Time Scales Analysis
Lecture 18
Improper Integrals

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Definition

An improper integral of the first kind is said to be *absolutely convergent* provided the integral

$$\int_a^{\infty} |f(t)|\Delta t \tag{1}$$

of the modulus of the function f is convergent. If an improper integral of the first kind is convergent but not absolutely convergent, then it is called *conditionally convergent*.

Theorem

If the integral

$$\int_a^\infty f(t)\Delta t \tag{2}$$

is absolutely convergent, then it is convergent.

Proof.

Let (2) be absolutely convergent. Then the integral (1) is convergent. Suppose that $\varepsilon > 0$ is arbitrarily chosen. Hence, employing the Cauchy criterion, it follows that there exists $A > a$ such that for any $A_1, A_2 > A$, we have

$$\left| \int_{A_1}^{A_2} |f(t)|\Delta t \right| < \varepsilon.$$

From here, for any $A_1, A_2 > A$, we have

$$\left| \int_{A_1}^{A_2} f(t)\Delta t \right| \leq \left| \int_{A_1}^{A_2} |f(t)|\Delta t \right| < \varepsilon,$$

Theorem

An integral (2) with $f(t) \geq 0$ for all $t \geq a$ is convergent if and only if there exists a constant $M > 0$ such that

$$\int_a^A f(t)\Delta t \leq M \quad \textit{whenever} \quad A \geq a.$$

Proof.

1. Let $F(A) = \int_a^A f(t)\Delta t \leq M$ whenever $A \geq a$. Then

$$\int_a^\infty f(t)\Delta t = \lim_{A\to\infty} F(A) \leq M.$$

Therefore, the integral (2) is convergent.

2. Let the integral (2) be convergent. Assume that the function $F(A)$, $A \geq a$, is unbounded. Then

$$\int_a^\infty f(t)\Delta t = \lim_{A\to\infty} F(A) = \infty,$$

which is a contradiction.

□

Theorem

Let the inequalities $0 \leq f(t) \leq g(t)$ be satisfied for all $t \in [a, \infty)$. Then the convergence of the improper integral

$$\int_a^\infty g(t)\Delta t \tag{3}$$

implies the convergence of the improper integral (2), while the divergence of the improper integral (2) implies the divergence of the improper integral (3).

Proof.

Since $0 \leq f(t) \leq g(t)$ for any $t \in [a, \infty)$, we get

$$0 \leq \int_a^A f(t)\Delta t \leq \int_a^A g(t)\Delta t \quad \text{for any} \quad A \in [a, \infty),$$

which completes the proof. □

Example

Let $\mathbb{T} = \mathbb{Z}$. Consider the integral

$$I = \int_1^\infty \log \frac{t+1}{t}(t^6 + 7t^3 + 100)\Delta t.$$

Here, $\sigma(t) = t + 1$ and

$$\log \frac{t+1}{t}(t^6 + 7t^3 + 100) \geq \log \frac{t+1}{t} \quad \text{for any} \quad t \in [1, \infty)$$

and

$$I \geq \int_1^\infty \log \frac{t+1}{t} \Delta t.$$

Note that

$$\begin{aligned} (\log t)^\Delta &= \frac{\log \sigma(t) - \log t}{\sigma(t) - t} \\ &= \log(t+1) - \log t \end{aligned}$$

Example

Therefore,

$$\begin{aligned}\int_1^{\infty} \log \frac{t+1}{t} \Delta t &= \lim_{A\to\infty} \int_1^{A} (\log t)^{\Delta} \Delta t \\ &= \lim_{A\to\infty} \log t \Big|_{t=1}^{t=A} \\ &= \infty.\end{aligned}$$

Hence, the improper integral I is divergent.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the integral

$$I = \int_1^\infty \frac{1}{t^3(t^2+5)(t^2+7t+1)} \Delta t.$$

Here, $\sigma(t) = 2t$ and

$$\frac{1}{t^3(t^2+5)(t^2+7t+1)} \leq \frac{1}{t^3} \quad \text{for any} \quad t \in [1, \infty).$$

Also,

$$\begin{aligned} \left(\frac{1}{t^2}\right)^\Delta &= -\frac{(t^2)^\Delta}{t^2(\sigma(t))^2} \\ &= -\frac{\sigma(t)+t}{4t^4} \\ &= -\frac{3}{4t^3}, \end{aligned}$$

