

If $\sigma(t) = t$, then

$$\begin{aligned}
&|e_f(t,t_0) - e_f(s,t_0) - f(t)e_f(t,t_0)(t-s)| \\
&= |e_f(t,t_0) - e_f(t,t_0)e_f(s,t) - f(t)e_f(t,t_0)(t-s)| \\
&= |e_f(t,t_0)||1 - e_f(s,t) - f(t)(t-s)| \\
&= |e_f(t,t_0)|\Big|1 - \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)
\end{aligned} \tag{1}$$

$\square$

Proof.

$$+\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - f(t)(t-s)\Big|$$

$$\leq |e_f(t,t_0)|\left(\Big|1-\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)\Big|\right.$$

$$\left.+\Big|\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - f(t)(t-s)\Big|\right)$$

$$\leq |e_f(t,t_0)|\left(\Big|1-\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)\Big|\right.$$

$$\left.+\Big|\int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau\Big|\right).$$

$\square$

Proof.

Since $\sigma(t) = t$ and $f \in \mathrm{C}_{\mathrm{rd}}$, we get

$$\lim_{r \to t} \xi_{\mu(r)}(f(r)) = \xi_0(f(t)).$$

Therefore, there exists a neighbourhood U_1 of t such that

$$|\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t))| < \frac{\varepsilon}{3|e_f(t, t_0)|} \quad \text{for all} \quad \tau \in U_1.$$

Let $s \in U_1$. Then

$$|e_f(t, t_0)| \left| \int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau \right| \leq \frac{\varepsilon}{3}|t - s|. \tag{2}$$

□

Proof.

Also, using that

$$\lim_{z\to 0} \frac{1 - z - e^{-z}}{z} = 0,$$

we conclude that there exists a neighbourhood U_2 of t so that if $s \in U_2$ and $s \leq t$, then we have

$$\left| \frac{1 - \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)}{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \right| < \varepsilon^*,$$

where

$$\varepsilon^* = \min\left\{1, \frac{\varepsilon}{1 + 3|f(t)||e_f(t,t_0)|}\right\}.$$

Let $s \in U = U_1 \cap U_2$. Then $\square$

Proof.

$$
\begin{aligned}
&|e_f(t,t_0)|\left|1-\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)\right| \\
&= \quad |e_f(t,t_0)\frac{\left|1-\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau - e_f(s,t)\right|}{\left|\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau\right|}\left|\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau\right| \\
&\leq \quad |e_f(t,t_0)|\varepsilon^*\left|\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau\right|
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
&\leq |e_f(t,t_0)|\varepsilon^* \left\{ \left| \int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau \right| + |f(t)||t-s| \right\} \\
&\leq |e_f(t,t_0)| \left| \int_s^t (\xi_{\mu(\tau)}(f(\tau)) - \xi_0(f(t)))\Delta\tau \right| + |e_f(t,t_0)|\varepsilon^* |f(t)||t-s| \\
&\leq \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t-s| \\
&= \frac{2\varepsilon}{3}|t-s|.
\end{aligned}
$$

$\square$

Proof.

From the last inequality and from (1) and (2), we conclude that

$$\begin{aligned} |e_f(t,t_0) - e_f(s,t_0) - f(t)e_f(t,t_0)(t-s)| &< \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t- \\ &= \varepsilon|t-s|, \end{aligned}$$

which completes the proof. □

Corollary

Let $f \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then $e_f(\cdot, t_0)$ is a solution to the Cauchy problem

$$y^{\Delta}(t) = f(t)y(t), \quad y(t_0) = 1. \tag{3}$$

Corollary

Let $f \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then $e_f(\cdot, t_0)$ is the unique solution of the problem (3).

Proof.

Let y be any solution of the problem (3). Then

$$\begin{aligned}\left(\frac{y}{e_f(\cdot, t_0)}\right)^{\Delta}(t) &= \frac{y^{\Delta}(t)e_f(t, t_0) - y(t)e_f^{\Delta}(t, t_0)}{e_f(\sigma(t), t_0)e_f(t, t_0)} \\ &= \frac{f(t)y(t)e_f(t, t_0) - y(t)f(t)e_f(t, t_0)}{e_f(\sigma(t), t_0)e_f(t, t_0)} \\ &= 0.\end{aligned}$$

Consequently, $y = ce_f(\cdot, t_0)$, where c is a constant. Thus,

$$1 = y(t_0) = ce_f(t_0, t_0) = c.$$

Therefore, $y = e_f(\cdot, t_0)$. $\square$

Theorem

Let $f \in \mathcal{R}$. Then

$$e_f(\sigma(t), s) = (1 + \mu(t) f(t)) e_f(t, s).$$

Proof.

