Time Scales Analysis
Lecture 17
The Taylor Formula. Improper Integrals

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Definition

Let $s, t \in \mathbb{T}$. We define the time scales *monomials* recursively by

$$g_0(t,s) = h_0(t,s) = 1$$

and

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\Delta\tau, \quad h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau, \quad k \in \mathbb{N}_0.$$

Remark

We have

$$\begin{aligned} g_1(t,s) &= \int_s^t g_0(\sigma(\tau),s)\Delta\tau \\ &= \int_s^t \Delta\tau \\ &= t-s, \\ h_1(t,s) &= \int_s^t h_0(\tau,s)\Delta\tau \\ &= \int_s^t \Delta\tau \\ &= t-s, \end{aligned}$$

Remark

$$\begin{aligned} g_2(t,s) &= \int_s^t g_1(\sigma(\tau),s)\Delta\tau \\ &= \int_s^t (\sigma(\tau)-s)\Delta\tau, \\ h_2(t,s) &= \int_s^t h_1(\tau,s)\Delta\tau \\ &= \int_s^t (\tau-s)\Delta\tau, \end{aligned}$$

and so on. Moreover,

$$g_k^{\Delta}(t,s) = g_{k-1}(\sigma(t),s), \quad h_k^{\Delta}(t,s) = h_{k-1}(t,s), \quad k \in \mathbb{N}.$$

Lemma

Let $n \in \mathbb{N}$. *If* f *is* n *times differentiable and* p_k, $0 \le k \le n-1$, *are differentiable at some* $t \in \mathbb{T}$ *with*

$$p_{k+1}^{\Delta}(t) = p_k^{\sigma}(t) \quad \textit{for all} \quad 0 \le k \le n-2, \quad n \in \mathbb{N} \setminus \{1\},$$

then

$$\left(\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} p_k \right)^{\Delta} (t) = (-1)^{n-1} f^{\Delta^n}(t) p_{n-1}^{\sigma}(t) + f(t) p_0^{\Delta}(t).$$

Proof.

We have

$$\left(\sum_{k=0}^{n-1}(-1)^k f^{\Delta^k} p_k\right)^{\Delta}(t) = \sum_{k=0}^{n-1}(-1)^k \left(f^{\Delta^k} p_k\right)^{\Delta}(t)$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1}(-1)^k \left(f^{\Delta^{k+1}}(t)p_k^{\sigma}(t) + f^{\Delta^k}(t)p_k^{\Delta}(t)\right) \\
&= \sum_{k=0}^{n-1}(-1)^k f^{\Delta^{k+1}}(t)p_k^{\sigma}(t) + \sum_{k=0}^{n-1}(-1)^k f^{\Delta^k}(t)p_k^{\Delta}(t) \\
&= \sum_{k=0}^{n-2}(-1)^k f^{\Delta^{k+1}}(t)p_k^{\sigma}(t) + (-1)^{n-1} f^{\Delta^n}(t)p_{n-1}^{\sigma}(t) \\
&\quad + \sum_{k=1}^{n-1}(-1)^k f^{\Delta^k}(t)p_k^{\Delta}(t) + f^{\Delta^0}(t)p_0^{\Delta}(t)
\end{aligned}$$

Proof.

$$
\begin{aligned}
&= \sum_{k=0}^{n-2}(-1)^k f^{\Delta^{k+1}}(t)p_{k+1}^{\Delta}(t) + (-1)^{n-1} f^{\Delta^n}(t)p_{n-1}^{\sigma}(t)\\
&\quad + \sum_{k=0}^{n-2}(-1)^{k+1} f^{\Delta^{k+1}}(t)p_{k+1}^{\Delta}(t) + f(t)p_0^{\Delta}(t)\\
&= (-1)^{n-1} f^{\Delta^n}(t)p_{n-1}^{\sigma}(t) + f(t)p_0^{\Delta}(t),
\end{aligned}
$$

completing the proof. □

Lemma

The functions $g_n(t,s)$ *satisfy for all* $t \in \mathbb{T}$ *the relationship*

$$g_n(\rho^k(t), t) = 0 \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and all} \quad 0 \leq k \leq n-1.$$

Proof.

