Time Scales Analysis
Lecture 20
Multidimensional Time Scales

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Example

Let $\Lambda^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0^2$ and $f : \Lambda^3 \to \mathbb{R}$ be defined by

$$f(t) = t_1 t_2 + t_3, \quad t = (t_1, t_2, t_3) \in \Lambda^3.$$

We will find

$$f^{\sigma}(t), \quad f_1^{\sigma_1}(t), \quad f_2^{\sigma_2}(t), \quad f_3^{\sigma_3}(t), \quad f_{12}^{\sigma_1\sigma_2}(t), \quad f_{13}^{\sigma_1\sigma_3}(t), \quad f_{23}^{\sigma_2\sigma_3}(t)$$

and

$$g(t) = f^{\sigma}(t) + f_2^{\sigma_2}(t) - 3f_{23}^{\sigma_2\sigma_3}(t), \quad t \in \Lambda^3.$$

Example

We have

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2,$$

$$\sigma_3(t_3) = \left(\sqrt{t_3} + 1\right)^2, \quad t_3 \in \mathbb{T}_3.$$

Hence,

$$f^\sigma(t) \quad = \quad f(\sigma_1(t_1), \sigma_2(t_2), \sigma_3(t_3))$$

Example

$$
\begin{aligned}
&= \sigma_1(t_1)\sigma_2(t_2) + \sigma_3(t_3) \\
&= (t_1+1)(t_2+1) + \left(\sqrt{t_3}+1\right)^2 \\
&= t_1 t_2 + t_1 + t_2 + 1 + t_3 + 2\sqrt{t_3} + 1 \\
&= t_1 t_2 + t_1 + t_2 + t_3 + 2\sqrt{t_3} + 2, \\
f_1^{\sigma_1}(t) &= f(\sigma_1(t_1), t_2, t_3) \\
&= \sigma_1(t_1)t_2 + t_3
\end{aligned}
$$

Example

$$
\begin{aligned}
&= (t_1+1)t_2+t_3 \\
&= t_1t_2+t_2+t_3, \\
f_2^{\sigma_2}(t) &= f(t_1,\sigma_2(t_2),t_3) \\
&= t_1\sigma_2(t_2)+t_3 \\
&= t_1(t_2+1)+t_3 \\
&= t_1t_2+t_1+t_3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\sigma_3}(t) &= f(t_1, t_2, \sigma_3(t_3)) \\
&= t_1 t_2 + \sigma_3(t_3) \\
&= t_1 t_2 + \left(\sqrt{t_3} + 1\right)^2 \\
&= t_1 t_2 + t_3 + 2\sqrt{t_3} + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\sigma_1\sigma_2}(t) &= f(\sigma_1(t_1), \sigma_2(t_2), t_3) \\
&= \sigma_1(t_1)\sigma_2(t_2) + t_3 \\
&= (t_1+1)(t_2+1) + t_3 \\
&= t_1 t_2 + t_1 + t_2 + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\sigma_1\sigma_3}(t) &= f(\sigma_1(t_1), t_2, \sigma_3(t_3)) \\
&= \sigma_1(t_1)t_2 + \sigma_3(t_3) \\
&= (t_1+1)t_2 + \left(\sqrt{t_3}+1\right)^2 \\
&= t_1 t_2 + t_2 + t_3 + 2\sqrt{t_3} + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\sigma_2\sigma_3}(t) &= f(t_1, \sigma_2(t_2), \sigma_3(t_3)) \\
&= t_1\sigma_2(t_2) + \sigma_3(t_3) \\
&= t_1(t_2+1) + \left(\sqrt{t_3}+1\right)^2 \\
&= t_1 t_2 + t_1 + t_3 + 2\sqrt{t_3} + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f^{\sigma}(t) + f_2^{\sigma_2}(t) - 3f_{23}^{\sigma_2\sigma_3}(t) \\
&= t_1 t_2 + t_1 + t_2 + t_3 \\
&\quad +2\sqrt{t_3} + 2 + t_1 t_2 + t_1 + t_3 - 3t_1 t_2 - 3t_1 - 3t_3 - 6\sqrt{t_3} - 3 \\
&= -t_1 t_2 - t_1 + t_2 - t_3 - 4\sqrt{t_3} - 1.
\end{aligned}
$$