We have

$$\begin{aligned} e_f(\sigma(t), s) - e_f(t, s) &= \mu(t) e_f^{\Delta}(t, s) \\ &= \mu(t) f(t) e_f(t, s), \end{aligned}$$

which completes the proof. $\square$

Theorem

Let $f \in \mathcal{R}$. Then

$$e_f(t,s) = \frac{1}{e_f(s,t)} = e_{\ominus f}(s,t).$$

Proof.

We have

$$\begin{aligned} e_f(t,s) &= e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{-\int_t^s \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= \frac{1}{e^{\int_t^s \xi_{\mu(\tau)}(f(\tau))\Delta\tau}} \\ &= \frac{1}{e_f(s,t)}. \end{aligned}$$

□

Proof.

Now, we fix $t_0 \in \mathbb{T}$ and consider the problem

$$y^{\Delta}(t) = (\ominus f)(t)y(t), \quad y(t_0) = 1.$$

Its solution is $e_{\ominus f}(t,s)$. Also,

$$\begin{aligned}
\left(\frac{1}{e_f(\cdot,s)}\right)^{\Delta}(t) &= -\frac{e_f^{\Delta}(t,s)}{e_f(\sigma(t),s)e_f(t,s)} \\
&= -\frac{f(t)e_f(t,s)}{(1+\mu(t)f(t))e_f(t,s)e_f(t,s)}
\end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= -\frac{f(t)}{(1+\mu(t)f(t))e_f(t,s)} \\
&= (\ominus f)(t)\frac{1}{e_f(t,s)}.
\end{aligned}
$$

Therefore,

$$\frac{1}{e_f(t,s)} = e_{\ominus f}(t,s),$$

completing the proof. □

Theorem

Let $f, g \in \mathcal{R}$. Then

$$e_f(t,s)e_g(t,s) = e_{f\oplus g}(t,s).$$

Proof.

We have

$$\begin{aligned} e_f(t,s)e_g(t,s) &= e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} e^{\int_s^t \xi_{\mu(\tau)}(g(\tau))\Delta\tau} \\ &= e^{\int_s^t (\xi_{\mu(\tau)}(f(\tau)) + \xi_{\mu(\tau)}(g(\tau)))\Delta\tau} \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= e^{\int_s^t \frac{1}{\mu(\tau)}(\mathrm{Log}(1+\mu(\tau)f(\tau))+\mathrm{Log}(1+\mu(\tau)g(\tau)))\Delta\tau} \\
&= e^{\int_s^t \frac{1}{\mu(\tau)}\mathrm{Log}((1+\mu(\tau)f(\tau))(1+\mu(\tau)g(\tau)))\Delta\tau} \\
&= e^{\int_s^t \frac{1}{\mu(\tau)}\mathrm{Log}(1+\mu(\tau)(f(\tau)+g(\tau)+\mu(\tau)f(\tau)g(\tau)))\Delta\tau} \\
&= e^{\int_s^t \xi_{\mu(\tau)}((f\oplus g)(\tau))\Delta\tau} \\
&= e_{f\oplus g}(t,s),
\end{aligned}
$$

completing the proof. □

Theorem

Let $f, g \in \mathcal{R}$. Then

$$\frac{e_f(t,s)}{e_g(t,s)} = e_{f \ominus g}(t,s).$$

Proof.

We have

$$\begin{aligned}
\frac{e_f(t,s)}{e_g(t,s)} &= e_f(t,s)e_{\ominus g}(t,s) \\
&= e_{f \oplus (\ominus g)}(t,s) \\
&= e_{f \ominus g}(t,s),
\end{aligned}$$

completing the proof. $\square$

Theorem

Let $f \in \mathcal{R}$. Then

$$e_f(t,\sigma(s))e_f(s,r) = \frac{1}{1+\mu(s)f(s)}e_f(t,r).$$

Proof.

