Time Scales Analysis
Lecture 16
Hyperbolic Functions. The Taylor Formula

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Definition

Let $f, g \in \mathrm{C_{rd}}$ and suppose that

$$2f + \mu(f^2 - g^2) = 0. \tag{1}$$

We define the hyperbolic functions ch_{fg} and sh_{fg} by

$$\mathrm{ch}_{fg} = \frac{e_{f+g} + e_{f-g}}{2} \quad \text{and} \quad \mathrm{sh}_{fg} = \frac{e_{f+g} - e_{f-g}}{2}.$$

Theorem

If $f, g \in \mathrm{C}_{\mathrm{rd}}$ *satisfy* (1), *then*

$$\mathrm{ch}_{fg}^{\Delta} = f\,\mathrm{ch}_{fg} + g\,\mathrm{sh}_{fg}, \quad \mathrm{sh}_{fg}^{\Delta} = g\,\mathrm{ch}_{fg} + f\,\mathrm{sh}_{fg}$$

and

$$\mathrm{ch}_{fg}^2 - \mathrm{sh}_{fg}^2 = 1.$$

Proof.

We have

$$\mathrm{ch}_{fg}^{\Delta} = \left(\frac{e_{f+g} + e_{f-g}}{2}\right)^{\Delta}$$

$\square$

Proof.

$$
\begin{aligned}
&= \frac{e_{f+g}^{\Delta} + e_{f-g}^{\Delta}}{2} \\
&= \frac{(f+g)e_{f+g} + (f-g)e_{f-g}}{2} \\
&= f\frac{e_{f+g} + e_{f-g}}{2} + g\frac{e_{f+g} - e_{f-g}}{2} \\
&= f \operatorname{ch}_{fg} + g \operatorname{sh}_{fg}
\end{aligned}
$$

and $\square$

Proof.

$$
\begin{aligned}
\mathrm{sh}_{fg}^{\Delta} &= \left(\frac{e_{f+g}-e_{f-g}}{2}\right)^{\Delta} \\
&= \frac{e_{f+g}^{\Delta}-pe_{f-g}^{\Delta}}{2} \\
&= \frac{(f+g)e_{f+g}-(f-g)e_{f-g}}{2} \\
&= f\frac{e_{f+g}-e_{f-g}}{2}+g\frac{e_{f+g}+e_{f-g}}{2} \\
&= f\,\mathrm{sh}_{fg}+g\,\mathrm{ch}_{fg}\,.
\end{aligned}
$$

Next, $\square$

Proof.

$$\begin{aligned}
\mathrm{ch}_{fg}^2 - \mathrm{sh}_{fg}^2 &= \left(\frac{e_{f+g}+e_{f-g}}{2}\right)^2 - \left(\frac{e_{f+g}-e_{f-g}}{2}\right)^2 \\
&= \frac{e_{f+g}^2 + 2e_{f+g}e_{f-g} + e_{f-g}^2}{4} \\
&\quad - \frac{e_{f+g}^2 - 2e_{f+g}e_{f-g} + e_{f-g}^2}{4} \\
&= e_{f+g}e_{f-g} \\
&= e_{(f+g)\oplus(f-g)}.
\end{aligned}$$

$\square$

Proof.

Note that

$$
\begin{aligned}
(f+g)\oplus(f-g) &= f+g+f-g+\mu(f+g)(f-g)\\
&= 2f+\mu(f^2-g^2)\\
&= 0.
\end{aligned}
$$

Therefore,

$$\mathrm{ch}_{fg}^2-\mathrm{sh}_{fg}^2=1,$$

completing the proof. □

Example

Let $f, g \in \mathrm{C_{rd}}$ satisfy (1). We will prove that

1. $\operatorname{sh}_{fg}(t,s) = -\operatorname{sh}_{fg}(s,t)$,
2. $\operatorname{sh}_{fg}(t,s) = \operatorname{sh}_{fg}(t,r)\operatorname{ch}_{fg}(r,s) - \operatorname{ch}_{fg}(t,r)\operatorname{sh}_{fg}(s,r)$,
3. $\operatorname{sh}_{fg}(t,r) = \operatorname{sh}_{fg}(t,s)\operatorname{ch}_{fg}(s,r) + \operatorname{ch}_{fg}(t,s)\operatorname{sh}_{fg}(s,r)$.