Example

whereupon

$$\frac{1}{t^3} = -\frac{4}{3}\left(\frac{1}{t^2}\right)^{\Delta}.$$

Hence,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{t^3}\Delta t &= \lim_{A\to\infty}\int_1^{A} \frac{1}{t^3}\Delta t \\
&= -\frac{4}{3}\lim_{A\to\infty}\int_1^{A}\left(\frac{1}{t^2}\right)^{\Delta}\Delta t \\
&= -\frac{4}{3}\lim_{A\to\infty}\frac{1}{t^2}\Big|_{t=1}^{A} \\
&= \frac{4}{3}.
\end{aligned}$$

Therefore, the integral I is convergent.

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Consider the integral

$$I = \int_1^\infty \frac{\sin^3 t + e_1(t,1) + \cos^3 t + 5}{t} \Delta t.$$

Here, $\sigma(t) = 3t$ and

$$\frac{\sin^3 t + e_1(t,1) + \cos^3 t + 5}{t} \geq \frac{1}{t} \quad \text{for any} \quad t \in [1, \infty),$$

whereupon

$$I \geq \int_1^\infty \frac{1}{t} \Delta t.$$

Also,

Example

$$
\begin{aligned}
(\log t)^{\Delta} &= \frac{\log(\sigma(t)) - \log t}{\sigma(t) - t} \\
&= \frac{\log(3t) - \log t}{2t} \\
&= \frac{\log 3}{2t},
\end{aligned}
$$

from where

$$\frac{1}{t} = \frac{2}{\log 3}(\log t)^{\Delta}.$$

Hence,

Example

$$\int_1^\infty \frac{1}{t}\Delta t = \frac{2}{\log 3}\lim_{A\to\infty}\int_1^A (\log t)^\Delta \Delta t$$
$$= \frac{2}{\log 3}\lim_{A\to\infty}\log t\Big|_{t=1}^{t=A}$$
$$= \infty.$$

Therefore, the considered integral is divergent.

Theorem

Let $|f(t)| \leq g(t)$ for all $t \in \mathbb{T}$ with $t \geq a$. Then the convergence of the integral $\int_a^\infty g(t)\Delta t$ implies the convergence of the integral $\int_a^\infty f(t)\Delta t$.

Proof.

Since $\int_a^\infty g(t)\Delta t$ is convergent, using Theorem 4, we have that the integral $\int_a^\infty |f(t)|\Delta t$ is convergent. Therefore, the integral $\int_a^\infty f(t)\Delta t$ is absolutely convergent. From here and from Theorem 2, it follows that the integral $\int_a^\infty f(t)\Delta t$ is convergent. □

Theorem (Comparison Criterion)

Let $\int_a^\infty f(t)\Delta t$ *and* $\int_a^\infty g(t)\Delta t$ *be improper integrals of the first kind with positive integrands. Suppose that the limit*

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = L \tag{4}$$

exists (finite) and is not zero. Then the integrals are simultaneously convergent or divergent.

Proof.

Let $\varepsilon \in (0, L)$ be arbitrarily chosen. From (4), it follows that there exists $A_0 > a$ such that

$$L - \varepsilon \leq \frac{f(t)}{g(t)} \leq L + \varepsilon \quad \text{for any} \quad t \geq A_0,$$

from where

$$(L - \varepsilon)g(t) \leq f(t) \leq (L + \varepsilon)g(t) \quad \text{for any} \quad t \geq A_0.$$

Hence,

$$(L - \varepsilon) \int_{A_0}^{\infty} g(t)\Delta t \leq \int_{A_0}^{\infty} f(t)\Delta t \leq (L + \varepsilon) \int_{A_0}^{\infty} g(t)\Delta t. \qquad (5)$$

□

Proof.

Let $\int_a^\infty g(t)\Delta t$ be convergent. Then $\int_{A_0}^\infty g(t)\Delta t$ is convergent. Hence,

$$(L+\varepsilon)\int_{A_0}^\infty g(t)\Delta t$$

is convergent. From here and from Theorem 4, using (5), we obtain that $\int_{A_0}^\infty f(t)\Delta t$ is convergent. Therefore, $\int_a^\infty f(t)\Delta t$ is convergent.
Let $\int_a^\infty f(t)\Delta t$ be convergent. Then $\int_{A_0}^\infty f(t)\Delta t$ is convergent. From here and from Theorem 4, using (5), we obtain that $\int_{A_0}^\infty g(t)\Delta t$ is convergent. Therefore, $\int_a^\infty g(t)\Delta t$ is convergent. □

Proof.

