Time Scales Analysis
Lecture 19

Improper Integrals. Multidimensional Time Scales

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Remark

All theorems for improper integrals of the first kind have exact analogues for improper integrals of the second kind.

1. For the existence of the improper integral of the second kind, it is necessary and sufficient that for any given $\varepsilon > 0$*, there exists* $b_0 < b$ *such that*

$$\left| \int_{c_1}^{c_2} f(t)\Delta t \right| < \varepsilon$$

for any $c_1, c_2 \in \mathbb{T}$ *satisfying the inequalities* $b_0 < c_1 < b$ *and* $b_0 < c_2 < b$*.*

Remark

2. Suppose that $f(t) \geq 0$. Then, for any $c \in [a, b]$,

$$F(c) = \int_a^c f(t)\Delta t$$

does not decrease as c increases, and the improper integral of the second kind is convergent if and only if f is bounded, in which case the value of the integral is $\lim_{c\to b-} F(c)$.
3. Let the limit

$$\lim_{t\to b-} \frac{f(t)}{g(t)} = L$$

exist (finite) and suppose it is not zero. Then the integrals $\int_a^b f(t)\Delta t$ and $\int_a^b g(t)\Delta t$ are simultaneously convergent or divergent.
Similar definitions are made and entirely similar results are obtained for integrals of the second kind that are improper at the lower limit of integration.

Example

Let $\mathbb{T}$ be an arbitrary time scale, $a, b \in \mathbb{T}$ with $a < b$, and suppose that b is left-dense. Let $p \geq 1$. We prove that the integral

$$\int_a^b \frac{\Delta t}{(b-t)^p} \tag{1}$$

is divergent.

1. Let $p = 1$. Let us choose points $t_n \in \mathbb{T}$ for $n \in \mathbb{N}_0$ such that

$$a = t_0 < t_1 < \ldots < b \quad \text{and} \quad \lim_{n\to\infty} t_n = b. \tag{2}$$

Example

We set

$$\tau_n = \frac{1}{b - t_n} \quad \text{for any} \quad n \in \mathbb{N}_0. \tag{3}$$

Then $\lim_{n\to\infty} \tau_n = \infty$, $t_n = b - \frac{1}{\tau_n}$, and

$$\begin{aligned} t_{n+1} - t_n &= \frac{1}{\tau_n} - \frac{1}{\tau_{n+1}} \\ &= \frac{\tau_{n+1} - \tau_n}{\tau_n \tau_{n+1}} \quad \text{for all} \quad n \in \mathbb{N}_0. \end{aligned}$$

Hence,

Example

$$
\begin{aligned}
\int_a^b \frac{\Delta t}{b-t} &= \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} \frac{\Delta t}{b-t} \\
&\geq \sum_{n=0}^{\infty} \frac{1}{b-t_n} \int_{t_n}^{t_{n+1}} \Delta t \\
&= \sum_{n=0}^{\infty} \frac{t_{n+1}-t_n}{b-t_n} \\
&= \sum_{n=0}^{\infty} \tau_n \frac{\tau_{n+1}-\tau_n}{\tau_n \tau_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{\tau_{n+1}-\tau_n}{\tau_{n+1}} \\
&= \infty.
\end{aligned}
$$

Example

2. Let $p > 1$. There exists $d \in [a, b)$ such that

$$0 < b - t < 1 \quad \text{for} \quad t \in [d, b).$$

Then

$$(b - t)^p < b - t \quad \text{for} \quad t \in [d, b).$$

Hence,

$$\begin{aligned}
\int_a^b \frac{\Delta t}{(b-t)^p} &= \int_a^d \frac{\Delta t}{(b-t)^p} + \int_d^b \frac{\Delta t}{(b-t)^p} \\
&> \int_a^d \frac{\Delta t}{(b-t)^p} + \int_d^b \frac{\Delta t}{b-t} \\
&= \infty.
\end{aligned}$$

Example

Let $\mathbb{T}$ be a time scale as before. Let $p < 1$ and suppose that for some $\alpha \in \left[1, \frac{1}{p}\right)$,

$$\frac{1}{b - t_{k+1}} = O\left(\frac{1}{(b - t_k)^\alpha}\right) \quad \text{as} \quad k \to \infty.$$

We prove that the improper integral is convergent. Let τ_n be as before. Then

$$\tau_{k+1} = O(\tau_k^\alpha) \quad \text{as} \quad k \to \infty.$$

Hence, $\tau_{k+1} \le K\tau_k^\alpha$ for all $k \in \mathbb{N}_0$, where $K > 0$ is a constant. Then