Definition

Let $f : \Lambda^n \to \mathbb{R}$. We introduce the following notations.

$$\begin{aligned} f^{\rho}(t) &= f\left(\rho_1(t_1), \rho_2(t_2), \ldots, \rho_n(t_n)\right), \\ f_i^{\rho_i}(t) &= f\left(t_1, \ldots, t_{i-1}, \rho_i(t_i), t_{i+1}, \ldots, t_n\right), \\ f_{i_1 i_2 \ldots i_l}^{\rho_{i_1}\rho_{i_2}\cdots\rho_{i_l}}(t) &= f\left(\ldots, \rho_{i_1}(t_{i_1}), \ldots, \rho_{i_2}(t_{i_2}), \ldots, \rho_{i_l}(t_{i_l}), \ldots\right), \end{aligned}$$

where $1 \leq i_1 < i_2 < \ldots < i_l \leq n$, $i_m \in \mathbb{N}$, $m \in \{1, 2, \ldots, n\}$.

Example

Let $\Lambda^3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}$ and consider the function $f : \Lambda^3 \to \mathbb{R}$ defined by

$$f(t) = t_1 - t_2 - 2t_3 + t_3^2, \quad t = (t_1, t_2, t_3) \in \Lambda^3.$$

Then

$$\mathbb{T}_1 = \mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{Z}, \quad \mathbb{T}_3 = \mathbb{R},$$

$$\rho_1(t_1) = t_1 - 1, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = t_2 - 1, \quad t_2 \in \mathbb{T}_2,$$

$$\rho_3(t_3) = t_3, \quad t_3 \in \mathbb{T}_3.$$

Example

We will find

$$f^{\rho}(t), \quad f_1^{\rho_1}(t), \quad f_2^{\rho_2}(t), \quad f_3^{\rho_3}(t), \quad f_{12}^{\rho_1\rho_2}(t), \quad f_{13}^{\rho_1\rho_3}(t), \quad f_{23}^{\rho_2\rho_3}(t)$$

and

$$g(t) = f^{\rho}(t) - f_1^{\rho_1}(t).$$

We have

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t), \rho_2(t), \rho_3(t)) \\ &= \rho_1(t_1) - \rho_2(t_2) - 2\rho_3(t_3) + \rho_3^2(t_3) \end{aligned}$$

Example

$$
\begin{aligned}
&= t_1 - 1 - (t_2 - 1) - 2t_3 + t_3^2 \\
&= t_1 - t_2 - 2t_3 + t_3^2, \\
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) - t_2 - 2t_3 + t_3^2 \\
&= t_1 - t_2 - 2t_3 + t_3^2 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 - \rho_2(t_2) - 2t_3 + t_3^2 \\
&= t_1 - t_2 - 2t_3 + t_3^2 + 1, \\
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 - t_2 - 2\rho_3(t_3) + \rho_3^2(t_3) \\
&= t_1 - t_2 - 2t_3 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) - \rho_2(t_2) - 2t_3 + t_3^2 \\
&= t_1 - 1 - (t_2 - 1) - 2t_3 + t_3^2 \\
&= t_1 - t_2 - 2t_3 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) - t_2 - 2\rho_3(t_3) + \rho_3^2(t_3) \\
&= t_1 - 1 - t_2 - 2t_3 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 - \rho_2(t_2) - 2\rho_3(t_3) + \rho_3^2(t_3) \\
&= t_1 - t_2 - 2t_3 + t_3^2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f^{\rho}(t) - f_1^{\rho_1}(t) \\
&= t_1 - t_2 - 2t_3 + t_3^2 - (t_1 - t_2 - 2t_3 + t_3^2 - 1) \\
&= 1.
\end{aligned}
$$