Example

We have

$$
\begin{aligned}
\operatorname{sh}_{fg}(t,s) &= \frac{e_{f+g}(t,s) - e_{f-g}(t,s)}{2} \\
&= \frac{\frac{1}{e_{f+g}(s,t)} - \frac{1}{e_{f-g}(s,t)}}{2} \\
&= \frac{e_{f-g}(s,t) - e_{f+g}(s,t)}{2e_{f+g}(s,t)e_{f-g}(s,t)} \\
&= -\frac{e_{f+g}(s,t) - e_{f-g}(s,t)}{2} \\
&= -\operatorname{sh}_{fg}(s,t).
\end{aligned}
$$

Example

We have

$$\begin{aligned}
&\mathrm{sh}_{fg}(t,r)\,\mathrm{ch}_{fg}(r,s)-\mathrm{ch}_{fg}(t,r)\,\mathrm{sh}_{fg}(s,r)\\
&= \mathrm{sh}_{fg}(t,r)\,\mathrm{ch}_{fg}(r,s)+\mathrm{ch}_{fg}(t,r)\,\mathrm{sh}_{fg}(r,s)\\
&= \frac{e_{f+g}(t,r)-e_{f-g}(t,r)}{2}\,\frac{e_{f+g}(r,s)+e_{f-g}(r,s)}{2}\\
&\quad+\frac{e_{f+g}(t,r)+e_{f-g}(t,r)}{2}\,\frac{e_{f+g}(r,s)+e_{f-g}(r,s)}{2}\\
&= \frac{1}{4}\Big(e_{f+g}(t,r)e_{f+g}(r,s)+e_{f+g}(t,r)e_{f-g}(r,s)
\end{aligned}$$

Example

$$
\begin{aligned}
&-e_{f-g}(t,r)e_{f+g}(r,s)-e_{f-g}(t,r)e_{f-g}(r,s)\Big) \\
&+\frac{1}{4}\Big(e_{f+g}(t,r)e_{f+g}(r,s)-e_{f+g}(t,r)e_{f-g}(r,s) \\
&-e_{f-g}(t,r)e_{f+g}(r,s)-e_{f-g}(t,r)e_{f-g}(r,s)\Big) \\
= \quad & \frac{e_{f+g}(t,s)-e_{f-g}(t,s)}{2} \\
= \quad & \mathrm{sh}_{fg}(t,s).
\end{aligned}
$$

Example

We have

$$
\begin{aligned}
&\mathrm{sh}_{fg}(t,s)\,\mathrm{ch}_{fg}(s,r) + \mathrm{ch}_{fg}(t,s)\,\mathrm{sh}_{fg}(s,r) \\
&= \quad \frac{e_{f+g}(t,s) - e_{f-g}(t,s)}{2}\,\frac{e_{f+g}(s,r) + e_{f-g}(s,r)}{2} \\
&\quad + \frac{e_{f+g}(t,s) + e_{f-g}(t,s)}{2}\,\frac{e_{f+g}(s,r) - e_{f-g}(s,r)}{2} \\
&= \quad \frac{1}{4}\Big(e_{f+g}(t,s)e_{f+g}(s,r) + e_{f+g}(t,s)e_{f-g}(s,r) \\
&\quad - e_{f-g}(t,s)e_{f+g}(s,r) - e_{f-g}(t,s)e_{f-g}(s,r)\Big)
\end{aligned}
$$

Example

$$
\begin{aligned}
& +\frac{1}{4}\Big(e_{f+g}(t,s)e_{f+g}(s,r) - e_{f+g}(t,s)e_{f-g}(s,r) \\
& +e_{f-g}(t,s)e_{f+g}(s,r) - e_{f-g}(t,s)e_{f-g}(s,r)\Big) \\
= \quad & \frac{e_{f+g}(t,r) - e_{f-g}(t,r)}{2} \\
= \quad & \mathrm{sh}_{fg}(t,r).
\end{aligned}
$$

Definition

Let $f \in \mathrm{C}_{\mathrm{rd}}$ and $\mu f^2 \in \mathcal{R}$. Define the *trigonometric functions* $\cos_f$ and $\sin_f$ by

$$\cos_f = \frac{e_{if} + e_{-if}}{2} \quad \text{and} \quad \sin_f = \frac{e_{if} - e_{-if}}{2i}.$$

Theorem

Let $f \in \mathrm{C}_{\mathrm{rd}}$ and $\mu f^2 \in \mathcal{R}$. Then

$$\cos_f^\Delta = -f \sin_f \quad \textit{and} \quad \sin_f^\Delta = f \cos_f$$

and

$$\cos_f^2 + \sin_f^2 = e_{\mu f^2}.$$

Proof.