We have

$$\begin{aligned} e_f(t,\sigma(s))e_f(s,r) &= e^{\int_{\sigma(s)}^{t}\xi_{\mu(\tau)}(f(\tau))\Delta\tau}e^{\int_{r}^{s}\xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\ &= e^{\int_{\sigma(s)}^{s}\xi_{\mu(\tau)}(f(\tau))\Delta\tau+\int_{s}^{t}\xi_{\mu(\tau)}(f(\tau))\Delta\tau+\int_{r}^{s}\xi_{\mu(\tau)}(f(\tau))\Delta\tau} \end{aligned}$$

□

Proof.

$$
\begin{aligned}
&= e^{-\xi_{\mu(s)}(f(s))\mu(s)+\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\
&= e^{-\operatorname{Log}(1+f(s)\mu(s))} e^{\int_r^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \\
&= \frac{1}{1+\mu(s)f(s)} e_f(t,r),
\end{aligned}
$$

completing the proof. □

Theorem

Let $f, g \in \mathcal{R}$. Then

$$e^{\Delta}_{f \ominus g}(t, t_0) = \frac{(f(t) - g(t))e_f(t, t_0)}{e_g(\sigma(t), t_0)}.$$

Proof.

We have

$$e^{\Delta}_{f \ominus g}(t, t_0) \quad = \quad \left(\frac{e_f(t, t_0)}{e_g(t, t_0)} \right)^{\Delta}$$

$\square$

Proof.

$$
\begin{aligned}
&= \frac{e_f^{\Delta}(t,t_0)e_g(t,t_0) - e_f(t,t_0)e_g^{\Delta}(t,t_0)}{e_g(t,t_0)e_g(\sigma(t),t_0)} \\
&= \frac{f(t)e_f(t,t_0)e_g(t,t_0) - g(t)e_f(t,t_0)e_g(t,t_0)}{e_g(t,t_0)e_g(\sigma(t),t_0)} \\
&= \frac{(f(t)-g(t))e_f(t,t_0)}{e_g(\sigma(t),t_0)},
\end{aligned}
$$

completing the proof. □

Theorem

Let $f \in \mathcal{R}$ *and* $a, b, c \in \mathbb{T}$. *Then*

$$(e_f(c,\dot{}))^\Delta(t) = -f(t)e_f(c,\sigma(t))$$

and

$$\int_a^b f(t)e_f(c,\sigma(t))\Delta t = e_f(c,a) - e_f(c,b).$$

Proof.

We have

$$f(t)e_f(c,\sigma(t)) \quad = \quad f(t)e_{\ominus f}(\sigma(t),c)$$

□

Proof.

$$
\begin{aligned}
&= f(t)(1+\mu(t)\ominus f(t))e_{\ominus f}(t,c) \\
&= f(t)\left(1-\frac{\mu(t)f(t)}{1+\mu(t)f(t)}\right)e_{\ominus f}(t,c) \\
&= \frac{f(t)}{1+\mu(t)f(t)}e_{\ominus f}(t,c) \\
&= \frac{f(t)}{1+\mu(t)f(t)}e_{\ominus f}(t,c) \\
&= -(\ominus f)(t)e_{\ominus f}(t,c) \\
&= -e^{\Delta}_{\ominus f}(t,c) \\
&= -e^{\Delta}_{f}(c,t).
\end{aligned}
$$

Proof.

Hence,

$$\begin{aligned}\int_a^b f(t)e_f(c,\sigma(t))\Delta t &= -\int_a^b e_f^{\Delta}(c,t)\Delta t \\ &= -e_f(c,t)\Big|_{t=a}^{t=b} \\ &= e_f(c,a) - e_f(c,b),\end{aligned}$$

which completes the proof. $\square$

Example

Assume $1 + \mu(t)\frac{2}{t} \neq 0$ and $1 + \mu(t)\frac{5}{t} \neq 0$ for all $t \in \mathbb{T} \cap (0, \infty)$. Let $t_0 \in \mathbb{T} \cap (0, \infty)$. We will evaluate the integral

$$I = \int_{t_0}^{t} \frac{e_{\frac{5}{s}}(s, t_0)}{s e^{\sigma}_{\frac{2}{s}}(s, t_0)} \Delta s.$$

We have

$$\begin{aligned} \left(\frac{5}{s} - \frac{2}{s}\right) \frac{e_{\frac{5}{s}}(s, t_0)}{e^{\sigma}_{\frac{2}{s}}(s, t_0)} &= \frac{3}{s} \frac{e_{\frac{5}{s}}(s, t_0)}{e^{\sigma}_{\frac{2}{s}}(s, t_0)} \\ &= e^{\Delta}_{\frac{5}{s} \ominus \frac{2}{s}}(s, t_0) \end{aligned}$$