Example

$$
\begin{aligned}
\int_a^b \frac{\Delta t}{(b-t)^p} &= \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{\Delta t}{(b-t)^p} \\
&\le \sum_{k=0}^{\infty} \frac{1}{(b-t_{k+1})^p} \int_{t_k}^{t_{k+1}} \Delta t \\
&= \sum_{k=0}^{\infty} \frac{t_{k+1}-t_k}{(b-t_{k+1})^p} \\
&= \sum_{k=0}^{\infty} \frac{\tau_{k+1}-\tau_k}{\tau_k \tau_{k+1}^{1-p}}
\end{aligned}
$$

Example

$$
\begin{aligned}
&\leq \quad K^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \frac{\tau_{k+1} - \tau_k}{\tau_{k+1}^{\frac{1}{\alpha}+1-p}} \\
&\leq \quad K^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \int_{\tau_k}^{\tau_{k+1}} \frac{\mathrm{d}t}{t^{\frac{1}{\alpha}+1-p}} \\
&= \quad K^{\frac{1}{\alpha}} \int_{\tau_0}^{\infty} \frac{\mathrm{d}t}{t^{\frac{1}{\alpha}+1-p}} \\
&< \quad \infty.
\end{aligned}
$$

Example

Let $\mathbb{T}$ be a time scale as before. We consider the integral

$$I = \int_a^b \frac{\Delta t}{(t^4 + t^2 + 1)(b - t)^{\frac{1}{2}}}.$$

We have

$$I \leq \int_a^b \frac{\Delta t}{(b - t)^{\frac{1}{2}}} < \infty.$$

Example

Let $\mathbb{T}$ be a time scale as before. Assume $a = 0, b = 2, \frac{1}{2}, 1 \in \mathbb{T}$. We consider the integral

$$I = \int_0^2 \frac{t^{\alpha-1}}{|1-t|} \Delta t.$$

We have

$$I = \int_0^{\frac{1}{2}} \frac{t^{\alpha-1}}{1-t} \Delta t + \int_{\frac{1}{2}}^1 \frac{t^{\alpha-1}}{1-t} \Delta t + \int_1^2 \frac{t^{\alpha-1}}{t-1} \Delta t.$$

Since $\int_{\frac{1}{2}}^1 \frac{t^{\alpha-1}}{1-t} \Delta t$ is divergent for all $\alpha \in \mathbb{R}$, we conclude that the integral I is divergent.

Example

Let $\mathbb{T}$ be a time scale as before. Assume $a = 0, b = 1, \frac{1}{2} \in \mathbb{T}$. Consider the integral

$$I = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Delta t.$$

We have

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} t^{\alpha-1}(1-t)^{\beta-1}\Delta t + \int_{\frac{1}{2}}^1 t^{\alpha-1}(1-t)^{\beta-1}\Delta t \\ &= I_1 + I_2. \end{aligned}$$

Example

Note that I_1 is convergent for $\alpha > 0$ and divergent for $\alpha \leq 0$. Also, I_2 is convergent for $\beta > 0$ and divergent for $\beta \leq 0$. Therefore, I is convergent for $\alpha > 0$ and $\beta > 0$, and I is divergent for $\alpha \leq 0$ or $\beta \leq 0$.

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, 2, \ldots, n\}$, we denote by $\mathbb{T}_i$ a time scale.

Definition

The set

$$\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n = \{t = (t_1, t_2, \ldots, t_n) : \; t_i \in \mathbb{T}_i, i = 1, 2, \ldots, n\}$$

is called an *n-dimensional time scale*.

Example

$(\mathbb{R}, \mathbb{Z}, \mathbb{N})$ is a 3-dimensional time scale.

Example

$(\mathbb{Z}, \mathbb{N}_0^2, 2^{\mathbb{N}}, \mathbb{N})$ is a 4-dimensional time scale.

Example

$(3^{\mathbb{N}}, 4^{\mathbb{N}}, \mathbb{Q})$ is not a 3-dimensional time scale because $\mathbb{Q}$ is not a time scale.

Remark

We equip Λ^n with the metric

$$d(t,s) = \left(\sum_{i=1}^{n} |t_i - s_i|^2 \right)^{\frac{1}{2}} \quad \textit{for} \quad t, s \in \Lambda^n.$$

The set Λ^n with this metric is a complete metric space. Therefore, we have for Λ^n the fundamental concepts such as open balls, neighbourhoods of points, open sets, compact sets, and so on. Also, we have for functions $f : \Lambda^n \to \mathbb{R}$ the concepts of the limit, continuity, and properties of continuous functions on general metric spaces.