Let $\int_a^\infty f(t)\Delta t$ be divergent. Hence, $\int_{A_0}^\infty f(t)\Delta t$ is divergent. Thus, using (5), it follows that $\int_{A_0}^\infty g(t)\Delta t$ is divergent. Therefore, $\int_a^\infty g(t)\Delta t$ is divergent.

Let $\int_a^\infty g(t)\Delta t$ be divergent. Hence, $\int_{A_0}^\infty g(t)\Delta t$ is divergent. Thus, using (5), it follows that $\int_{A_0}^\infty f(t)\Delta t$ is divergent. Therefore, $\int_a^\infty f(t)\Delta t$ is divergent. □

Example

Let $\mathbb{T} = \mathbb{Z}$. Consider the integral

$$I = \int_1^{\infty} \frac{t^4}{(t^2 + 11t + 30)(t^4 + t^3 + t^2 + 1)} \Delta t.$$

Define

$$f(t) = \frac{t^4}{(t^2 + 11t + 30)(t^4 + t^3 + t^2 + 1)}, \quad g(t) = \frac{1}{t^2 + 11t + 30}$$

and

$$J = \int_1^{\infty} \frac{1}{t^2 + 11t + 30} \Delta t.$$

We have

Example

$$\lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{t\to\infty}\frac{\frac{t^4}{(t^2+11t+30)(t^4+t^3+t^2+1)}}{\frac{1}{t^2+11t+30}}$$

$$= \lim_{t\to\infty}\frac{t^4}{t^4+t^3+t^2+1}$$

$$= 1.$$

Thus, using Theorem 13, it follows that the integrals I and J are simultaneously convergent or divergent. Note that

Example

$$
\begin{aligned}
\left(\frac{1}{t+5}\right)^{\Delta} &= -\frac{(t+5)^{\Delta}}{(t+5)(\sigma(t)+5)} \\
&= -\frac{1}{(t+5)(t+6)} \\
&= -\frac{1}{t^2+11t+30}.
\end{aligned}
$$

Example

Therefore,

$$\begin{aligned} J &= -\lim_{A\to\infty}\int_1^A \left(\frac{1}{t+5}\right)^{\Delta} \Delta t \\ &= -\lim_{A\to\infty} \frac{1}{t+5}\Big|_{t=1}^{t=A} \\ &= \frac{1}{6}. \end{aligned}$$

Consequently, the integral I is convergent.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the integral

$$I = \int_1^\infty (t^2 + 2t + 3)\Delta t.$$

Let

$$f(t) = t^2 + 2t + 3, \quad g(t) = t^2, \quad J = \int_1^\infty g(t)\Delta t.$$

We have that

$$\begin{aligned} \lim_{t\to\infty} \frac{f(t)}{g(t)} &= \lim_{t\to\infty} \frac{t^2 + 2t + 3}{t^2} \\ &= 1. \end{aligned}$$

Example

Thus, employing Theorem 13, it follows that the integrals I and J are simultaneously convergent or divergent. Since

$$\begin{aligned} J &= \lim_{A\to\infty} \int_1^A t^2 \Delta t \\ &= \frac{1}{7} \lim_{A\to\infty} \int_1^A (t^3)^{\Delta} \Delta t \\ &= \frac{1}{7} \lim_{A\to\infty} t^3 \Big|_{t=1}^{t=A} \\ &= \infty, \end{aligned}$$

we conclude that I is divergent.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the integral

$$I = \int_1^\infty \frac{e^{-t} - e^{-2t}}{t^2}(t+1)\Delta t.$$

We set

$$f(t) = \frac{e^{-t} - e^{-2t}}{t^2}(t+1), \quad g(t) = \frac{e^{-t} - e^{-2t}}{t}$$

and

$$J = \int_1^\infty \frac{e^{-t} - e^{-2t}}{t}\Delta t.$$

Then

Example

$$\lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{t\to\infty}\frac{\frac{e^{-t}-e^{-2t}}{t^2}(t+1)}{\frac{e^{-t}-e^{-2t}}{t}}$$