Example

$$
\begin{aligned}
&= e^{\Delta}_{\frac{\frac{3}{s}}{1+\mu(s)\frac{2}{s}}}(s, t_0) \\
&= e^{\Delta}_{\frac{3}{s+2\mu(s)}}(s, t_0).
\end{aligned}
$$

Consequently,

$$
\frac{1}{s}\frac{e_{\frac{5}{s}}(s, t_0)}{e^{\sigma}_{\frac{2}{s}}(s, t_0)} = \frac{1}{3}e^{\Delta}_{\frac{3}{s+2\mu(s)}}(s, t_0).
$$

Hence,

$$
\begin{aligned}
I &= \frac{1}{3}\int_{t_0}^{t} e^{\Delta}_{\frac{3}{s+2\mu(s)}}(s, t_0)\Delta s \\
&= \frac{1}{3}e_{\frac{3}{s+2\mu(s)}}(s, t_0)\Big|_{s=t_0}^{s=t} = \frac{1}{3}e_{\frac{3}{t+2\mu(t)}}(t, t_0) - \frac{1}{3}.
\end{aligned}
$$

Example

Let $\alpha \in \mathbb{R}$. Suppose that the exponentials

$$e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0) \quad \text{and} \quad e_{\frac{\alpha}{t}}(t,t_0)$$

exist for all $t \in \mathbb{T} \cap (0,\infty)$. We will prove that

1. $$\frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0)}{e_{\frac{\alpha}{t}}(t,t_0)} = e_{\frac{\alpha-1}{\sigma(t)}}(t,t_0),$$

2. $$\frac{e_{\frac{\alpha-1}{\sigma(t)}}(t,t_0)}{e_{\frac{\alpha}{t}}(t,t_0)} = e_{-\frac{1}{\sigma(t)}}(t,t_0) = \frac{t_0}{t}$$

for all $t, t_0 \in \mathbb{T} \cap (0,\infty)$.

Example

1. We have

$$\left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right) \ominus \frac{\alpha}{t} = \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right) \oplus \left(\ominus\frac{\alpha}{t}\right)$$

$$= \frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)} + \left(\ominus\frac{\alpha}{t}\right) + \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right)\left(\ominus\frac{\alpha}{t}\right)\mu(t)$$

$$= \frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)} - \frac{\frac{\alpha}{t}}{1+\frac{\alpha}{t}\mu(t)}$$

Example

$$
\begin{aligned}
&+\left(\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}\right)\left(-\frac{\frac{\alpha}{t}}{1+\frac{\alpha}{t}\mu(t)}\right)\mu(t)\\
= \quad & \frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}-\frac{\alpha}{t+\alpha\mu(t)}\\
&-\left(\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}\right)\frac{\alpha}{t+\alpha\mu(t)}\mu(t)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right)\left(1 - \frac{\alpha\mu(t)}{t+\alpha\mu(t)}\right) - \frac{\alpha}{t+\alpha\mu(t)} \\
&= \left(\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}\right)\frac{t}{t+\alpha\mu(t)} - \frac{\alpha}{1+\alpha\mu(t)} \\
&= \frac{\alpha^2\sigma(t) - t(\alpha-1)^2}{t\sigma(t)}\,\frac{t}{t+\alpha\mu(t)} - \frac{\alpha}{t+\alpha\mu(t)}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{\alpha^2(t+\mu(t)) - t\alpha^2 + 2\alpha t - t}{\sigma(t)} \frac{1}{t+\alpha\mu(t)} - \frac{\alpha}{t+\alpha\mu(t)} \\
&= \frac{\alpha^2\mu(t) + 2\alpha t - t}{\sigma(t)(t+\alpha\mu(t))} - \frac{\alpha}{t+\alpha\mu(t)} \\
&= \frac{\alpha^2\mu(t) + 2\alpha t - t - \alpha\mu(t) - \alpha t}{\sigma(t)(t+\alpha\mu(t))}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{\alpha\mu(t)(\alpha-1)+(\alpha-1)t}{\sigma(t)(t+\alpha\mu(t))} \\
&= \frac{(\alpha-1)(t+\alpha\mu(t))}{\sigma(t)(t+\alpha\mu(t))} \\
&= \frac{\alpha-1}{\sigma(t)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0)}{e_{\frac{\alpha}{t}}(t,t_0)} &= e_{\left(\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}\right)\ominus\frac{\alpha}{t}}(t,t_0) \\
&= e_{\frac{\alpha-1}{t}}(t,t_0).
\end{aligned}
$$