Example

Let $\Lambda^3 = 2^{\mathbb{N}} \times \mathbb{N} \times \mathbb{Z}$ and $f : \Lambda^3 \to \mathbb{R}$ be defined by

$$f(t) = t_1 + t_2 + t_3, \quad t = (t_1, t_2, t_3) \in \Lambda^3.$$

We will find

$$f^{\rho}(t), \quad f_1^{\rho_1}(t), \quad f_2^{\rho_2}(t), \quad f_3^{\rho_3}(t), \quad f_{12}^{\rho_1\rho_2}(t), \quad f_{13}^{\rho_1\rho_3}(t), \quad f_{23}^{\rho_2\rho_3}(t)$$

and

$$g(t) = f(t) - f_1^{\rho_1}(t) - f_{23}^{\rho_2\rho_3}(t), \quad t \in \Lambda^3.$$

We have

$$\mathbb{T}_1 = 2^{\mathbb{N}}, \quad \mathbb{T}_2 = \mathbb{N}, \quad \mathbb{T}_3 = \mathbb{Z}.$$

1. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 \neq 2$, $t_2 \neq 1$. Then

$$\rho_1(t_1) = \frac{t_1}{2}, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = t_2 - 1, \quad t_2 \in \mathbb{T}_2, \quad \rho_3(t_3) = t_3 - 1, \quad t_3 \in$$

Hence,

Example

$$
\begin{aligned}
f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\
&= \rho_1(t_1) + \rho_2(t_2) + \rho_3(t_3) \\
&= \frac{t_1}{2} + t_2 - 1 + t_3 - 1 \\
&= \frac{t_1}{2} + t_2 + t_3 - 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3 \\
&= \frac{t_1}{2} + t_2 + t_3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3 \\
&= t_1 + t_2 + t_3 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3(t_3) \\
&= t_1 + t_2 + t_3 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3 \\
&= \frac{t_1}{2} + t_2 + t_3 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3(t_3) \\
&= \frac{t_1}{2} + t_2 + t_3 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3(t_3) \\
&= t_1 + t_2 + t_3 - 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f(t) - f_1^{\rho_1}(t) - f_{23}^{\rho_2\rho_3}(t) \\
&= t_1 + t_2 + t_3 - \left(\frac{t_1}{2} + t_2 + t_3\right) - (t_1 + t_2 + t_3 - 2) \\
&= -\frac{t_1}{2} - t_2 - t_3 + 2.
\end{aligned}
$$

Example

2. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 = 2$, $t_2 \neq 1$. Then

$$\rho_1(t_1) = 2, \quad \rho_2(t_2) = t_2 - 1, \quad t_2 \in \mathbb{T}_2, \quad \rho_3(t_3) = t_3 - 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

$$\begin{aligned} f^\rho(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3(t_3) \\ &= 2 + t_2 - 1 + t_3 - 1 \\ &= t_2 + t_3, \end{aligned}$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3 \\
&= t_2 + t_3 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3 \\
&= 2 + t_2 - 1 + t_3 \\
&= t_2 + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3(t_3) \\
&= 2 + t_2 + t_3 - 1 \\
&= t_2 + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3 \\
&= 2 + t_2 + t_3 - 1 \\
&= t_2 + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3(t_3) \\
&= 2 + t_2 - 1 + t_3 \\
&= t_2 + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3} &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3(t_3) \\
&= 2 + t_2 - 1 + t_3 - 1 \\
&= t_2 + t_3,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f(t) - f_1^{\rho_1}(t) - f_{23}^{\rho_2\rho_3}(t) \\
&= 2 + t_2 + t_3 - (t_2 + t_3 + 2) - (t_2 + t_3) \\
&= -t_2 - t_3.
\end{aligned}
$$

Example

3. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 \neq 2$, $t_2 = 1$. Then

$$\rho_1(t_1) = \frac{t_1}{2}, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = 1, \quad \rho_3(t_3) = t_3 - 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3(t_3) \\ &= \frac{t_1}{2} + 1 + t_3 - 1 \\ &= \frac{t_1}{2} + t_3, \end{aligned}$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3 \\
&= \frac{t_1}{2} + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_2) \\
&= t_1 + \rho_2(t_2) + t_3 \\
&= t_1 + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3(t_3) \\
&= t_1 + 1 + t_3 - 1 \\
&= t_1 + t_3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3 \\
&= \frac{t_1}{2} + t_3 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3(t_3) \\
&= \frac{t_1}{2} + 1 + t_3 - 1 \\
&= \frac{t_1}{2} + t_3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3(t_3) \\
&= t_1 + 1 + t_3 - 1 \\
&= t_1 + t_3,
\end{aligned}
$$