Let $n \in \mathbb{N}$ be arbitrarily chosen. Then

$$\begin{aligned} g_n(\rho^0(t), t) &= g_n(t,t) \\ &= \int_t^t g_{n-1}(\sigma(\tau), t)\Delta\tau \\ &= 0. \end{aligned}$$

$\square$

Proof.

Assume that

$$g_{n-1}(\rho^k(t), t) = 0 \quad \text{and} \quad g_n(\rho^k(t), t) = 0$$

for some $0 \le k < n-1$. We will prove that

$$g_n(\rho^{k+1}(t), t) = 0.$$

We have the following cases.
Let $\rho^k(t)$ be left-dense. Then

$$\rho^{k+1}(t) = \rho(\rho^k(t)) = \rho^k(t).$$

Consequently, using the induction assumption, we have

$$g_n(\rho^{k+1}(t), t) = g_n(\rho^k(t), t) = 0.$$

□

Proof.

Let $\rho^k(t)$ be left-scattered. Then

$$\rho(\rho^k(t)) < \rho^k(t),$$

and there is no $s \in \mathbb{T}$ such that $\rho^{k+1}(t) < s < \rho^k(t)$. Hence,

$$\sigma(\rho^{k+1}(t)) = \rho^k(t).$$

Therefore

$$g_n(\sigma(\rho^{k+1}(t)), t) = g_n(\rho^{k+1}(t), t) + \mu(\rho^{k+1}(t)) g_n^{\Delta}(\rho^{k+1}(t), t),$$

i.e.,

$$g_n(\rho^k(t), t) = g_n(\rho^{k+1}(t), t) + \mu(\rho^{k+1}(t)) g_n^{\Delta}(\rho^{k+1}(t), t),$$

whereupon $\square$

Proof.

$$\begin{aligned} g_n(\rho^{k+1}(t),t) &= g_n(\rho^k(t),t) - \mu(\rho^{k+1}(t))g_n^{\Delta}(\rho^{k+1}(t),t) \\ &= g_n(\rho^k(t),t) - \mu(\rho^{k+1}(t))g_{n-1}(\sigma(\rho^{k+1}(t)),t) \\ &= g_n(\rho^k(t),t) - \mu(\rho^{k+1}(t))g_{n-1}(\rho^k(t),t) \\ &= 0. \end{aligned}$$

The proof is complete. □

Lemma

Let $n \in \mathbb{N}$ and suppose that f is $n-1$ times differentiable at $\rho^{n-1}(t)$. Then

$$f(t) = \sum_{k=0}^{n-1} (-1)^k f^{\Delta^k}(\rho^{n-1}(t)) g_k(\rho^{n-1}(t), t).$$

Proof.

Since,

$$\begin{aligned} \sum_{k=0}^{0} (-1)^k f^{\Delta^k}(\rho^0(t)) g_k(\rho^0(t), t) &= (-1)^0 f^{\Delta^0}(t) g_0(t, t) \\ &= f(t), \end{aligned}$$

the claim holds for $n = 1$. □

Proof.

Assume that

$$f(t) = \sum_{k=0}^{m-1} (-1)^k f^{\Delta^k}(\rho^{m-1}(t)) g_k(\rho^{m-1}(t), t)$$

holds for some $m \in \mathbb{N}$. We will prove that

$$f(t) = \sum_{k=0}^{m} (-1)^k f^{\Delta^k}(\rho^{m}(t)) g_k(\rho^{m}(t), t).$$

Let $\rho^{m-1}(t)$ be left-dense. Then

$$\rho^m(t) = \rho(\rho^{m-1}(t)) = \rho^{m-1}(t).$$

Thus, using the induction assumption, we obtain □

Proof.

$$\begin{aligned}
\sum_{k=0}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^m(t),t)
&= \sum_{k=0}^{m-1}(-1)^k f^{\Delta}[k(\rho^m(t))g_k(\rho^m(t),t)+(-1)^m f^{\Delta^m}(\rho^m(t))g_m(\rho^m(t),t)\\
&= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^{m-1}(t))g_k(\rho^{m-1}(t),t)\\
&\quad +(-1)^m f^{\Delta^m}(\rho^{m-1}(t))g_m(\rho^{m-1}(t),t)\\
&= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^{m-1}(t))g_k(\rho^{m-1}(t),t)\\
&= f(t),
\end{aligned}$$

Proof.