Example

We have

$$\begin{aligned}
\frac{\alpha-1}{\sigma(t)} \ominus \frac{\alpha}{t} &= \frac{\alpha-1}{\sigma(t)} \oplus \left(\ominus \frac{\alpha}{t}\right) \\
&= \frac{\alpha-1}{\sigma(t)} + \left(\ominus \frac{\alpha}{t}\right) + \frac{\alpha-1}{\sigma(t)} \left(\ominus \frac{\alpha}{t}\right) \mu(t) \\
&= \frac{\alpha-1}{\sigma(t)} - \frac{\frac{\alpha}{t}}{1+\mu(t)\frac{\alpha}{t}} - \frac{\alpha-1}{\sigma(t)} \frac{\frac{\alpha}{t}}{1+\mu(t)\frac{\alpha}{t}} \mu(t)
\end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{\alpha-1}{\sigma(t)} - \frac{\alpha}{t+\alpha\mu(t)} - \frac{\alpha-1}{\sigma(t)}\frac{\alpha}{t+\alpha\mu(t)}\mu(t) \\
&= \frac{\alpha-1}{\sigma(t)}\left(1-\frac{\alpha\mu(t)}{t+\alpha\mu(t)}\right) - \frac{\alpha}{t+\alpha\mu(t)} \\
&= \frac{t(\alpha-1)}{\sigma(t)(t+\alpha\mu(t))} - \frac{\alpha}{t+\alpha\mu(t)}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{\alpha t - t - \alpha\mu(t) - \alpha t}{\sigma(t)(t + \alpha\mu(t))} \\
&= -\frac{t + \alpha\mu(t)}{\sigma(t)(t + \alpha\mu(t))} \\
&= -\frac{1}{\sigma(t)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{e_{\frac{\alpha-1}{\sigma(t)}}(t, t_0)}{e_{\frac{\alpha}{t}}(t, t_0)} &= e_{\frac{\alpha-1}{\sigma(t)} \ominus \frac{\alpha}{t}}(t, t_0) \\
&= e_{-\frac{1}{\sigma(t)}}(t, t_0)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= e^{\int_{t_0}^{t} \xi_{\mu(\tau)}\left(-\frac{1}{\sigma(\tau)}\right)\Delta\tau} \\
&= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} \operatorname{Log}\left(1-\frac{\mu(\tau)}{\sigma(\tau)}\right)\Delta\tau} \\
&= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} \operatorname{Log} \frac{\tau}{\sigma(\tau)}\Delta\tau} \\
&= e^{-\operatorname{Log}\tau\Big|_{\tau=t_0}^{\tau=t}} \\
&= e^{-\operatorname{Log}\frac{t}{t_0}} \\
&= \frac{t_0}{t}.
\end{aligned}
$$

Definition

Let $f \in \mathrm{C}_{\mathrm{rd}}$ and $-\mu f^2 \in \mathcal{R}$. Define the *hyperbolic functions* $\cosh_f$ and $\sinh_f$ by

$$\cosh_f = \frac{e_f + e_{-f}}{2} \quad \text{and} \quad \sinh_f = \frac{e_f - e_{-f}}{2}.$$

Theorem

Let $f \in \mathrm{C}_{\mathrm{rd}}$ and $-\mu f^2 \in \mathcal{R}$. Then

$$\cosh_f^{\Delta} = f \sinh_f, \quad \sinh_f^{\Delta} = f \cosh_f \quad \textit{and} \quad \cosh_f^2 - \sinh_f^2 = e_{-\mu f^2}.$$

Proof.

We have

$$\begin{aligned}
\cosh_f^{\Delta} &= \left(\frac{e_f + e_{-f}}{2}\right)^{\Delta} \\
&= \frac{e_f^{\Delta} + e_{-f}^{\Delta}}{2} \\
&= \frac{f e_f - f e_{-f}}{2} \\
&= f\,\frac{e_f - e_{-f}}{2} \\
&= f \sinh_f,
\end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
\sinh_f^{\Delta} &= \left(\frac{e_f - e_{-f}}{2} \right)^2 \\
&= \frac{e_f^{\Delta} - e_{-f}^{\Delta}}{2} \\
&= \frac{f e_f + f e_{-f}}{2} \\
&= f \frac{e_f + e_{-f}}{2} \\
&= f \cosh_f ,
\end{aligned}
$$