$$= \lim_{t\to\infty}\frac{t+1}{t}$$

$$= 1.$$

Using Theorem 13, we conclude that the integrals I and J are simultaneously convergent or divergent. Note that

$$(e^{-t})^{\Delta} = \frac{e^{-\sigma(t)}-e^{-t}}{\sigma(t)-t}$$

$$= \frac{e^{-2t}-e^{-t}}{t}.$$

Hence,

Example

$$\begin{aligned} J &= -\lim_{A\to\infty}\int_1^A (e^{-t})^{\Delta}\Delta t \\ &= -\lim_{A\to\infty} e^{-t}\Big|_{t=1}^{t=A} \\ &= \frac{1}{e}. \end{aligned}$$

Consequently, the integral I is convergent.

Theorem

Let f be integrable from a to any point $A \in \mathbb{T}$, $A > a$. Suppose that the integral

$$F(A) = \int_a^A f(t)\Delta t$$

is bounded for any $A \geq a$. Suppose that g is monotone on $[a, \infty)$ and $\lim_{t\to\infty} g(t) = 0$. Then the improper integral of the first kind of the form

$$\int_a^{\infty} f(t)g(t)\Delta t \tag{6}$$

is convergent.

Proof.

Let $A_1, A_2 \in \mathbb{T}$, $A_2 > A_1 \geq a$. By the mean value theorem, Theorem **??**, it follows that there exists Λ between $\inf_{A \in [A_1, A_2]} F(A)$ and $\sup_{A \in [A_1, A_2]} F(A)$ such that

$$\int_{A_1}^{A_2} f(t)g(t)\Delta t = (g(A_1) - g(A_2))\Lambda + g(A_2) \int_{A_1}^{A_2} f(t)\Delta t. \qquad (7)$$

Let $M > 0$ be a constant such that

$$|F(A)| \leq M \quad \text{on} \quad [a, \infty).$$

□

Proof.

From (7), we get

$$\int_{A_1}^{A_2} f(t)g(t)\Delta t = (g(A_1) - g(A_2))\Lambda + g(A_2)(F(A_2) - F(A_1))$$

and

$$\begin{aligned}\left|\int_{A_1}^{A_2} f(t)g(t)\Delta t\right| &= |(g(A_1) - g(A_2))\Lambda + g(A_2)(F(A_2) - F(A_1))| \\ &\leq |g(A_1)||\Lambda| + |g(A_2)||\Lambda| + |g(A_2)|(|F(A_2)| + |F(A_1)|) \\ &\leq M|g(A_1)| + 3M|g(A_2)| \\ &= M(|g(A_1)| + 3|g(A_2)|). \end{aligned} \tag{8}$$

□

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since $g(t) \to 0$ as $t \to \infty$, there exists $A_3 > a$ such that

$$|g(A)| < \frac{\varepsilon}{4M} \quad \text{for any} \quad A > A_3.$$

Hence, using (8), for $A_1, A_2 > A_3$, we get

$$\begin{aligned} \left| \int_{A_1}^{A_2} f(t)g(t)\Delta t \right| &< M\left(\frac{\varepsilon}{4M} + \frac{3\varepsilon}{4M}\right) \\ &= \varepsilon. \end{aligned}$$

From here and from Cauchy's criterion, Theorem **??**, it follows that the integral (6) is convergent. □

Example

Let $\mathbb{T} = \mathbb{Z}$. Consider the integral

$$I = \int_1^\infty \frac{t \sin t}{(t^2 + 3t + 2)(t^2 + 1)} \Delta t.$$

Let

$$f(t) = \frac{\sin t}{t^2 + 3t + 2}, \quad g(t) = \frac{t}{t^2 + 1}.$$

We have

$$\begin{aligned} g^\Delta(t) &= \frac{t^\Delta(t^2+1) - t(t^2+1)^\Delta}{(t^2+1)((\sigma(t))^2+1)} \\ &= \frac{t^2+1-t(\sigma(t)+t)}{(t^2+1)((t+1)^2+1)} \\ &= \frac{t^2+1-t(2t+1)}{(t^2+1)(t^2+2t+2)} \end{aligned}$$