We have

$$\begin{aligned}\cos_f^{\Delta} &= \left(\frac{e_{if}+e_{-if}}{2}\right)^{\Delta}\\ &= \frac{e_{if}^{\Delta}+e_{-if}^{\Delta}}{2}\\ &= \frac{ife_{if}-ife_{-if}}{2}\\ &= -f\frac{e_{if}-e_{-if}}{2i}\\ &= -f\sin_f,\end{aligned}$$

□

Proof.

$$
\begin{aligned}
\sin_f^\Delta &= \left(\frac{e_{if} - e_{-if}}{2i}\right)^\Delta \\
&= \frac{e_{if}^\Delta - e_{-if}^\Delta}{2i} \\
&= \frac{ife_{if} + ife_{-if}}{2i} \\
&= f\frac{e_{if} + e_{-if}}{2} \\
&= f\cos_f,
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
\cos_f^2 + \sin_f^2 &= \left(\frac{e_{if} + e_{-if}}{2}\right)^2 + \left(\frac{e_{if} - e_{-if}}{2}\right)^2 \\
&= \frac{e_{if}^2 + 2e_{if}e_{-if} + e_{-if}^2}{4} - \frac{e_{if} - 2e_{if}e_{-if} + e_{-if}^2}{4} \\
&= e_{if}e_{-if} \\
&= e_{(if)\oplus(-if)}.
\end{aligned}
$$

Since

$$(if) \oplus (-if) = (if) + (-if) + \mu(if)(-if) = \mu f^2,$$

we get the desired result. $\square$

Example

Let $f \in \mathrm{C}_{\mathrm{rd}}$ and $\mu f^2 \in \mathcal{R}$. We simplify

$$A = \frac{\cos_f(s, t_0)\cos_f(t, t_0) + \sin_f(s, t_0)\sin_f(t, t_0)}{\cos_f^2(s, t_0) + \sin_f^2(s, t_0)}.$$

We have

$$\begin{aligned} & \cos_f(s, t_0)\cos_f(t, t_0) + \sin_f(s, t_0)\sin_f(t, t_0) \\ = \quad & \frac{e_{if}(s, t_0) + e_{-if}(s, t_0)}{2}\,\frac{e_{if}(t, t_0) + e_{-if}(t, t_0)}{2} \end{aligned}$$

Example

$$
\begin{aligned}
& +\frac{e_{if}(s,t_0)-e_{-if}(s,t_0)}{2i}\frac{e_{if}(t,t_0)-e_{-if}(t,t_0)}{2i} \\
= \quad & \frac{1}{4}\Big(e_{if}(s,t_0)e_{if}(t,t_0)+e_{-if}(s,t_0)e_{if}(t,t_0) \\
& +e_{if}(s,t_0)e_{-if}(t,t_0)+e_{-if}(s,t_0)e_{-if}(t,t_0)\Big) \\
& -\frac{1}{4}\Big(e_{if}(s,t_0)e_{if}(t,t_0)-e_{if}(s,t_0)e_{-if}(t,t_0) \\
& -e_{-if}(s,t_0)e_{if}(t,t_0)+e_{-if}(s,t_0)e_{-if}(t,t_0)\Big) \\
= \quad & \frac{e_{-if}(s,t_0)e_{if}(t,t_0)+e_{if}(s,t_0)e_{-if}(t,t_0)}{2},
\end{aligned}
$$

Example

$$
\begin{aligned}
&\cos_f^2(s,t_0)+\sin_f^2(s,t_0)\\
&= \left(\frac{e_{if}(s,t_0)+e_{-if}(s,t_0)}{2}\right)^2+\left(\frac{e_{if}(s,t_0)-e_{-if}(s,t_0)}{2i}\right)^2\\
&= \frac{e_{if}^2(s,t_0)+2e_{if}(s,t_0)e_{-if}(s,t_0)+e_{-if}^2(s,t_0)}{4}\\
&\quad -\frac{e_{if}^2(s,t_0)-2e_{if}(s,t_0)e_{-if}(s,t_0)+e_{-if}^2(s,t_0)}{4}\\
&= e_{if}(s,t_0)e_{-if}(s,t_0).
\end{aligned}
$$