Let $\rho^{m-1}(t)$ be left-scattered. Then

$$\rho^m(t) = \rho(\rho^{m-1}(t)) < \rho^{m-1}(t),$$

and there is no $s \in \mathbb{T}$ such that

$$\rho^m(t) < s < \rho^{m-1}(t).$$

Also,

$$\sigma(\rho^m(t)) = \rho^{m-1}(t).$$

Hence,

$$g_k(\sigma(\rho^m(t)), t) = g_k(\rho^{m-1}(t), t).$$

Therefore,

$$g_k(\rho^{m-1}(t), t) \quad = \quad g_k(\sigma(\rho^m(t)), t)$$

□

Proof.

$$
\begin{aligned}
&= g_k(\rho^m(t), t) + \mu(\rho^m(t)) g_k^{\Delta}(\rho^m(t), t) \\
&= g_k(\rho^m(t), t) + \mu(\rho^m(t)) g_{k-1}(\sigma(\rho^m(t)), t) \\
&= g_k(\rho^m(t), t) + \mu(\rho^m(t)) g_{k-1}(\rho^{m-1}(t), t),
\end{aligned}
$$

whereupon

$$
g_k(\rho^m(t), t) = g_k(\rho^{m-1}(t), t) - \mu(\rho^m(t)) g_{k-1}(\rho^{m-1}(t), t).
$$

Consequently, $\square$

Proof.

$$\sum_{k=0}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^m(t),t)$$

$$= f(\rho^m(t)) + \sum_{k=1}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^m(t),t)$$

$$= f(\rho^m(t)) + \sum_{k=1}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^{m-1}(t),t)$$

$$+ \sum_{k=1}^{m}(-1)^{k-1} f^{\Delta^k}(\rho^m(t))\mu(\rho^m(t))g_{k-1}(\rho^{m-1}(t),t)$$

$\square$

Proof.

$$
\begin{aligned}
&= f(\rho^m(t)) + \sum_{k=1}^{m-1} (-1)^k f^{\Delta^k}(\rho^m(t)) g_k(\rho^{m-1}(t), t) \\
&\quad + (-1)^m f^{\Delta^m}(\rho^m(t)) g_m(\rho^{m-1}(t), t) \\
&\quad + \sum_{k=0}^{m-1} (-1)^k f^{\Delta^{k+1}}(\rho^m(t)) \mu(\rho^m(t)) g_k(\rho^{m-1}(t), t) \\
&= \sum_{k=0}^{m-1} (-1)^k f^{\Delta^k}(\rho^m(t)) g_k(\rho^{m-1}(t), t) \\
&\quad + \sum_{k=0}^{m-1} (-1)^k \mu(\rho^m(t)) f^{\Delta^{k+1}}(\rho^m(t)) g_k(\rho^{m-1}(t), t)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&= \sum_{k=0}^{m-1}(-1)^k \left(f^{\Delta^k}(\rho^m(t)) + \mu(\rho^m(t))(f^{\Delta^k})^{\Delta}(\rho^m(t))\right) g_k(\rho^{m-1}(t), t) \\
&= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\sigma(\rho^m(t)))g_k(\rho^{m-1}(t), t) \\
&= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^{m-1}(t))g_k(\rho^{m-1}(t), t) \\
&= f(t).
\end{aligned}
$$

This completes the proof. □

Theorem (Taylor's Formula)

Let $n \in \mathbb{N}$. Suppose that f is n times differentiable on $\mathbb{T}^{\kappa^n}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$ and $t \in \mathbb{T}$. Then

$$\begin{aligned} f(t) &= \sum_{k=0}^{n-1} (-1)^k g_k(\alpha, t) f^{\Delta^k}(\alpha) \\ &\quad + \int_{\alpha}^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}(\sigma(\tau), t) f^{\Delta^n}(\tau) \Delta\tau. \end{aligned}$$

Proof.