Example

$$= \frac{t^2 + 1 - 2t^2 - t}{(t^2+1)(t^2+2t+2)}$$

$$= \frac{1 - t - t^2}{(t^2+1)(t^2+2t+2)}.$$

Therefore, the function g is monotone on $[1, \infty)$. Moreover,

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{t}{t^2+1} = 0$$

and

$$\left|\int_1^\infty f(t)\Delta t\right| \leq \int_1^\infty |f(t)|\Delta t$$

$$= \int_1^\infty \frac{|\sin t|}{(t+1)(t+2)}\Delta t$$

Example

$$
\begin{aligned}
&\leq \int_1^{\infty} \frac{1}{(t+1)(t+2)} \Delta t \\
&= -\int_1^{\infty} \left(\frac{1}{t+1}\right)^{\Delta} \Delta t \\
&= -\frac{1}{t+1}\Big|_{t=1}^{t=\infty} \\
&= \frac{1}{2}.
\end{aligned}
$$

Thus, using Theorem 23, it follows that the integral I is convergent.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the integral

$$I = \int_1^{\infty} \frac{\sin^2 t + \cos t + 3}{t^2(t^2+1)} \Delta t.$$

Let

$$f(t) = \frac{\sin^2 t + \cos t + 3}{t^2}, \quad g(t) = \frac{1}{t^2+1}.$$

Then

$$\begin{aligned} \left(\frac{1}{t^2+1}\right)^{\Delta} &= -\frac{(t^2+1)^{\Delta}}{(t^2+1)((\sigma(t))^2+1)} \\ &= -\frac{\sigma(t)+t}{(t^2+1)(4t^2+1)} \\ &= -\frac{3t}{(t^2+1)(4t^2+1)}. \end{aligned}$$

Example

Therefore, the function g is monotone on $[1,\infty)$. Moreover,

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{1}{t^2+1} = 0$$

and

$$\begin{aligned}
\left|\int_1^\infty f(t)\Delta t\right| &\leq \int_1^\infty |f(t)|\Delta t \\
&= \int_1^\infty \frac{|\sin^2 t + \cos t + 3|}{t^2}\Delta t \\
&\leq \int_1^\infty \frac{\sin^2 t + |\cos t| + 3}{t^2}\Delta t
\end{aligned}$$

Example

$$
\begin{aligned}
&\leq 5\int_1^\infty \frac{1}{t^2}\Delta t \\
&= -10\int_1^\infty \left(\frac{1}{t}\right)^\Delta \Delta t \\
&= -10\frac{1}{t}\Big|_{t=1}^{t=\infty} \\
&= 10.
\end{aligned}
$$

Therefore, using Theorem 23, it follows that the integral I is convergent.

Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Consider the integral

$$I = \int_1^{\infty} \frac{1}{t(t^{10} + t^{11} + t^{12} + 1)} \Delta t.$$

We set

$$f(t) = \frac{1}{t^{10} + t^{11} + t^{12} + 1}, \quad g(t) = \frac{1}{t}.$$

Then

$$g^{\Delta}(t) = -\frac{t^{\Delta}}{t\sigma(t)} = -\frac{1}{3t^2}.$$

Therefore, the function g is a monotonic function on $[1, \infty)$. Also,

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{1}{t} = 0$$

and

Example

$$\begin{aligned}\left|\int_1^\infty f(t)\Delta t\right| &\leq \int_1^\infty |f(t)|\Delta t \\ &\leq \int_1^\infty \frac{1}{t^{10}+t^{11}+t^{12}+1}\Delta t \\ &\leq \int_1^\infty \frac{1}{t^2}\Delta t \\ &= -3\int_1^\infty \left(\frac{1}{t}\right)^\Delta \Delta t \\ &= -3\frac{1}{t}\Big|_{t=1}^{t=\infty} \\ &= 3.\end{aligned}$$

Hence, using Theorem 23, it follows that the integral I is convergent.

Example

Let $\mathbb{T}$ be a time scale of the form

$$\mathbb{T} = \{t_k : \; k \in \mathbb{N}_0\} \quad \text{with} \quad 0 < t_0 < t_1 < \dots \quad \text{and} \quad \lim_{k\to\infty} t_k = \infty. \qquad (9)$$

Suppose that $f : [t_0, \infty) \to \mathbb{R}$ is nonincreasing with $\int_{t_0}^{\infty} f(t)\Delta t < \infty$. Assume that $g : \mathbb{T} \to \mathbb{R}_+$ satisfies

$$g(t_k) \leq Kf(t_{k+1}) \quad \text{for all} \quad k \in \mathbb{N}_0,$$

where $K > 0$ is a constant. We will prove that $\int_{t_0}^{\infty} g(t)\Delta t < \infty$.