Hence,

Example

$$
\begin{aligned}
A &= \frac{1}{2}\frac{e_{-if}(s,t_0)e_{if}(t,t_0)+e_{if}(s,t_0)e_{-if}(t,t_0)}{e_{if}(s,t_0)e_{-if}(s,t_0)} \\
&= \frac{1}{2}\left(\frac{e_{if}(t,t_0)}{e_{if}(s,t_0)}+\frac{e_{-if}(t,t_0)}{e_{-if}(s,t_0)}\right) \\
&= \frac{1}{2}(e_{if}(t,s)+e_{-if}(t,s)) \\
&= \cos_f(t,s).
\end{aligned}
$$

Definition

Let $f, g \in \mathrm{C_{rd}}$ and assume

$$2f + \mu(f^2 + g^2) = 0. \tag{2}$$

Define the trigonometric functions c_{fg} and s_{fg} by

$$\mathrm{c}_{fg} = \frac{e_{f+ig} + e_{f-ig}}{2} \quad \text{and} \quad \mathrm{s}_{fg} = \frac{e_{f+ig} - e_{f-ig}}{2i}.$$

Theorem

Let $f, g \in \mathrm{C}_{\mathrm{rd}}$ *satisfy* (2). *Then*

$$\mathrm{c}_{fg}^{\Delta} = f\,\mathrm{c}_{fg} - g\,\mathrm{s}_{fg} \quad \text{and} \quad \mathrm{s}_{fg}^{\Delta} = g\,\mathrm{c}_{fg} + f\mathrm{s}_{fg}$$

and

$$\mathrm{c}_{fg}^{2} + \mathrm{s}_{fg}^{2} = 1.$$

Proof.

We have

$$\begin{aligned} \mathrm{c}_{fg}^{\Delta} &= \left(\frac{e_{f+ig} + e_{f-ig}}{2}\right)^{\Delta} \\ &= \frac{e_{f+ig}^{\Delta} + e_{f-ig}^{\Delta}}{2} \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= \frac{(f+ig)e_{f+ig} + (f-ig)e_{f-ig}}{2} \\
&= f\frac{e_{f+ig} + e_{f-ig}}{2} + ig\frac{e_{f+ig} - e_{f-ig}}{2} \\
&= f\,\mathrm{c}_{fg} - g\frac{e_{f+ig} - e_{f-ig}}{2i} \\
&= f\,\mathrm{c}_{fg} - g\,\mathrm{s}_{fg},
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
s_{fg}^{\Delta} &= \left(\frac{e_{f+ig} - e_{f-ig}}{2i}\right)^{\Delta} \\
&= \frac{e_{f+ig}^{\Delta} - e_{f-ig}^{\Delta}}{2i} \\
&= \frac{(f+ig)e_{f+ig} - (f-ig)e_{f-ig}}{2i} \\
&= f\frac{e_{f+ig} - e_{f-ig}}{2i} + g\frac{e_{f+ig} + e_{f-ig}}{2} \\
&= f\, s_{fg} + g\, c_{fg},
\end{aligned}
$$

and □

Proof.

$$
\begin{aligned}
c_{fg}^2 + s_{fg}^2 &= \left(\frac{e_{f+ig} + e_{f-ig}}{2}\right)^2 + \left(\frac{e_{f+ig} - e_{f-ig}}{2i}\right)^2 \\
&= \frac{e_{f+ig}^2 + 2e_{f+ig}e_{f-ig} + e_{f-ig}^2}{4} \\
&\quad - \frac{e_{f+ig}^2 - 2e_{f+ig}e_{f-ig} + e_{f-ig}^2}{4} \\
&= e_{(f+ig)\oplus(f-ig)}.
\end{aligned}
$$

Since

$\square$

Proof.

$$
\begin{aligned}
(f+ig)\oplus(f-ig) &= f+ig+f-ig+\mu(f+ig)(f-ig) \\
&= 2f+\mu(f^2+g^2) \\
&= 0,
\end{aligned}
$$

we get

$$
c_{fg}^2+s_{fg}^2=1,
$$

completing the proof. $\square$

Example

Let $f, g \in \mathrm{C}_{\mathrm{rd}}$ satisfy (2). We will prove that

1. $\mathsf{c}_{fg}(s,t) = \mathsf{c}_{fg}(t,s)$,
2. $\mathsf{c}_{fg}(t,s) = \mathsf{c}_{fg}(t,r)\,\mathsf{c}_{fg}(s,r) + \mathsf{s}_{fg}(t,r)\,\mathsf{s}_{fg}(s,r)$,
3. $\mathsf{c}_{fg}(t,r) = \mathsf{c}_{fg}(t,s)\,\mathsf{c}_{fg}(s,r) - \mathsf{s}_{fg}(t,s)\,\mathsf{s}_{fg}(s,r)$.