We note that, applying Lemma 2 for $p_k = g_k$, we have

$$\begin{aligned}\left(\sum_{k=0}^{n-1}(-1)^k g_k(\cdot,t) f^{\Delta^k}\right)^{\Delta}(\tau) &= (-1)^{n-1} f^{\Delta^n}(\tau) g_{n-1}(\sigma(\tau),t) + f(\tau) g_0^{\Delta}(\tau,t)\\ &= (-1)^{n-1} f^{\Delta^n}(\tau) g_{n-1}(\sigma(\tau),t) \quad \text{for all} \quad \tau \in \mathbb{T}^{\kappa^n}.\end{aligned}$$

We integrate the last relation from α to $\rho^{n-1}(t)$ to obtain □

Proof.

$$\int_{\alpha}^{\rho^{n-1}(t)} \left(\sum_{k=0}^{n-1}(-1)^k g_k(\cdot,t) f^{\Delta^k}\right)^{\Delta}(\tau)\Delta\tau$$

$$= \int_{\alpha}^{\rho^{n-1}(t)} (-1)^{n-1} f^{\Delta^n}(\tau) g_{n-1}(\sigma(\tau),t)\Delta\tau,$$

i.e.,

$$\sum_{k=0}^{n-1}(-1)^k g_k(\rho^{n-1}(t),t) f^{\Delta^k}(\rho^{n-1}(t)) - \sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t) f^{\Delta^k}(\alpha$$

$$= \int_{\alpha}^{\rho^{n-1}(t)} (-1)^{n-1} f^{\Delta^n}(\tau) g_{n-1}(\sigma(\tau),t)\Delta\tau.$$

$\square$

Proof.

Hence, applying Lemma 4, we get

$$f(t)-\sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t)f^{\Delta^k}(\alpha)=\int_{\alpha}^{\rho^{n-1}(t)}(-1)^{n-1}f^{\Delta^n}(\tau)g_{n-1}(\sigma(\tau),t)\Delta\tau,$$

completing the proof. $\square$

Theorem

The functions g_n and h_n satisfy the relationship

$$h_n(t,s) = (-1)^n g_n(s,t)$$

for all $t \in \mathbb{T}$ and all $s \in \mathbb{T}^{\kappa^n}$.

Proof.

Let $t \in \mathbb{T}$ and $s \in \mathbb{T}^{\kappa^n}$ be arbitrarily chosen. We apply the previous theorem for $\alpha = s$ and $f(\tau) = h_n(\tau, s)$. We observe that $\square$

Proof.

$$f^{\Delta^k}(\tau) = h_{n-k}(\tau, s), \quad 0 \le k \le n.$$

Hence,

$$f^{\Delta^k}(s) = h_{n-k}(s, s) = 0, \quad 0 \le k \le n-1$$

and

$$f^{\Delta^n}(s) = h_0(s, s) = 1, \quad f^{\Delta^{n+1}}(\tau) = 0.$$

From here, using Taylor's formula, we get

$$\begin{aligned} f(t) &= h_n(t, s) \\ &= \sum_{k=0}^{n} (-1)^k g_k(\alpha, t) f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^n(t)} (-1)^n g_n(\sigma(\tau), t) f^{\Delta^{n+1}}(\tau) \Delta\tau \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&= \sum_{k=0}^{n} (-1)^k g_k(s,t) f^{\Delta^k}(s) + \int_s^{\rho^n(t)} (-1)^n g_n(\sigma(\tau),t) f^{\Delta^{n+1}}(\tau)\Delta\tau \\
&= \sum_{k=0}^{n-1} (-1)^k g_k(s,t) f^{\Delta^k}(s) + (-1)^n g_n(s,t) f^{\Delta^n}(s) \\
&= (-1)^n g_n(s,t) f^{\Delta^n}(s) \\
&= (-1)^n g_n(s,t),
\end{aligned}
$$

i.e.,

$$h_n(t,s) = (-1)^n g_n(s,t),$$

completing the proof. □

From these theorems, the following result is clear.