Example

We have

$$\begin{aligned} \int_{t_0}^{\infty} g(t)\Delta t &= \sum_{k=0}^{\infty} g(t_k)\mu(t_k) \\ &\leq K \sum_{k=0}^{\infty} f(t_{k+1})\mu(t_k) \\ &\leq K \sum_{k=0}^{\infty} f(t_k)\mu(t_k) \\ &= K \int_{t_0}^{\infty} f(t)\Delta t \\ &< \infty. \end{aligned}$$

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. We will prove that the integral

$$\int_1^{\infty} \frac{1}{t^p} \Delta t$$

is divergent for $p \in [0, 1]$ and convergent for $p > 1$.
Let $t_k = 2^k$, $k \in \mathbb{N}_0$. Then

$$t_{k+1} = 2^{k+1}, \quad \mu(t_k) = t_{k+1} - t_k = 2^{k+1} - 2^k$$

and

Example

$$\begin{aligned}
\int_1^\infty \frac{1}{t^p}\Delta t &= \sum_{k=0}^{\infty} \frac{1}{t_k^p}\mu(t_k) \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{kp}}(2^{k+1} - 2^k) \\
&= 2\sum_{k=0}^{\infty} \frac{1}{2^{k(p-1)}} - \sum_{k=0}^{\infty} \frac{1}{2^{k(p-1)}} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{k(p-1)}} \begin{cases} = \infty & \text{if} \quad p \in [0,1] \\ < \infty & \text{if} \quad p > 1. \end{cases}
\end{aligned}$$

Theorem

Let $\mathbb{T}$ be a time scale that satisfies (9). If $f : [t_0, \infty) \to \mathbb{R}$ is nonincreasing, then

$$\int_{t_0}^{\infty} f(t)\nabla t \leq \int_{t_0}^{\infty} f(t)\mathrm{d}t \leq \int_{t_0}^{\infty} f(t)\Delta t,$$

where the first and last integrals are taken over $\mathbb{T}$, while the middle integral is taken over the interval $[t_0, \infty)$ of $\mathbb{R}$.

Proof.

Since $\mathbb{T}$ is a time scale that satisfies (9), we have

$$\int_{t_0}^{\infty} f(t)\nabla t = \sum_{k=0}^{\infty} f(t_{k+1})(t_{k+1} - t_k) \quad \text{and} \quad \int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} f(t_k)(t_{k+1} - t$$

□

Proof.

Because the function f is nonincreasing on $[t_0, \infty)$, we get

$$f(t_{k+1})(t_{k+1} - t_k) \leq \int_{t_k}^{t_{k+1}} f(t)\mathrm{d}t \leq f(t_k)(t_{k+1} - t_k)$$

for all $k \in \mathbb{N}_0$. Hence,

$$\sum_{k=0}^{\infty} f(t_{k+1})(t_{k+1} - t_k) \leq \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} f(t)\mathrm{d}t \leq \sum_{k=0}^{\infty} f(t_k)(t_{k+1} - t_k),$$

which completes the proof. □

Corollary

Let $\mathbb{T}$ *be a time scale that satisfies* (9) *and assume that* $f : [t_0, \infty) \to \mathbb{R}_+$ *is nonincreasing.*

1. *If* $\int_{t_0}^{\infty} f(t)\mathrm{d}t = \infty$, *then* $\int_{t_0}^{\infty} f(t)\Delta t = \infty$.
2. *If* $\int_{t_0}^{\infty} f(t)\mathrm{d}t < \infty$, *then* $\int_{t_0}^{\infty} f(t)\nabla t < \infty$.

Theorem

Let $\mathbb{T}$ *be a time scale that satisfies* (9). *Then*

$$\int_{t_0}^{\infty} \frac{1}{t^p} \Delta t = \infty \quad \text{if} \quad 0 \le p \le 1. \tag{10}$$

Proof.