Example

We have

$$
\begin{aligned}
c_{fg}(t,s) &= \frac{e_{f+ig}(t,s)+e_{f-ig}(t,s)}{2} \\
&= \frac{\frac{1}{e_{f+ig}(s,t)}+\frac{1}{e_{f-ig}(s,t)}}{2} \\
&= \frac{e_{f+ig}(s,t)+e_{f-ig}(s,t)}{2e_{f+ig}(s,t)e_{f-ig}(s,t)} \\
&= \frac{e_{f+ig}(s,t)+e_{f-ig}(s,t)}{2} \\
&= c_{fg}(s,t).
\end{aligned}
$$

Example

We have

$$\begin{aligned} c_{fg}(t,r)\,c_{fg}(s,r) &= \frac{e_{f+ig}(t,r)+e_{f-ig}(t,r)}{2}\,\frac{e_{f+ig}(s,r)+e_{f-ig}(s,r)}{2} \\ &= \frac{1}{4}\Big(e_{f+ig}(t,r)e_{f+ig}(s,r)+e_{f+ig}(t,r)e_{f-ig}(s,r) \\ &\quad +e_{f-ig}(t,r)e_{f+ig}(s,r)+e_{f-ig}(t,r)e_{f-ig}(s,r)\Big), \end{aligned}$$

Example

$$
\begin{aligned}
s_{fg}(t,r)\, s_{fg}(s,r) &= \frac{e_{f+ig}(t,r) - e_{f-ig}(t,r)}{2i} \frac{e_{f+ig}(s,r) - e_{f-ig}(s,r)}{2i} \\
&= -\frac{1}{4}\Big(e_{f+ig}(t,r) e_{f+ig}(s,r) - e_{f+ig}(t,r) e_{f-ig}(s,r) \\
&\quad - e_{f-ig}(t,r) e_{f+ig}(s,r) + e_{f-ig}(t,r) e_{f-ig}(s,r) \Big).
\end{aligned}
$$

Hence,

Example

$$
\begin{aligned}
& \mathrm{c}_{fg}(t,r)\,\mathrm{c}_{fg}(s,r)+\mathrm{s}_{fg}(t,r)\,\mathrm{s}_{fg}(s,r) \\
&= \frac{1}{2}\left(e_{f+ig}(t,r)e_{f-ig}(s,r)+e_{f-ig}(t,r)e_{f+ig}(s,r)\right) \\
&= \frac{1}{2}\left(e_{f+ig}(t,r)e_{f+ig}(r,s)+e_{f-ig}(t,r)e_{f-ig}(r,s)\right) \\
&= \frac{1}{2}(e_{f+ig}(t,s)+e_{f-ig}(t,s)) \\
&= \mathrm{c}_{fg}(t,s).
\end{aligned}
$$

Example

We have

$$\begin{aligned} c_{fg}(t,s)\,c_{fg}(s,r) &= \frac{e_{f+ig}(t,s)+e_{f-ig}(t,s)}{2}\,\frac{e_{f+ig}(s,r)+e_{f-ig}(s,r)}{2} \\ &= \frac{1}{4}\Big(e_{f+ig}(t,r)+e_{f-ig}(t,r)+e_{f+ig}(t,s)e_{f-ig}(s,r) \\ &\quad +e_{f-ig}(t,s)e_{f+ig}(s,r)\Big), \end{aligned}$$

Example

$$
\begin{aligned}
s_{fg}(t,s)\, s_{fg}(s,r) &= \frac{e_{f+ig}(t,s) - e_{f-ig}(t,s)}{2i} \frac{e_{f+ig}(s,r) - e_{f-ig}(s,r)}{2i} \\
&= -\frac{1}{4}\Big(e_{f+ig}(t,r) + e_{f-ig}(t,r) - e_{f+ig}(t,s) e_{f-ig}(s,r) \\
&\quad - e_{f-ig}(t,s) e_{f+ig}(s,r) \Big).
\end{aligned}
$$

Example

Hence,

$$\begin{aligned} c_{fg}(t,s)\,c_{fg}(s,r) - s_{fg}(t,s)\,s_{fg}(s,r) &= \frac{1}{2}\left(e_{f+ig}(t,r) + e_{f-ig}(t,r)\right) \\ &= c_{fg}(t,r). \end{aligned}$$