Definition

Let σ_i, $i \in \{1, 2, \ldots, n\}$, be the forward jump operator in $\mathbb{T}_i$. The operator $\sigma : \Lambda^n \to \mathbb{R}^n$ defined by

$$\sigma(t) = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_n(t))$$

is said to be the *forward jump operator* in Λ^n.

Example

Let $n = 4$ and $\Lambda^4 = \mathbb{R} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}_0^2$. Then

$$\mathbb{T}_1 = \mathbb{R}, \quad \mathbb{T}_2 = \mathbb{Z}, \quad \mathbb{T}_3 = \mathbb{Z}, \quad \mathbb{T}_4 = \mathbb{N}_0^2.$$

We have that

$$\sigma_1(t_1) = t_1, \quad t_1 \in \mathbb{R}, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{Z}$$

and

$$\sigma_3(t_3) = t_3 + 1, \quad t_3 \in \mathbb{Z}, \quad \sigma_4(t_4) = \left(\sqrt{t_4} + 1\right)^2, \quad t_4 \in \mathbb{N}_0^2.$$

Hence,

Example

$$\sigma(t) = \sigma(t_1, t_2, t_3, t_4) = \left(t_1, t_2 + 1, t_3 + 1, \left(\sqrt{t_4} + 1\right)^2\right),$$

$$(t_1, t_2, t_3, t_4) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3 \times \mathbb{T}_4.$$

Example

Let $n = 3$ and $\Lambda^3 = \mathbb{R} \times \mathbb{N}_0^2 \times 3\mathbb{Z}$. Then

$$\sigma_1(t_1) = t_1, \quad t_1 \in \mathbb{T}_1 = \mathbb{R}, \quad \sigma_2(t_2) = \left(\sqrt{t_2} + 1\right)^2, \quad t_2 \in \mathbb{T}_2 = \mathbb{N}_0^2,$$

and

$$\sigma_3(t_3) = t_3 + 3, \quad t_3 \in \mathbb{T}_3 = 3\mathbb{Z}.$$

Hence,

$$\sigma(t) = \sigma(t_1, t_2, t_3) = \left(t_1, \left(\sqrt{t_2} + 1\right)^2, t_3 + 3\right), \quad (t_1, t_2, t_3) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3.$$

Example

Let $n = 2$ and $\Lambda^2 = 2\mathbb{Z} \times 2^{\mathbb{N}}$. Here

$$\mathbb{T}_1 = 2\mathbb{Z}, \quad \mathbb{T}_2 = 2^{\mathbb{N}}.$$

Then

$$\sigma_1(t_1) = t_1 + 2, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = 2t_2, \quad t_2 \in \mathbb{T}_2.$$

Hence,

$$\sigma(t) = \sigma(t_1, t_2) = (t_1 + 2, 2t_2), \quad (t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Definition

Let ρ_i, $i \in \{1, 2, \ldots, n\}$, be the backward jump operator in $\mathbb{T}_i$. The operator $\rho : \Lambda^n \to \mathbb{R}^n$ defined by

$$\rho(t) = (\rho_1(t_1), \rho_2(t_2), \ldots, \rho_n(t_n)), \quad t = (t_1, t_2, \ldots, t_n) \in \Lambda^n,$$

is said to be the *backward jump operator* in Λ^n.

Example

Let $n = 4$ and $\Lambda^4 = \mathbb{R} \times \mathbb{Z} \times 3^{\mathbb{N}} \times \mathbb{N}_0^3$. Here,

$$\mathbb{T}_1 = \mathbb{R}, \quad \mathbb{T}_2 = \mathbb{Z}, \quad \mathbb{T}_3 = 3^{\mathbb{N}}, \quad \mathbb{T}_4 = \mathbb{N}_0^3.$$

Then

$$\rho_1(t_1) = t_1, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = t_2 - 1, \quad t_2 \in \mathbb{T}_2,$$

$$\rho_3(t_3) = \frac{t_3}{3}, \quad t_3 \in \mathbb{T}_3 \setminus \{3\}, \quad \rho(3) = 3,$$

$$\rho_4(t_4) = \left(\sqrt[3]{t_4} - 1\right)^3, \quad t_4 \in \mathbb{T}_4 \setminus \{0\}, \quad \rho_4(0) = 0.$$