Let $f(t) = \frac{1}{t^p}$. Then f is nonincreasing on $[t_0, \infty)$ and

$$\int_{t_0}^{\infty} \frac{1}{t^p} \mathrm{d}t = \infty.$$

Thus, employing Corollary 37, we get (10). □

Theorem

Let $\mathbb{T}$ *be a time scale satisfying* (9). *Then*

$$\int_{t_0}^{\infty} \frac{\nabla t}{t^p} < \infty \quad \textit{if} \quad p > 1. \tag{11}$$

Proof.

Let $f(t) = \frac{1}{t^p}$. Then f is nonincreasing on $[t_0, \infty)$. Since

$$\int_{t_0}^{\infty} \frac{1}{t^p} \mathrm{d}t < \infty \quad \text{for} \quad p > 1,$$

using Corollary 6, we get (11). □

Theorem

Let $\mathbb{T}$ *be a time scale satisfying* (9). *Then*

$$\int_{t_0}^{\infty} \frac{\nabla t}{t^p} = \infty \quad \textit{if} \quad 0 \leq p \leq 1. \tag{12}$$

Proof.

. Firstly, we prove that

$$\int_{t_0}^{\infty} \frac{\nabla t}{t} = \infty. \tag{13}$$

We have

Proof.

$$\int_{t_0}^{\infty} \frac{\nabla t}{t} = \sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_{k+1}}.$$

Assume that

$$\sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_{k+1}} < \infty.$$

Then

$$\lim_{k \to \infty} \frac{t_{k+1} - t_k}{t_{k+1}} = 0,$$

whereupon

$$\lim_{k \to \infty} \frac{t_{k+1}}{t_k} = 1.$$

Since $t_k < t_{k+1}$ for any $k \in \mathbb{N}_0$, there exists $N \in \mathbb{N}$ such that

$$\frac{t_{k+1}}{t_k} < 2 \quad \text{for any} \quad k > N.$$

Proof.

We note that by Theorem 38, we have

$$\sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_k} = \infty.$$

On the other hand, we have

$$\sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_k} = \sum_{k=0}^{N-1} \frac{t_{k+1} - t_k}{t_k} + \sum_{k=N}^{\infty} \frac{t_{k+1} - t_k}{t_k}$$

□

Proof.

$$\begin{aligned} &= \sum_{k=0}^{N-1} \frac{t_{k+1}-t_k}{t_k} + \sum_{k=N}^{\infty} \frac{t_{k+1}-t_k}{t_{k+1}} \frac{t_{k+1}}{t_k} \\ &< \sum_{k=0}^{N-1} \frac{t_{k+1}-t_k}{t_k} + 2\sum_{k=N}^{\infty} \frac{t_{k+1}-t_k}{t_{k+1}} \\ &< \infty, \end{aligned}$$

which is a contradiction. Therefore

$$\sum_{k=0}^{\infty} \frac{t_{k+1}-t_k}{t_{k+1}} = \infty,$$

from where (13) follows. □

Proof.

Let $p \in [0,1)$. Since $\lim_{k\to\infty} t_k = \infty$, there exists $t_l \in \mathbb{T}$ so that $t_l > 1$. Hence,

$$\int_{t_0}^{\infty} \frac{1}{t^p}\nabla t = \int_{t_0}^{t_l} \frac{1}{t^p}\nabla t + \int_{t_l}^{\infty} \frac{1}{t^p}\nabla t. \tag{14}$$

Because $t_l > 1$ and $p \in [0,1)$, we have

$$\int_{t_l}^{\infty} \frac{1}{t}\nabla t = \infty \quad \text{and} \quad \int_{t_l}^{\infty} \frac{1}{t^p}\nabla t \geq \int_{t_l}^{\infty} \frac{1}{t}\nabla t.$$

Thus, using (14), we get the desired result (12). This completes the proof. □

In the following situation, the ordinary Riemann integral of f on $[a, b]$ cannot exist since a Riemann integrable function from a to b must be bounded on $[a, b)$.

Definition

Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$. Suppose that b is left-dense. Assume that the function f is defined in the interval $[a, b)$. Suppose that f is integrable on any interval $[a, c]$ with $c < b$ and is unbounded on $[a, b)$. The formal expression

$$\int_a^b f(t)\Delta t \tag{15}$$

is called the *improper integral of the second kind*.