Example

$$\begin{aligned} g(t) &= f(t) - f_1^{\rho_1}(t) - f_{23}^{\rho_2\rho_3}(t) \\ &= t_1 + t_2 + t_3 - \left(\frac{t_1}{2} + t_3 + 1\right) - (t_1 + t_3) \\ &= -\frac{t_1}{2} - t_3. \end{aligned}$$

Example

4. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 = 2$, $t_2 = 1$. Then

$$\rho_1(t_1) = 2, \quad \rho_2(t_2) = 1, \quad \rho_3(t_3) = t_3 - 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3(t_3) \\ &= 2 + 1 + t_3 - 1 \\ &= t_3 + 2, \end{aligned}$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3 \\
&= 2 + 1 + t_3 \\
&= t_3 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3 \\
&= 2 + 1 + t_3 \\
&= t_3 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3(t_3) \\
&= 2 + 1 + t_3 - 1 \\
&= t_3 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3 \\
&= 2 + 1 + t_3 \\
&= t_3 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3(t_3) \\
&= 2 + 1 + t_3 - 1 \\
&= t_3 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3(t_3) \\
&= 2 + 1 + t_3 - 1 \\
&= t_3 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f(t) - f_1^{\rho_1}(t) - f_{23}^{\rho_2\rho_3}(t) \\
&= t_1 + t_2 + t_3 - (t_3 + 3) - (t_3 + 2) \\
&= 2 + 1 + t_3 - t_3 - 3 - t_3 - 2 \\
&= -t_3 - 2.
\end{aligned}
$$

Example

Let $\Lambda^3 = \mathbb{N}_0 \times 2\mathbb{N} \times \mathbb{N}$ and $f : \Lambda^3 \to \mathbb{R}$ be defined by

$$f(t) = t_1 + t_2 + t_3^2, \quad t = (t_1, t_2, t_3) \in \Lambda^3.$$

We will find

$$f^{\rho}(t), \quad f_1^{\rho_1}(t), \quad f_2^{\rho_2}(t), \quad f_3^{\rho_3}(t), \quad f_{12}^{\rho_1\rho_2}(t), \quad f_{13}^{\rho_1\rho_3}(t), \quad f_{23}^{\rho_2\rho_3}(t)$$

and

$$g(t) = f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t), \quad t \in \Lambda^3.$$

Here,

$$\mathbb{T}_1 = \mathbb{N}_0, \quad \mathbb{T}_2 = 2\mathbb{N}, \quad \mathbb{T}_3 = \mathbb{N}.$$

Example

1. Let $(t_1, t_2, t_3) \in \Lambda^3$, $t_1 \neq 0$, $t_2 \neq 2$, $t_3 \neq 1$. Then

$$\rho_1(t_1) = t_1 - 1, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = t_2 - 2, \quad t_2 \in \mathbb{T}_2,$$

$$\rho_3(t_3) = t_3 - 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

Example

$$
\begin{aligned}
f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\
&= \rho_1(t_1) + \rho_2(t_2) + \rho_3^2(t_3) \\
&= t_1 - 1 + t_2 - 2 + (t_3 - 1)^2 \\
&= t_1 + t_2 - 3 + t_3^2 - 2t_3 + 1 \\
&= t_1 + t_2 - 2t_3 + t_3^2 - 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3^2 \\
&= t_1 - 1 + t_2 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3^2 \\
&= t_1 + t_2 + t_3^2 - 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3^2(t_3) \\
&= t_1 + t_2 + (t_3 - 1)^2 \\
&= t_1 + t_2 - 2t_3 + t_3^2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3^2 \\
&= t_1 - 1 + t_2 - 2 + t_3^2 \\
&= t_1 + t_2 + t_3^2 - 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3^2(t_3) \\
&= t_1 - 1 + t_2 + (t_3 - 1)^2 \\
&= t_1 + t_2 - 2t_3 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3^2(t_3) \\
&= t_1 + t_2 - 2 + (t_3 - 1)^2 \\
&= t_1 + t_2 - 2t_3 + t_3^2 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t) \\
&= t_1 + t_2 + t_3^2 - 1 + t_1 + t_2 + t_3^2 - 2 + t_1 + t_2 - 2t_3 + t_3^2 + 1 \\
&= 3t_1 + 3t_2 - 2t_3 + 3t_3^2 - 2.
\end{aligned}
$$