Hence,

Example

$$
\begin{aligned}
\rho(t) &= (\rho_1(t_1), \rho_2(t_2), \rho_3(t_3), \rho_4(t_4)) \\
&= \begin{cases}
\left(t_1, t_2 - 1, \frac{t_3}{3}, (\sqrt[3]{t_4} - 1)^3\right) \\
\quad \text{if} \quad t_1 \in \mathbb{T}_1, \quad t_2 \in \mathbb{T}_2, \quad t_3 \in \mathbb{T}_3 \setminus \{3\}, \quad t_4 \in \mathbb{T}_4 \setminus \{0\}, \\
(t_1, t_2 - 1, 3, (\sqrt[3]{t_4} - 1)^3) \\
\quad \text{if} \quad t_1 \in \mathbb{T}_1, \quad t_2 \in \mathbb{T}_2, \quad t_3 = 3, \quad t_4 \in \mathbb{T}_4 \setminus \{0\}
\end{cases}
\end{aligned}
$$

Example

$$\rho(t) = \begin{cases} \left(t_1, t_2 - 1, \frac{t_3}{3}, 0\right) \\ \quad \text{if} \quad t_1 \in \mathbb{T}_1, \quad t_2 \in \mathbb{T}_2, \quad t_3 \in \mathbb{T}_3 \setminus \{3\}, \quad t_4 = 0, \\ (t_1, t_2 - 1, 3, 0) \\ \quad \text{if} \quad t_1 \in \mathbb{T}_1, \quad t_2 \in \mathbb{T}_2, \quad t_3 = 3, \quad t_4 = 0. \end{cases}$$

Example

Let $n = 2$ and $\Lambda^2 = (4\mathbb{Z}, \mathbb{R})$. Here,

$$\mathbb{T}_1 = 4\mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{R}.$$

Then

$$\rho_1(t_1) = t_1 - 4, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = t_2, \quad t_2 \in \mathbb{T}_2.$$

Hence,

$$\rho(t) = (\rho_1(t_1), \rho_2(t_2)) = (t_1 - 4, t_2), \quad t = (t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Example

Let $n = 3$ and $\Lambda^3 = \mathbb{Z} \times \mathbb{Z} \times 7^{\mathbb{N}}$. Here,

$$\mathbb{T}_1 = \mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{Z}, \quad \mathbb{T}_3 = 7^{\mathbb{N}}.$$

Then

$$\rho_1(t_1) = t_1 - 1, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = t_2 - 1, \quad t_2 \in \mathbb{T}_2,$$

$$\rho_3(t_3) = \frac{t_3}{7}, \quad t_3 \in \mathbb{T}_3 \setminus \{7\}, \quad \rho_3(7) = 7.$$

Hence,

Example

$$
\begin{aligned}
\rho(t) &= (\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\
&= \begin{cases} \left(t_1 - 1, t_2 - 1, \frac{t_3}{7}\right) & \text{if} \quad t_1 \in \mathbb{T}_1, \quad t_2 \in \mathbb{T}_2, \quad t_3 \in \mathbb{T}_3 \setminus \{7\}, \\ (t_1 - 1, t_2 - 1, 7) & \text{if} \quad t_2 \in \mathbb{T}_1, \quad t_2 \in \mathbb{T}_2, \quad t_3 = 7. \end{cases}
\end{aligned}
$$

Definition

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, we write

$$x \geq y$$

whenever

$$x_i \geq y_i \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

In a similar way, we understand $x > y$ and $x < y$ and $x \leq y$.

Definition

Let $t \in \Lambda^n$, $t = (t_1, t_2, \ldots, t_n)$.
1. If $\sigma(t) > t$, then t is called *strictly right-scattered*.
2. If $\sigma(t) \geq t$ and there are $j, l \in \{1, 2, \ldots, n\}$ such that $\sigma_j(t_j) > t_j$ and $\sigma_l(t_l) = t_l$, then t is called *right-scattered*.
3. If $t < \sup \Lambda^n$ and $\sigma(t) = t$, then t is called *right-dense*.

Definition

4. If $\rho(t) < t$, then t is called *strictly left-scattered*.
5. If $\rho(t) \leq t$ and there are $l, j \in \{1, 2, \ldots, n\}$ such that $\rho_j(t_j) < t_j$ and $\rho_l(t_l) = t_l$, then t is called *left-scattered*.
6. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, then t is called *left-dense*.