$\square$

Proof.

and

$$\begin{aligned}
\cosh_f^2 - \sinh_f^2 &= \left(\frac{e_f + e_{-f}}{2}\right)^2 - \left(\frac{e_f - e_{-f}}{2}\right)^2 \\
&= \frac{e_f^2 + 2e_f e_{-f} + e_{-f}^2 - e_f^2 + 2e_f e_{-f} - e_{-f}^2}{4} \\
&= e_f e_{-f} \\
&= e_{f \oplus (-f)} \\
&= e_{-\mu f^2}.
\end{aligned}$$

The proof is complete. □

Example

Let $\mathbb{T} = \mathbb{Z}$. We compute $\cosh_{\frac{1}{2}}(t, t_0)$ and $\sinh_{\frac{1}{2}}(t, t_0)$ for $t, t_0 \in \mathbb{T}$. Here, $\mu(t) = 1$ and $f = \frac{1}{2}$. Then

$$1 - \mu^2 f^2 = 1 - \frac{1}{4} = \frac{3}{4} \neq 0.$$

Therefore, $\cosh_{\frac{1}{2}}(t, t_0)$ and $\sinh_{\frac{1}{2}}(t, t_0)$ are well defined. We have

$$e_{\frac{1}{2}}(t, t_0) = \left(\frac{3}{2}\right)^{t-t_0} \quad \text{and} \quad e_{-\frac{1}{2}}(t, t_0) = \left(\frac{1}{2}\right)^{t-t_0}.$$

Hence,

Example

$$
\begin{aligned}
\cosh_{\frac{1}{2}}(t, t_0) &= \frac{e_{\frac{1}{2}}(t, t_0) + e_{-\frac{1}{2}}(t, t_0)}{2} \\
&= \frac{\frac{3^{t-t_0}}{2^{t-t_0}} + \frac{1}{2^{t-t_0}}}{2} \\
&= \frac{3^{t-t_0} + 1}{2^{1+t-t_0}}, \\
\sinh_{\frac{1}{2}}(t, t_0) &= \frac{e_{\frac{1}{2}}(t, t_0) - e_{-\frac{1}{2}}(t, t_0)}{2} \\
&= \frac{3^{t-t_0} - 1}{2^{1+t-t_0}}.
\end{aligned}
$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We compute $\cosh_2(t, t_0)$ and $\sinh_2(t, t_0)$ for $t, t_0 \in \mathbb{T}$. Here, $\mu(t) = t$ and $f = 2$. Then

$$1 - \mu^2(t) f^2 = 1 - 4t^2 \neq 0 \quad \text{for any} \quad t \in \mathbb{T}.$$

Therefore, $\cosh_2(t, t_0)$ and $\sinh_2(t, t_0)$ are well defined. We have

$$e_2(t, t_0) = \prod_{s \in [t_0, t)} (1 + 2s), \quad e_{-2}(t, t_0) = \prod_{s \in [t_0, t)} (1 - 2s).$$

Hence,

Example

$$
\begin{aligned}
\cosh_2(t, t_0) &= \frac{e_2(t, t_0) + e_{-2}(t, t_0)}{2} \\
&= \frac{1}{2}\left(\prod_{s\in[t_0,t)} (1+2s) + \prod_{s\in[t_0,t)} (1-2s) \right), \\
\sinh_2(t, t_0) &= \frac{e_2(t, t_0) - e_{-2}(t, t_0)}{2} \\
&= \frac{1}{2}\left(\prod_{s\in[t_0,t)} (1+2s) - \prod_{s\in[t_0,t)} (1-2s) \right).
\end{aligned}
$$

Example

Let $\mathbb{T} = \mathbb{N}_0^2$. We compute $\cosh_2(t, t_0)$ and $\sinh_2(t, t_0)$ for $t, t_0 \in \mathbb{T}$. Here,

$$\sigma(t) = (\sqrt{t} + 1)^2, \quad \mu(t) = 2\sqrt{t} + 1, \quad f = 2,$$

and

$$1 - \mu^2(t) f^2 = 1 - 4(2\sqrt{t} + 1)^2 \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

Therefore, $\cosh_2(t, t_0)$ and $\sinh_2(t, t_0)$ are well defined. Note that

Example

$$
\begin{aligned}
e_2(t, t_0) &= \prod_{s\in[t_0,t)} \left(1 + 2(2\sqrt{s} + 1)\right) \\
&= \prod_{s\in[t_0,t)} (3 + 4\sqrt{s}), \\
e_{-2}(t, t_0) &= \prod_{s\in[t_0,t)} \left(1 - 2(2\sqrt{s} + 1)\right) \\
&= \prod_{s\in[t_0,t)} (-1 - 4\sqrt{s}).
\end{aligned}
$$