Example

2. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 = 0$, $t_2 \neq 2$, $t_3 \neq 1$. Then

$$\rho_1(t_1) = 0, \quad \rho_2(t_2) = t_2 - 2, \quad t_2 \in \mathbb{T}_2, \quad \rho_3(t_3) = t_3 - 1, \quad t_3 \in \mathbb{T}_3$$

and

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3^2(t_3) \\ &= 0 + t_2 - 2 + (t_3 - 1)^2 \end{aligned}$$

Example

$$
\begin{aligned}
&= t_2 - 2 + t_3^2 - 2t_3 + 1 \\
&= t_2 - 2t_3 + t_3^2 - 1, \\
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3^2 \\
&= t_2 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3^2 \\
&= t_2 + t_3^2 - 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3^2(t_3) \\
&= t_2 + (t_3 - 1)^2 \\
&= t_2 - 2t_3 + t_3^2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3^2 \\
&= t_2 + t_3^2 - 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3^2(t_3) \\
&= t_2 + (t_3 - 1)^2 \\
&= t_2 - 2t_3 + t_3^2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= \rho_2(t_2) + \rho_3^2(t_3) \\
&= t_2 - 2 + (t_3 - 1)^2 \\
&= t_2 + t_3^2 - 2t_3 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t) \\
&= t_2 + t_3^2 + t_2 + t_3^2 - 2 + t_2 - 2t_3 + t_3^2 + 1 \\
&= 3t_2 - 2t_3 + 3t_3^2 - 1.
\end{aligned}
$$

Example

2. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 \neq 0$, $t_2 = 2$, $t_3 \neq 1$. Then

$$\rho_1(t_1) = t_1 - 1, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = 2, \quad t_2 \in \mathbb{T}_2, \quad \rho_3(t_3) = t_3 - 1, \quad t_3 \in \mathbb{T}$$

and

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3^2(t_3) \\ &= t_1 - 1 + 2 + (t_3 - 1)^2 \\ &= t_1 + 1 + t_3^2 - 2t_3 + 1 \\ &= t_1 - 2t_3 + t_3^2 + 2, \end{aligned}$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3^2 \\
&= t_1 - 1 + 2 + t_3^2 \\
&= t_1 + t_3^2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3^2 \\
&= t_1 + t_3^2 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3^2(t_3) \\
&= t_1 + 2 + (t_3 - 1)^2 \\
&= t_1 + 2 + t_3^2 - 2t_3 + 1 \\
&= t_1 - 2t_3 + t_3^2 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3^2 \\
&= t_1 - 1 + 2 + t_3^2 \\
&= t_1 + t_3^2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3^2(t_3) \\
&= t_1 - 1 + 2 + (t_3 - 1)^2 \\
&= t_1 + 1 + t_3^2 - 2t_3 + 1 \\
&= t_1 - 2t_3 + t_3^2 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3^2(t_3) \\
&= t_1 + 2 + (t_3 - 1)^2 \\
&= t_1 + 2 + t_3^2 - 2t_3 + 1 \\
&= t_1 - 2t_3 + t_3^2 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t) \\
&= t_1 + t_3^2 + 1 + t_1 + t_3^2 + 2 + t_1 - 2t_3 + t_3^2 + 3 \\
&= 3t_1 - 2t_3 + 3t_3^2 + 6.
\end{aligned}
$$

Example

3. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 \neq 0$, $t_2 \neq 2$, $t_3 = 1$. Then

$$\rho_1(t_1) = t_1 - 1, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = t_2 - 2, \quad t_2 \in \mathbb{T}_2, \quad \rho_3(t_3) = 1.$$

Hence,

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3^2(t_3) \\ &= t_1 - 1 + t_2 - 2 + 1 \\ &= t_1 + t_2 - 2, \end{aligned}$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3^2 \\
&= t_1 - 1 + t_2 + 1 \\
&= t_1 + t_2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3^2 \\
&= t_1 + t_2 - 2 + 1 \\
&= t_1 + t_2 - 1,
\end{aligned}
$$