Definition

7. If t is strictly right-scattered and strictly left-scattered, then t is said to be *strictly isolated*.
8. If t is right-dense and left-dense, then t is said to be *dense*. 9. If t is right-scattered and left-scattered, then t is said to be *isolated*.

Example

Let $\Lambda^3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Then, for $t = (t_1, t_2, t_3) \in \Lambda^3$, we have

$$\sigma(t) = (\sigma_1(t_1), \sigma_2(t_2), \sigma_3(t_3)) = (t_1 + 1, t_2 + 1, t_3 + 1) > (t_1, t_2, t_3),$$

i.e., all points of Λ^3 are strictly right-scattered. Also,

$$\rho(t) = (\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) = (t_1 - 1, t_2 - 1, t_3 - 1) < (t_1, t_2, t_3),$$

i.e., all points of Λ^3 are strictly left-scattered. Consequently, all points of Λ^3 are strictly isolated.

Example

Let $\Lambda^4 = 2\mathbb{Z} \times \mathbb{R} \times 2^{\mathbb{N}} \times \left(\frac{1}{3}\right)^{\mathbb{N}}$. Then

$$\begin{aligned} \sigma(t) &= (\sigma_1(t_1), \sigma_2(t_2), \sigma_3(t_3), \sigma_4(t_4)) \\ &= (t_1 + 2, t_2, 2t_3, 3t_4) \geq (t_1, t_2, t_3, t_4), \quad t_4 \neq \frac{1}{3}. \end{aligned}$$

Therefore, all points $(t_1, t_2, t_3, t_4) \in \Lambda^4$, $t_4 \neq \frac{1}{3}$, are right-scattered. We note that

Example

$$\left(\sigma_1(t_1), \sigma_2(t_2), \sigma_3(t_3), \sigma_4\left(\frac{1}{3}\right)\right) = \left(t_1 + 1, t_2, 2t_3, \frac{1}{3}\right) \geq \left(t_1, t_2, t_3, \frac{1}{3}\right).$$

From here, all points $\left(t_1, t_2, t_3, \frac{1}{3}\right)$ are right-scattered. Also,

$$\begin{aligned} \rho(t) &= (\rho_1(t_1), \rho_2(t_2), \rho_3(t_3), \rho_4(t_4)) \\ &= \left(t_1 - 2, t_2, \frac{t_3}{2}, \frac{t_4}{3}\right) \leq (t_1, t_2, t_3, t_4) \quad \text{if} \quad t_3 \neq 2, \end{aligned}$$

i.e., all points $(t_1, t_2, t_3, t_4) \in \Lambda^4$, $t_3 \neq 2$, are left-scattered. We note that

Example

$$\begin{aligned}\rho(t) &= (\rho_1(t_1), \rho_2(t_2), \rho_3(2), \rho_4(t_4)) \\ &= \left(t_1 - 2, t_2, 2, \frac{t_4}{3}\right) \\ &\leq (t_1, t_2, 2, t_4).\end{aligned}$$

Therefore, all points $(t_1, t_2, 2, t_4) \in \Lambda^4$ are left-scattered. Moreover, the points

$$(t_1, t_2, t_3, t_4) \in \Lambda^4$$

are isolated.

Example

Let $\Lambda^3 = \mathbb{N} \times \mathbb{H} \times \mathbb{R}$. Then

$$\begin{aligned} \sigma(t) &= (\sigma_1(t_1), \sigma_2(H_n), \sigma_3(t_3)) \\ &= (t_1 + 1, H_{n+1}, t_3) \\ &\geq (t_1, H_n, t_3) \quad \text{for} \quad (t_1, H_n, t_3) \in \Lambda^3, \end{aligned}$$

i.e., any point $(t_1, H_n, t_3) \in \Lambda^3$ is right-scattered. Also,

Example

$$
\begin{aligned}
\rho(t) &= (\rho_1(t_1), \rho_2(H_n), \rho_3(t_3)) \\
&= (t_1 - 1, H_{n-1}, t_3) \\
&\leq (t_1, H_n, t_3) \quad \text{for} \quad (t_1, H_n, t_3) \in \Lambda^3, \quad t_1 \neq 1, \quad n \neq 0.
\end{aligned}
$$