Theorem (Taylor's Formula)

Let $n \in \mathbb{N}$. Suppose that f is n times differentiable on $\mathbb{T}^{\kappa^n}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$ and $t \in \mathbb{T}$. Then

$$f(t) = \sum_{k=0}^{n-1} h_k(t, \alpha) f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau.$$

Corollary

If $\alpha \in [a, b]$, then

$$\sum_{k=0}^{\infty} \alpha^k h_k(t, \alpha)$$

is absolutely and uniformly convergent on the interval $a \leq t \leq b$.

Now, we assume that $\mathbb{T}$ is unbounded above.

Definition

Suppose that the real-valued function f is defined on $[a, \infty)$ and is integrable from a to any point $A \in \mathbb{T}$, $A \geq a$. Assume that the integral

$$F(A) = \int_a^A f(t)\Delta t$$

approaches a finite limit as $A \to \infty$. Then we call that limit the *improper integral of the first kind* from a to ∞ and write

$$\int_a^\infty f(t)\Delta t = \lim_{A\to\infty} \left\{ \int_a^A f(t)\Delta t \right\}. \tag{1}$$

Definition

In such a case, we say that the improper integral

$$\int_a^\infty f(t)\Delta t \tag{2}$$

exists or that it is *convergent*. If the limit does not exist, then the improper integral (2) is said to be not existent or *divergent*.

Example

Let $\mathbb{T} = \mathbb{Z}$. We consider

$$I = \int_1^\infty \frac{3t^2 + 3t + 1}{t^3(t+1)^3} \Delta t.$$

We have $\sigma(t) = t + 1$ and

$$\begin{aligned}
\left(\frac{1}{t^3}\right)^\Delta &= -\frac{(t^3)^\Delta}{t^3(\sigma(t))^3} \\
&= -\frac{(\sigma(t))^2 + t\sigma(t) + t^2}{t^3(t+1)^3} \\
&= -\frac{(t+1)^2 3 + t(t+1) + t^2}{t^3(t+1)^3} \\
&= -\frac{t^2 + 2t + 1 + t^2 + t + t^2}{t^3(t+1)^3}
\end{aligned}$$

$$3t^2 + 3t + 1$$

Example

Therefore,

$$\begin{aligned}
\int_1^\infty \frac{3t^2+3t+1}{t^3(t+1)^3}\Delta t &= \lim_{A\to\infty}\int_1^A \frac{3t^2+3t+1}{t^3(t+1)^3}\Delta t \\
&= -\lim_{A\to\infty}\int_1^A \left(\frac{1}{t^3}\right)^\Delta \Delta t \\
&= -\lim_{A\to\infty} \frac{1}{t^3}\Big|_{t=1}^{t=A} \\
&= -\lim_{A\to\infty}\left(\frac{1}{A^3}-1\right) \\
&= 1.
\end{aligned}$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We consider

$$I = \int_1^\infty \frac{1}{t^3} \Delta t.$$

Here, $\sigma(t) = 2t$ and

$$\begin{aligned} \left(\frac{1}{t^2}\right)^\Delta &= -\frac{(t^2)^\Delta}{t^2(\sigma(t))^2} \\ &= -\frac{\sigma(t)+t}{4t^4} \\ &= -\frac{2t+t}{4t^4} \\ &= -\frac{3}{4t^3}, \end{aligned}$$

whereupon

Example

$$\frac{1}{t^3} = -\frac{4}{3}\left(\frac{1}{t^2}\right)^{\Delta}.$$

Therefore

$$\begin{aligned}
I &= \lim_{A\to\infty}\int_1^A \frac{1}{t^2}\Delta t \\
&= -\frac{4}{3}\lim_{A\to\infty}\int_1^A \left(\frac{1}{t^2}\right)^{\Delta}\Delta t \\
&= -\frac{4}{3}\lim_{A\to\infty}\frac{1}{t^2}\Big|_{t=1}^{t=A} \\
&= -\frac{4}{3}\lim_{A\to\infty}\left(\frac{1}{A^2}-1\right) \\
&= \frac{4}{3}.
\end{aligned}$$