Example

$$\begin{aligned} f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\ &= t_1 + t_2 + \rho_3^2(t_3) \\ &= t_1 + t_2 + 1, \end{aligned}$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3^2 \\
&= t_1 - 1 + t_2 - 2 + 1 \\
&= t_1 + t_2 - 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3^2(t_3) \\
&= t_1 - 1 + t_2 + 1 \\
&= t_1 + t_2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3^2(t_3) \\
&= t_1 + t_2 - 2 + 1 \\
&= t_1 + t_2 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t) \\
&= t_1 + t_2 + t_1 + t_2 - 1 + t_1 + t_2 + 1 \\
&= 3t_1 + 3t_2.
\end{aligned}
$$

Example

3. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 = 0$, $t_2 = 2$, $t_3 \neq 1$. Then

$$\rho_1(t_1) = 0, \quad \rho_2(t_2) = 2, \quad \rho_3(t_3) = t_3 - 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3^2(t_3) \\ &= 0 + 2 + (t_3 - 1)^2 \\ &= -2t_3 + t_3^2 + 3, \end{aligned}$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3^2 \\
&= 2 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3^2 \\
&= 2 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3^2(t_3) \\
&= 2 + (t_3 - 1)^2 \\
&= -2t_3 + t_3^2 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3^2 \\
&= 2 + t_3^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3^2(t_3) \\
&= 2 + (t_3 - 1)^2 \\
&= -2t_3 + t_3^2 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3^2(t_3) \\
&= 2 + (t_3 - 1)^2 \\
&= -2t_3 + t_3^2 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t) \\
&= 2 + t_3^2 + 2 + t_3^2 + t_3^2 - 2t_3 + 3 \\
&= 3t_3^2 - 2t_3 + 7.
\end{aligned}
$$

Example

4. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 = 0$, $t_2 \neq 2$, $t_3 = 1$. Then

$$\rho_1(t_1) = 0, \quad \rho_2(t_2) = t_2 - 2, \quad t_2 \in \mathbb{T}_2, \quad \rho_3(t_3) = 1.$$

Hence,

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3^2(t_3) \\ &= t_2 - 2 + 1 \\ &= t_2 - 1, \end{aligned}$$

Example

$$
\begin{aligned}
f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\
&= \rho_1(t_1) + t_2 + t_3^2 \\
&= t_2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3^2 \\
&= t_2 - 2 + 1 \\
&= t_2 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3^2(t_3) \\
&= t_2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3^2 \\
&= t_2 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3^2(t_3) \\
&= t_2 + 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3^2(t_3) \\
&= t_2 - 2 + 1 \\
&= t_2 - 1,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t) \\
&= t_2 + 1 + t_2 - 1 + t_2 + 1 \\
&= 3t_2 + 1.
\end{aligned}
$$

Example

5. Let $t = (t_1, t_2, t_3) \in \Lambda^3$, $t_1 \neq 0$, $t_2 = 2$, $t_3 = 1$. Then

$$\rho_1(t_1) = t_1 - 1, \quad t_1 \in \mathbb{T}_1, \quad \rho_2(t_2) = 2, \quad \rho_3(t_3) = 1.$$

Hence,

$$\begin{aligned} f^{\rho}(t) &= f(\rho_1(t_1), \rho_2(t_2), \rho_3(t_3)) \\ &= \rho_1(t_1) + \rho_2(t_2) + \rho_3^2(t_3) \\ &= t_1 - 1 + 2 + 1 \\ &= t_1 + 2, \end{aligned}$$

Example

$$\begin{aligned} f_1^{\rho_1}(t) &= f(\rho_1(t_1), t_2, t_3) \\ &= \rho_1(t_1) + t_2 + t_3^2 \\ &= t_1 - 1 + 2 + 1 \\ &= t_1 + 2, \end{aligned}$$