Example

Therefore, all points $(t_1, H_n, t_3) \in \Lambda^3$, $t_1 \neq 1$, $n \neq 0$, are left-scattered. We note that

$$
\begin{aligned}
(\rho_1(1), \rho_2(H_n), \rho_3(t_3)) &= (1, H_{n-1}, t_3) \\
&\leq (1, H_n, t_3) \in \Lambda^3, \quad n \in \mathbb{N}, \\
(\rho_1(t_1), \rho_2(H_0), \rho_3(t_3)) &= (t_1 - 1, H_0, t_3) \leq (t_1, H_0, t_3), \quad t_1 \neq 1, \\
(\rho_1(1), \rho_2(H_0), \rho_3(t_3)) &= (1, H_0, t_3) \leq (1, H_0, t_3) \in \Lambda^3.
\end{aligned}
$$

Example

Therefore, the points

$$(t_1, H_0, t_3), \quad t_1 \neq 1; \quad (1, H_n, t_3), \quad n \in \mathbb{N}; \quad (1, H_0, t_3)$$

are left-scattered. Consequently, all points $(t_1, H_n, t_3) \in \Lambda^3$ are isolated.

Definition

The *graininess* function $\mu : \Lambda^n \to [0, \infty)^n$ is defined by

$$\mu(t) = (\mu_1(t_1), \mu_2(t_2), \dots, \mu_n(t_n)), \quad t = (t_1, t_2, \dots, t_n) \in \Lambda^n.$$

Example

Let $\Lambda^3 = 3\mathbb{Z} \times \mathbb{R} \times \mathbb{N}_0^4$. Then

$$\mathbb{T}_1 = 3\mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{R}, \quad \mathbb{T}_3 = \mathbb{N}_0^4,$$

$$\sigma_1(t_1) = t_1 + 3, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2, \quad t_2 \in \mathbb{T}_2,$$

$$\sigma_3(t_3) = \left(\sqrt[4]{t_3} + 1\right)^4, \quad t_3 \in \mathbb{T}_3.$$

Hence,

Example

$$
\begin{aligned}
\mu_1(t_1) &= \sigma_1(t_1) - t_1 = t_1 + 3 - t_1 = 3, \quad t_1 \in \mathbb{T}_1, \\
\mu_2(t_2) &= \sigma_2(t_2) - t_2 = t_2 - t_2 = 0, \quad t_2 \in \mathbb{T}_2, \\
\mu_3(t_3) &= \sigma_3(t_3) - t_3 \\
&= \left(\sqrt[4]{t_3} + 1\right)^4 - t_3 \\
&= t_3 + 4\sqrt[4]{t_3^3} + 6\sqrt[4]{t_3^2} + 4\sqrt[4]{t_3} + 1 - t_3
\end{aligned}
$$

Example

$$
\begin{aligned}
&= 6\sqrt{t_3}+4\sqrt[4]{t_3}+4\sqrt[4]{t_3^3}+1, \quad t_3\in\mathbb{T}_3,\\
\mu(t) &= (\mu_1(t_1),\mu_2(t_2),\mu_3(t_3))\\
&= \left(3,0,6\sqrt{t_3}+4\sqrt[4]{t_3}+4\sqrt[4]{t_3^3}+1\right), \quad t=(t_1,t_2,t_3)\in\Lambda^3.
\end{aligned}
$$

Example

Let $\Lambda^3 = \mathbb{Z} \times 2^{\mathbb{N}} \times 3^{\mathbb{N}}$. Then

$$\mathbb{T}_1 = \mathbb{Z}, \quad \mathbb{T}_2 = 2^{\mathbb{N}}, \quad \mathbb{T}_3 = 3^{\mathbb{N}},$$

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = 2t_2, \quad t_2 \in \mathbb{T}_2,$$

$$\sigma_3(t_3) = 3t_3, \quad t_3 \in \mathbb{T}_3.$$

Hence,

Example

$$
\begin{aligned}
\mu_1(t_1) &= \sigma_1(t_1) - t_1 = t_1 + 1 - t_1 = 1, \quad t_1 \in \mathbb{T}_1, \\
\mu_2(t_2) &= \sigma_2(t_2) - t_2 = 2t_2 - t_2 = t_2, \quad t_2 \in \mathbb{T}_2, \\
\mu_3(t_3) &= \sigma_3(t_3) - t_3 = 3t_3 - t_3 = 2t_3, \quad t_3 \in \mathbb{T}_3, \\
\mu(t) &= (\mu_1(t_1), \mu_2(t_2), \mu_3(t_3)) \\
&= (1, t_2, 2t_3), \quad t = (t_1, t_2, t_3) \in \Lambda^3.
\end{aligned}
$$