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. We consider

$$I = \int_1^\infty \frac{1}{\sqrt{t}} \Delta t.$$

Here, $\sigma(t) = 3t$ and

$$\begin{aligned}
(\sqrt{t})^\Delta &= \frac{\sqrt{\sigma(t)} - \sqrt{t}}{\sigma(t) - t} \\
&= \frac{\sqrt{\sigma(t)} - \sqrt{t}}{(\sqrt{\sigma(t)} - \sqrt{t})(\sqrt{\sigma(t)} + \sqrt{t})} \\
&= \frac{1}{\sqrt{\sigma(t)} + \sqrt{t}} \\
&= \frac{1}{(1+\sqrt{3})\sqrt{t}},
\end{aligned}$$

whereupon

Example

$$\frac{1}{\sqrt{t}} = (1+\sqrt{3})(\sqrt{t})^{\Delta}.$$

Therefore,

$$\begin{aligned} I &= \lim_{A\to\infty}\int_1^A \frac{1}{\sqrt{t}}\Delta t \\ &= (1+\sqrt{3})\lim_{A\to\infty}\int_1^A (\sqrt{t})^{\Delta}\Delta t \\ &= (1+\sqrt{3})\lim_{A\to\infty}\sqrt{t}\Big|_{t=1}^{t=A} \\ &= (1+\sqrt{3})\lim_{A\to\infty}(\sqrt{A}-1) \\ &= \infty. \end{aligned}$$

Consequently, this improper integral diverges to ∞.

Theorem (Cauchy's Criterion)

For the existence of the integral (1), *it is necessary and sufficient that for any given* $\varepsilon > 0$, *there exists* $A_0 > a$ *such that*

$$\left| \int_{A_1}^{A_2} f(t)\Delta t \right| < \varepsilon \tag{3}$$

for any $A_1, A_2 \in \mathbb{T}$ *satisfying the inequalities* $A_1 > A_0$ *and* $A_2 > A_0$.

Proof.

The convergence of the integral (1) is equivalent to the existence of the limit $\lim_{A\to\infty} F(A)$. Using Cauchy's criterion for the existence of the limit of a function, it follows that the existence of the integral (1) is equivalent to the condition (3). □

Example

Let $\mathbb{T} = 3\mathbb{Z}$. We prove that the integral

$$\int_1^\infty \frac{1}{t^2 + 11t + 28}\Delta t$$

is convergent. Note that $\sigma(t) = t + 3$ and

$$\begin{aligned}
\left(\frac{1}{t+4}\right)^\Delta &= -\frac{(t+4)^\Delta}{(t+4)(\sigma(t)+4)} \\
&= -\frac{1}{(t+4)(t+7)} \\
&= -\frac{1}{t^2+11t+28}.
\end{aligned}$$

Example

Let $\varepsilon > 0$ be arbitrarily chosen. We take

$$A > \max\left\{\frac{2-4\varepsilon}{\varepsilon}, 1\right\}.$$

Then, for any $A_1, A_2 > A$, we have

$$\begin{aligned} \left|\int_{A_1}^{A_2} \frac{1}{t^2+11t+28}\Delta t\right| &= \left|-\int_{A_1}^{A_2}\left(\frac{1}{t+4}\right)^{\Delta}\Delta t\right| \\ &= \left|\frac{1}{t+4}\Big|_{t=A_1}^{t=A_2}\right| \end{aligned}$$

Example

$$
\begin{aligned}
&= \left|\frac{1}{A_2+4}-\frac{1}{A_1+4}\right| \\
&\le \frac{1}{A_2+4}+\frac{1}{A_1+4} \\
&< \frac{2}{A+4} \\
&< \varepsilon.
\end{aligned}
$$