Hence,

Example

$$
\begin{aligned}
\cosh_2(t, t_0) &= \frac{1}{2}(e_2(t, t_0) + e_{-2}(t, t_0)) \\
&= \frac{1}{2}\left(\prod_{s\in[t_0,t)} (3 + 4\sqrt{s}) + \prod_{s\in[t_0,t)} (-1 - 4\sqrt{s})\right), \\
\sinh_2(t, t_0) &= \frac{1}{2}(e_2(t, t_0) - e_{-2}(t, t_0)) \\
&= \frac{1}{2}\left(\prod_{s\in[t_0,t)} (3 + 4\sqrt{s}) - \prod_{s\in[t_0,t)} (-1 - 4\sqrt{s})\right).
\end{aligned}
$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. We will compute $\cosh_2(t, t_0)$ and $\sinh_2(t, t_0)$ for $t, t_0 \in \mathbb{T}$. We have

$$\begin{aligned} \cosh_2(t, t_0) &= \frac{1}{2}\left(\prod_{s\in[t_0,t)}(1+4s) + \prod_{s\in[t_0,t)}(1-4s)\right), \\ \sinh_2(t, t_0) &= \frac{1}{2}\left(\prod_{s\in[t_0,t)}(1+4s) - \prod_{s\in[t_0,t)}(1-4s)\right). \end{aligned}$$

Theorem

Let $f \in \mathrm{C}_{\mathrm{rd}}$ *and* $-\mu f^2 \in \mathcal{R}$. *Then*

$$(\cosh_f(c,\cdot))^{\Delta}(t) = -f(t)\sinh_f(c,\sigma(t)).$$

Proof.

We have

$$\begin{aligned}
(\cosh_f(c,\cdot))^{\Delta}(t) &= \left(\frac{e_f(c,t)+e_{-f}(c,t)}{2}\right)^{\Delta} \\
&= \frac{e_f^{\Delta}(c,t)+e_{-f}^{\Delta}(c,t)}{2} \\
&= \frac{-f(t)e_f(c,\sigma(t))+f(t)e_{-f}(c,\sigma(t))}{2} \\
&= -f(t)\frac{e_f(c,\sigma(t))-e_{-f}(c,\sigma(t))}{2}
\end{aligned}$$

Theorem

Let $f \in \mathrm{C_{rd}}$ and $-\mu f^2 \in \mathcal{R}$. Then

$$\int_a^b f(t)\sinh_f(c,\sigma(t))\Delta t = \cosh_f(c,a) - \cosh_f(c,b).$$

Proof.

We have

$$f(t)\sinh_f(c,\sigma(t)) = \left(\cosh_f(c,\cdot)\right)^{\Delta}(t).$$

Hence,

$$\begin{aligned}
\int_a^b f(t)\sinh_f(c,\sigma(t)) &= -\int_a^b \left(\cosh_f(c,\cdot)\right)^{\Delta}(t)\Delta t \\
&= -\cosh_f(c,t)\Big|_{t=a}^{t=b} \\
&= \cosh_f(c,a) - \cosh_f(c,b),
\end{aligned}$$

Example

Let $f \in \mathrm{C}_{\mathrm{rd}}$ and $-\mu f^2 \in \mathcal{R}$. We will simplify

1. $A = \dfrac{\cosh_f(s,t_0)\cosh_f(t,t_0) - \sinh_f(s,t_0)\sinh_f(t,t_0)}{\cosh_f^2(s,t_0) - \sinh_f^2(s,t_0)}$,
2. $B = \dfrac{\cosh_f(s,t_0)\sinh_f(t,t_0) - \sinh_f(s,t_0)\cosh_f(t,t_0)}{\cosh_f^2(s,t_0) - \sinh_f^2(s,t_0)}$.