Example

Let $\Lambda^4 = 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{N}_0^2 \times 4^{\mathbb{N}}$. Then

$$\mathbb{T}_1 = 2\mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{Z}, \quad \mathbb{T}_3 = \mathbb{N}_0^2, \quad \mathbb{T}_4 = 4^{\mathbb{N}},$$

$$\sigma_1(t_1) = t_1 + 2, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2,$$

$$\sigma_3(t_3) = \left(\sqrt{t_3} + 1\right)^2, \quad t_3 \in \mathbb{T}_3, \quad \sigma_4(t_4) = 4t_4, \quad t_4 \in 4^{\mathbb{N}}.$$

Hence,

Example

$$\begin{aligned}
\mu_1(t_1) &= \sigma_1(t_1) - t_1 = t_1 + 2 - t_1 = 2, \quad t_1 \in \mathbb{T}_1, \\
\mu_2(t_2) &= \sigma_2(t_2) - t_2 = t_2 + 1 - t_2 = 1, \quad t_2 \in \mathbb{T}_2, \\
\mu_3(t_3) &= \sigma_3(t_3) - t_3 = \left(\sqrt{t_3} + 1\right)^2 - t_3 = 2\sqrt{t_3} + 1, \quad t_3 \in \mathbb{T}_3, \\
\mu_4(t_4) &= \sigma_4(t_4) - t_4 = 4t_4 - t_4 = 3t_4, \quad t_4 \in \mathbb{T}_4, \\
\mu(t) &= (\mu_1(t_1), \mu_2(t_2), \mu_3(t_3), \mu_4(t_4)) \\
&= (2, 1, 2\sqrt{t_3} + 1, 3t_4), \quad t = (t_1, t_2, t_3, t_4) \in \Lambda^4.
\end{aligned}$$

Definition

Let $f : \Lambda \to \mathbb{R}$. We introduce the following notations.

$$
\begin{aligned}
f^{\sigma}(t) &= f\left(\sigma_1(t_1), \sigma_2(t_2), \ldots, \sigma_n(t_n)\right), \\
f_i^{\sigma_i}(t) &= f\left(t_1, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n\right), \\
f_{i_1 i_2 \ldots i_l}^{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_l}}(t) &= f\left(\ldots, \sigma_{i_1}(t_{i_1}), \ldots, \sigma_{i_2}(t_{i_2}), \ldots, \sigma_{i_l}(t_{i_l}), \ldots\right),
\end{aligned}
$$

where $1 \le i_1 < i_2 < \ldots < i_l \le n$, $i_m \in \mathbb{N}$, $m \in \{1, 2, \ldots, l\}$, $l \in \mathbb{N}$.

Example

Let $\Lambda^2 = 2\mathbb{Z} \times 2^{\mathbb{N}}$. Here,

$$\mathbb{T}_1 = 2\mathbb{Z}, \quad \mathbb{T}_2 = 2^{\mathbb{N}},$$

$$\sigma_1(t_1) = t_1 + 2, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = 2t_2, \quad t_2 \in \mathbb{T}_2.$$

Let

$$f(t_1, t_2) = t_1^2 + t_2.$$

Hence,

$$\begin{aligned} f^\sigma(t) &= f(\sigma_1(t_1), \sigma_2(t_2)) \\ &= (\sigma_1(t_1))^2 + \sigma_2(t_2) \end{aligned}$$

Example

$$
\begin{aligned}
&= (t_1+2)^2 + 2t_2 \\
&= t_1^2 + 4t_1 + 2t_2 + 4, \\
f_1^{\sigma_1}(t) &= f(\sigma_1(t_1), t_2) \\
&= (\sigma_1(t_1))^2 + t_2 \\
&= (t_1+2)^2 + t_2
\end{aligned}
$$

Example

$$\begin{aligned} &= t_1^2 + 4t_1 + t_2 + 4, \\ f_2^{\sigma_2}(t) &= f(t_1, \sigma_2(t_2)) \\ &= t_1^2 + \sigma_2(t_2) \\ &= t_1^2 + 2t_2, \quad t \in \Lambda^2. \end{aligned}$$