Therefore, employing the Cauchy criterion, we conclude that the considered integral is convergent.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We investigate the integral

$$\int_1^\infty \frac{3t+7}{2(t+2)(t+3)(t+4)(2t+3)} \Delta t$$

for convergence. Here, $\sigma(t) = 2t$ and

$$\begin{aligned}
\left(\frac{1}{(t+3)(t+4)}\right)^\Delta &= -\frac{((t+3)(t+4))^\Delta}{(t+3)(t+4)(\sigma(t)+3)(\sigma(t)+4)} \\
&= -\frac{(t+3)^\Delta(t+4)+(\sigma(t)+3)(t+4)^\Delta}{2(t+2)(t+3)(t+4)(2t+3)} \\
&= -\frac{t+4+2t+3}{2(t+2)(t+3)(t+4)(2t+3)} \\
&= -\frac{3t+7}{2(t+2)(t+3)(t+4)(2t+3)}.
\end{aligned}$$

Example

Let $\varepsilon > 0$ be arbitrarily chosen. We take $A > 1$ such that

$$A^2 + 7A + 21 - \frac{2}{\varepsilon} > 0.$$

Then, for any $A_1, A_2 > A$, we have

$$\begin{aligned}
&\left| \int_{A_1}^{A_2} \frac{3t+7}{2(t+2)(t+3)(t+4)(2t+3)} \Delta t \right| \\
&= \left| - \int_{A_1}^{A_2} \left(\frac{1}{(t+3)(t+4)} \right)^{\Delta} \Delta t \right| \\
&= \left| - \frac{1}{(t+3)(t+4)} \Big|_{t=A_1}^{t=A_2} \right| \\
&= \left| - \frac{1}{(A_2+3)(A_2+4)} + \frac{1}{(A_1+3)(A_1+4)} \right|
\end{aligned}$$

Example

$$
\begin{aligned}
&\leq \frac{1}{(A_2+3)(A_2+4)} + \frac{1}{(A_1+3)(A_1+4)} \\
&< \frac{2}{(A+3)(A+4)} \\
&< \varepsilon.
\end{aligned}
$$

Thus, using the Cauchy criterion, it follows that the considered integral is convergent.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We investigate the integral

$$\int_1^\infty \frac{2\sin t - \sin(2t)}{2t^2}\Delta t$$

for convergence. Here, $\sigma(t) = 2t$ and

$$\begin{aligned}
\left(\frac{\sin t}{t}\right)^\Delta &= \frac{t(\sin t)^\Delta - t^\Delta \sin t}{t\sigma(t)} \\
&= \frac{t\frac{\sin(2t)-\sin t}{2t-t} - \sin t}{2t^2} \\
&= \frac{\sin(2t) - 2\sin t}{2t^2}.
\end{aligned}$$

Example

Let $\varepsilon > 0$ be arbitrarily chosen. We take $A > \frac{2}{\varepsilon}$. Then, for any $A_1, A_2 > A$, we have

$$\begin{aligned} \left| \int_{A_1}^{A_2} \frac{2\sin t - t\sin(2t)}{2t^2} \Delta t \right| &= \left| - \int_{A_1}^{A_2} \left(\frac{\sin t}{t} \right)^{\Delta} \Delta t \right| \\ &= \left| - \frac{\sin t}{t} \Big|_{t=A_1}^{t=A_2} \right| \\ &= \left| - \frac{\sin A_2}{A_2} + \frac{\sin A_1}{A_1} \right| \end{aligned}$$

Example

$$
\begin{aligned}
&\leq \frac{|\sin A_2|}{A_2} + \frac{|\sin A_1|}{A_1} \\
&\leq \frac{1}{A_2} + \frac{1}{A_1} \\
&< \frac{2}{A} \\
&< \varepsilon.
\end{aligned}
$$

Thus, using the Cauchy criterion, we conclude that the considered integral is convergent.