Example

We have

$$\cosh_f(s,t_0)\cosh_f(t,t_0)-\sinh_f(s,t_0)\sinh_f(t,t_0)$$

$$\begin{aligned}
&= \frac{e_f(s,t_0)+e_{-f}(s,t_0)}{2}\,\frac{e_f(t,t_0)+e_{-f}(t,t_0)}{2} \\
&\quad -\frac{e_f(s,t_0)-e_{-f}(s,t_0)}{2}\,\frac{e_f(t,t_0)-e_{-f}(t,t_0)}{2} \\
&= \frac{1}{4}\Big(e_f(s,t_0)e_f(t,t_0)+e_f(s,t_0)e_{-f}(t,t_0) \\
&\quad +e_{-f}(s,t_0)e_f(t,t_0)+e_{-f}(s,t_0)e_{-f}(t,t_0)
\end{aligned}$$

Example

$$
\begin{aligned}
&-e_f(s,t_0)e_f(t,t_0)+e_f(s,t_0)e_{-f}(t,t_0)\\
&+e_{-f}(s,t_0)e_f(t,t_0)-e_{-f}(s,t_0)e_{-f}(t,t_0)\Big)\\
= \quad & \frac{e_f(s,t_0)e_{-f}(t,t_0)+e_{-f}(s,t_0)e_f(t,t_0)}{2}.
\end{aligned}
$$

Example

Also,

$$
\begin{aligned}
\cosh_f^2(s,t_0)-\sinh_f^2(s,t_0) &= \left(\frac{e_f(s,t_0)+e_{-f}(s,t_0)}{2}\right)^2 \\
&\quad -\left(\frac{e_f(s,t_0)-e_{-f}(s,t_0)}{2}\right)^2 \\
&= \frac{e_f^2(s,t_0)+2e_f(s,t_0)e_{-f}(s,t_0)+e_{-f}^2(s,t_0)}{4} \\
&\quad -\frac{e_f^2(s,t_0)-2e_f(s,t_0)e_{-f}(s,t_0)+e_{-f}^2(s,t_0)}{4} \\
&= e_f(s,t_0)e_{-f}(s,t_0).
\end{aligned}
$$

Therefore,

Example

$$
\begin{aligned}
A &= \frac{1}{2}\frac{e_f(s,t_0)e_{-f}(t,t_0)+e_{-f}(s,t_0)e_f(t,t_0)}{e_f(s,t_0)e_{-f}(s,t_0)} \\
&= \frac{1}{2}\left(\frac{e_{-f}(t,t_0)}{e_{-f}(s,t_0)}+\frac{e_f(t,t_0)}{e_f(s,t_0)}\right) \\
&= \frac{1}{2}(e_f(t,s)+e_{-f}(t,s)) \\
&= \cosh_f(t,s).
\end{aligned}
$$

Example

We have

$$\cosh_f(s,t_0)\sinh_f(t,t_0) - \sinh_f(s,t_0)\cosh_f(t,t_0)$$

$$\begin{aligned}
&= \frac{e_f(s,t_0)+e_{-f}(s,t_0)}{2}\frac{e_f(t,t_0)-e_{-f}(t,t_0)}{2} \\
&\quad - \frac{e_f(s,t_0)-e_{-f}(s,t_0)}{2}\frac{e_f(t,t_0)+e_{-f}(t,t_0)}{2} \\
&= \frac{1}{4}\Big(e_f(s,t_0)e_f(t,t_0) - e_f(s,t_0)e_{-f}(t,t_0) \\
&\quad + e_{-f}(s,t_0)e_f(t,t_0) - e_{-f}(s,t_0)e_{-f}(t,t_0)\Big)
\end{aligned}$$

Example

$$-\frac{1}{4}\Big(e_f(s,t_0)e_f(t,t_0)+e_f(s,t_0)e_{-f}(t,t_0)$$

$$-e_{-f}(s,t_0)e_f(t,t_0)-e_{-f}(s,t_0)e_{-f}(t,t_0)\Big)$$

$$= \quad \frac{-e_f(s,t_0)e_{-f}(t,t_0)+e_{-f}(s,t_0)e_f(t,t_0)}{2}.$$

Hence,

Example

$$
\begin{aligned}
B &= \frac{1}{2}\frac{-e_f(s,t_0)e_{-f}(t,t_0)+e_{-f}(s,t_0)e_f(t,t_0)}{e_f(s,t_0)e_{-f}(s,t_0)} \\
&= \frac{1}{2}\left(-\frac{e_{-f}(t,t_0)}{e_{-f}(s,t_0)}+\frac{e_f(t,t_0)}{e_f(s,t_0)}\right) \\
&= \frac{1}{2}(e_f(t,s)-e_{-f}(t,s)) \\
&= \sinh_f(t,s).
\end{aligned}
$$