Example

Let $\Lambda^3 = 3\mathbb{Z} \times \mathbb{N} \times 4^{\mathbb{N}}$ and $f : \Lambda^3 \to \mathbb{R}$ be defined by

$$f(t) = t_1 t_2 + t_2 t_3, \quad t = (t_1, t_2, t_3) \in \Lambda^3.$$

We will find

$$f^{\sigma}(t), \quad f_1^{\sigma_1}(t), \quad f_2^{\sigma_2}(t), \quad f_3^{\sigma_3}(t), \quad f_{12}^{\sigma_1\sigma_2}(t), \quad f_{13}^{\sigma_1\sigma_3}(t), \quad f_{23}^{\sigma_2\sigma_3}(t)$$

and

$$g(t) = f^{\sigma}(t) + 2f_{12}^{\sigma_1\sigma_2}(t) - f_{23}^{\sigma_2\sigma_3}(t), \quad t \in \Lambda^3.$$

We have

Example

$$\mathbb{T}_1 = 3\mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{N}, \quad \mathbb{T}_3 = 4^{\mathbb{N}}$$

and

$$\sigma_1(t_1) = t_1 + 3, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2, \quad \sigma_3(t_3) = 4t_3, \quad t_3 \in$$

Hence,

$$\begin{aligned} f^{\sigma}(t) &= f(\sigma_1(t_1), \sigma_2(t_2), \sigma_3(t_3)) \\ &= \sigma_1(t_1)\sigma_2(t_2) + \sigma_2(t_2)\sigma_3(t_3) \\ &= (t_1 + 3)(t_2 + 1) + (t_2 + 1)4t_3 \end{aligned}$$

Example

$$
\begin{aligned}
&= t_1 t_2 + t_1 + 3t_2 + 3 + 4t_2 t_3 + 4t_3 \\
&= t_1 t_2 + 4t_2 t_3 + t_1 + 3t_2 + 4t_3 + 3, \\
f_1^{\sigma_1}(t) &= f(\sigma_1(t_1), t_2, t_3) \\
&= \sigma_1(t_1) t_2 + t_2 t_3 \\
&= (t_1 + 3) t_2 + t_2 t_3
\end{aligned}
$$

Example

$$
\begin{aligned}
&= t_1 t_2 + t_2 t_3 + 3t_2, \\
f_2^{\sigma_2}(t) &= f(t_1, \sigma_2(t_2)) \\
&= t_1 \sigma_2(t_2) + \sigma_2(t_2) t_3 \\
&= t_1(t_2 + 1) + (t_2 + 1) t_3 \\
&= t_1 t_2 + t_2 t_3 + t_1 + t_3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\sigma_3}(t) &= f(t_1, t_2, \sigma_3(t_3)) \\
&= t_1 t_2 + t_2 \sigma_3(t_3) \\
&= t_1 t_2 + 4 t_2 t_3, \\
f_{12}^{\sigma_1 \sigma_2}(t) &= f(\sigma_1(t_1), \sigma_2(t_2), t_3) \\
&= \sigma_1(t_1)\sigma_2(t_2) + \sigma_2(t_2) t_3
\end{aligned}
$$

Example

$$
\begin{aligned}
&= (t_1+3)(t_2+1)+(t_2+1)t_3 \\
&= t_1t_2+t_2t_3+t_1+3t_2+t_3+3, \\
f_{13}^{\sigma_1\sigma_3}(t) &= f(\sigma_1(t_1), t_2, \sigma_3(t_3)) \\
&= \sigma_1(t_1)t_2+t_2\sigma_3(t_3)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= (t_1+3)t_2 + 4t_2t_3 \\
&= t_1t_2 + 4t_2t_3 + 3t_2, \\
f_{23}^{\sigma_2\sigma_3}(t) &= f(t_1, \sigma_2(t_2), \sigma_3(t_3)) \\
&= t_1\sigma_2(t_2) + \sigma_2(t_2)\sigma_3(t_3) \\
&= t_1(t_2+1) + (t_2+1)4t_3
\end{aligned}
$$

Example

$$
\begin{aligned}
&= t_1 t_2 + 4t_2 t_3 + t_1 + 4t_3, \\
g(t) &= f^{\sigma}(t) + 2f_{12}^{\sigma_1\sigma_2}(t) - f_{23}^{\sigma_2\sigma_3}(t) \\
&= t_1 t_2 + 4t_2 t_3 + t_1 + 3t_2 + 4t_3 + 3 \\
&\quad +2t_1 t_2 + 2t_2 t_3 + 2t_1 + 6t_2 + 2t_3 + 6 - t_1 t_2 - 4t_2 t_3 - t_1 - 4t_3 \\
&= 2t_1 t_2 + 2t_2 t_3 + 2t_1 + 9t_2 + 2t_3 + 9, \quad t \in \Lambda^3.
\end{aligned}
$$