Definition

We say that the integral (15) is improper at $t = b$. We also say that f has a singularity at $t = b$. If the left-sided limit

$$\lim_{c \to b-} \int_a^c f(t)\Delta t \tag{16}$$

exists as a finite number, then the improper integral (15) is said to exist or be *convergent*. In such a case, we call this limit the value of the improper integral (15) and write

$$\int_a^b f(t)\Delta t = \lim_{c \to b-} \int_a^c f(t)\Delta t.$$

If the limit (16) does not exist, then the integral (15) is said to be not existent or *divergent*.

Example

Let $\mathbb{T} = [0,1] \cup 2^{\mathbb{N}}$, where $[0,1]$ is the real-valued interval. Define

$$f(t) = \begin{cases} \sqrt{1-t^2} & \text{for} \quad t \in [0,1] \\ \\ t^4 & \text{for} \quad t \in 2^{\mathbb{N}}. \end{cases}$$

Consider the integral

$$I = \int_0^8 \frac{1}{f(t)} \Delta t.$$

We have

$$\begin{aligned} I &= \int_0^1 \frac{1}{f(t)} \Delta t + \int_2^8 \frac{1}{f(t)} \Delta t \\ &= \int_0^1 \frac{1}{\sqrt{1-t^2}} \mathrm{d}t + \int_2^8 \frac{1}{t^4} \Delta t \end{aligned}$$

Example

$$
\begin{aligned}
&= \lim_{c \to 1-} \int_0^c \frac{1}{\sqrt{1-t^2}} \mathrm{d}t + \frac{1}{t^4}\mu(t)\Big|_{t=2} + \frac{1}{t^4}\mu(t)\Big|_{t=4} \\
&= \lim_{c \to 1-} \arcsin t \Big|_{t=0}^{t=c} + \frac{2}{16} + \frac{4}{256} \\
&= \frac{\pi}{2} + \frac{9}{64}.
\end{aligned}
$$

Therefore, the considered integral is convergent.

Example

Let $\mathbb{T} = \{-4, -2\} \cup [0,1]$, where $[0,1]$ is the real-valued interval. Consider the integral

$$I = \int_{-4}^{1} \frac{\Delta t}{\sqrt{1-t}}.$$

We have

$$\begin{aligned} I &= \int_{-4}^{-2} \frac{\Delta t}{\sqrt{1-t}} + \int_{0}^{1} \frac{\mathrm{d}t}{\sqrt{1-t}} \\ &= \frac{1}{\sqrt{1-t}}\mu(t)\Big|_{t=-4} + \lim_{c \to 1-} \int_{0}^{c} \frac{\mathrm{d}t}{\sqrt{1-t}} \\ &= \frac{2}{\sqrt{5}} - 2 \lim_{c \to 1-} \sqrt{1-t}\Big|_{t=0}^{t=c} \\ &= \frac{2}{\sqrt{5}} + 2. \end{aligned}$$

Therefore, the considered integral is convergent.

Example

Let $\mathbb{T} = \{-1, 0\} \cup [1, 2]$, where $[1, 2]$ is the real-valued interval. Consider the integral

$$I = \int_{-1}^{2} \frac{t^3}{\sqrt{4 - t^2}} \Delta t.$$

We have

$$\begin{aligned} I &= \int_{-1}^{0} \frac{t^3}{\sqrt{4-t^2}} \Delta t + \int_{1}^{2} \frac{t^3}{\sqrt{4-t^2}} \mathrm{d}t \\ &= \left. \frac{t^3 \mu(t)}{\sqrt{4-t^2}} \right|_{t=-1} - \lim_{c \to 2-} \int_{1}^{c} t^2 d\sqrt{4-t^2} \\ &= -\frac{1}{\sqrt{3}} - \lim_{c \to 2-} t^2 \sqrt{4-t^3} \Big|_{t=1}^{t=c} + 2 \lim_{c \to 2-} \int_{1}^{c} t\sqrt{4-t^2} \mathrm{d}t \end{aligned}$$

Example

$$
\begin{aligned}
&= -\frac{1}{\sqrt{3}} + \sqrt{3} - \lim_{c \to 2-} \int_1^c \sqrt{4 - t^2} \mathrm{d}(4 - t^2) \\
&= \frac{2\sqrt{3}}{3} - \lim_{c \to 2-} \frac{(4 - t^2)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_{t=1}^{t=c} \\
&= \frac{2\sqrt{3}}{3} + 2\sqrt{3} \\
&= \frac{8\sqrt{3}}{3}.
\end{aligned}
$$

Therefore, the considered integral is convergent.