Example

$$
\begin{aligned}
f_2^{\rho_2}(t) &= f(t_1, \rho_2(t_2), t_3) \\
&= t_1 + \rho_2(t_2) + t_3^2 \\
&= t_1 + 2 + 1 \\
&= t_1 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_3^{\rho_3}(t) &= f(t_1, t_2, \rho_3(t_3)) \\
&= t_1 + t_2 + \rho_3^2(t_3) \\
&= t_1 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{12}^{\rho_1\rho_2}(t) &= f(\rho_1(t_1), \rho_2(t_2), t_3) \\
&= \rho_1(t_1) + \rho_2(t_2) + t_3^2 \\
&= t_1 - 1 + 2 + 1 \\
&= t_1 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{13}^{\rho_1\rho_3}(t) &= f(\rho_1(t_1), t_2, \rho_3(t_3)) \\
&= \rho_1(t_1) + t_2 + \rho_3^2(t_3) \\
&= t_1 - 1 + 2 + 1 \\
&= t_1 + 2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{23}^{\rho_2\rho_3}(t) &= f(t_1, \rho_2(t_2), \rho_3(t_3)) \\
&= t_1 + \rho_2(t_2) + \rho_3^2(t_3) \\
&= t_1 + 2 + 1 \\
&= t_1 + 3,
\end{aligned}
$$

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\rho_2}(t) + f_3^{\rho_3}(t) \\
&= t_1 + 2 + t_1 + 3 + t_1 + 3 \\
&= 3t_1 + 8.
\end{aligned}
$$

Example

6. Let $t = (0, 2, 1)$. Then

$$\rho_1(t_1) = 0, \quad \rho_2(t_2) = 2, \quad \rho_3(t_3) = 1$$

and

$$\begin{aligned} f^{\rho}(t) &= f_1^{\rho_1}(t) = f_2^{\rho_2}(t) = f_3^{\rho_3}(t_3) \\ &= f_{12}^{\rho_1\rho_2}(t) = f_{13}^{\rho_1\rho_3}(t) = f_{23}^{\rho_2\rho_3}(t) \\ &= f(t) \\ &= 0 + 2 + 1 \\ &= 3, \end{aligned}$$

Example

Let $\Lambda^2 = \mathbb{N}_0 \times \mathbb{N}$ and $f : \Lambda^2 \to \mathbb{R}$ be defined by

$$f(t) = t_1 + t_2 + t_1^2 + t_2^2, \quad t = (t_1, t_2) \in \Lambda^2.$$

We will find

$$g(t) = f_1^{\rho_1}(t) + f_2^{\sigma_2}(t), \quad t \in \Lambda^2.$$

1. Let $t = (t_1, t_2) \in \Lambda^2$, $t_1 \neq 0$. Then

$$\rho_1(t_1) = t_1 - 1, \quad t_1 \in \mathbb{N}_0, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{N}.$$

Hence,

Example

$$
\begin{aligned}
g(t) &= f_1^{\rho_1}(t) + f_2^{\sigma_2}(t) \\
&= f(\rho_1(t_1), t_2) + f(t_1, \sigma_2(t_2)) \\
&= \rho_1(t_1) + t_2 + \rho_1^2(t_1) + t_2^2 + t_1 + \sigma_2(t_2) + t_1^2 + \sigma_2^2(t_2) \\
&= t_1 - 1 + t_2 + (t_1 - 1)^2 + t_2^2 + t_1 + t_2 + 1 + t_1^2 + (t_2 + 1)^2 \\
&= t_1 - 1 + t_2 + t_1^2 - 2t_1 + 1 + t_2^2 + t_1 + t_2 + 1 + t_1^2 + t_2^2 + 2t_2 + 1 \\
&= 2t_1^2 + 2t_2^2 + 4t_2 + 2.
\end{aligned}
$$

Example

2. Let $t = (0, t_2) \in \Lambda^2$. Then $\rho_1(t_1) = 0$ and

$$
\begin{aligned}
g(t) &= \rho_1(t_1) + t_2 + \rho_1^2(t_1) + t_2^2 + t_1 + \sigma_2(t_2) + t_1^2 + \sigma_2^2(t_2) \\
&= t_2 + t_2^2 + t_2 + 1 + (t_2 + 1)^2 \\
&= t_2^2 + 2t_2 + 1 + t_2^2 + 2t_2 + 1 \\
&= 2t_2^2 + 4t_2 + 2.
\end{aligned}
$$