Time Scales Analysis
Lecture 21
Partial Differentiation

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Definition

We set

$$\Lambda^{\kappa n} = \mathbb{T}_1^{\kappa} \times \mathbb{T}_2^{\kappa} \times \ldots \times \mathbb{T}_n^{\kappa},$$

$$\Lambda_i^{\kappa_i n} = \mathbb{T}_1 \times \ldots \times \mathbb{T}_{i-1} \times \mathbb{T}_i^{\kappa} \times \mathbb{T}_{i+1} \times \ldots \times \mathbb{T}_n, \quad i = 1, 2, \ldots, n,$$

$$\Lambda_{i_1 i_2 \ldots i_l}^{\kappa_{i_1} \kappa_{i_2} \ldots \kappa_{i_l} n} = \ldots \times \mathbb{T}_{i_1}^{\kappa} \times \ldots \times \mathbb{T}_{i_2}^{\kappa} \times \ldots \times \mathbb{T}_{i_l}^{\kappa} \times \ldots,$$

where $1 \le i_1 < i_2 < \ldots < i_l \le n$, $i_m \in \mathbb{N}$, $m = 1, 2, \ldots, l$.

Remark

If $(i_1, i_2, \ldots, i_l) = (1, 2, \ldots, n)$, *then*

$$\Lambda_{i_1 i_2 \ldots i_l}^{\kappa_1 \kappa_2 \ldots \kappa_l n} = \Lambda^{\kappa n}.$$

Definition

Assume that $f : \Lambda^n \to \mathbb{R}$ is a function and let $t \in \Lambda_i^{\kappa_i n}$. We define

$$\frac{\partial f(t_1, t_2, \ldots, t_n)}{\Delta_i t_i} = \frac{\partial f(t)}{\Delta_i t_i} = \frac{\partial f}{\Delta_i t_i}(t) = f_{t_i}^{\Delta_i}(t)$$

to be the number, provided it exists, with the property that for any $\varepsilon_i > 0$, there exists a neighbourhood

$$U_i = (t_i - \delta_i, t_i + \delta_i) \cap \mathbb{T}_i,$$

for some $\delta_i > 0$, such that

$$\Big| f(t_1, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)$$

$$- f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)\Big| \leq \varepsilon_i |\sigma_i(t_i) - s_i| \quad \text{for all} \quad s_i \in U_i. \quad (1)$$

Definition

We call $f_{t_i}^{\Delta_i}(t)$ the *partial delta derivative* (or *partial Hilger derivative* of f with respect to t_i at t. We say that f is *partial delta differentiable* (or *partial Hilger differentiable* with respect to t_i in $\Lambda_i^{\kappa_i n}$ if $f_{t_i}^{\Delta_i}(t)$ exists for all $t \in \Lambda_i^{\kappa_i n}$. The function $f_{t_i}^{\Delta_i} : \Lambda_i^{\kappa_i n} \to \mathbb{R}$ is said to be the *partial delta derivative* (or *partial Hilger derivative* with respect to t_i of f in $\Lambda_i^{\kappa_i n}$.

Theorem

The partial delta derivative is well defined.

Proof.

Let $t \in \Lambda^{\kappa_i n}$ for some $i \in \{1, 2, \ldots, n\}$. We assume that the partial derivative $f_{t_i}^{\Delta_i}(t)$ exists and

$$f_1(t) = f_{t_i}^{\Delta_i}(t), \quad f_2(t) = f_{t_i}^{\Delta_i}(t).$$

Let $\varepsilon_i > 0$ be arbitrarily chosen. Then there exists $\delta_i > 0$ such that for every

$$s_i \in (t_i - \delta_i, t_i + \delta_i) \cap \mathbb{T}_i,$$

we have

Proof.

$$\Big| f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i_1}, s_i, t_{i+1}, \ldots, t_n)$$

$$- f_1(t)(\sigma_i(t_i) - s_i)\Big| \leq \frac{\varepsilon_i}{2} |\sigma_i(t_i) - s_i| \quad (2)$$

and

$$\Big| f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)$$

$$- f_2(t)(\sigma_i(t_i) - s_i)\Big| \leq \frac{\varepsilon_i}{2} |\sigma_i(t_i) - s_i|. \quad (3)$$

From (2) and (3), for

$$s_i \in (t_i - \delta_i, t_i + \delta_i) \cap \mathbb{T}_i, \quad s_i \neq \sigma_i(t_i),$$

we obtain

Proof.

$$|f_1(t) - f_2(t)| = \Bigg| f_1(t) - \frac{f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i}$$

$$+ \frac{f(t_1, t_2, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i} + \frac{f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i}$$

$$- \frac{f(t_1, t_2, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i} - f_2(t) \Bigg|$$

□

Proof.

$$\leq \left| f_1(t) - \frac{f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, s_i, t_{i+}}{\sigma_i(t_i) - s_i} \right.$$

$$+ \left| f_2(t) - \frac{f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, s_i, t}{\sigma_i(t_i) - s_i} \right.$$

$$\leq \frac{\varepsilon_i}{2} + \frac{\varepsilon_i}{2} = \varepsilon_i.$$

Because $\varepsilon_i > 0$ was arbitrarily chosen, we conclude that $f_1(t) = f_2(t)$. □

Example

Let $f(t) = t_1 t_2 t_3$, $t = (t_1, t_2, t_3) \in \Lambda^3$. We will prove that

$$f_{t_1}^{\Delta_1}(t) = t_2 t_3.$$

Indeed, for every $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that for every $s_1 \in (t_1 - \delta_1, t_1 + \delta_1)$, $s_1 \in \mathbb{T}_1$, we have

$$\begin{aligned} & |f(\sigma_1(t_1), t_2, t_3) - f(s_1, t_2, t_3) - t_2 t_3(\sigma_1(t_1) - s_1)| \\ = \; & |\sigma_1(t_1) t_2 t_3 - s_1 t_2 t_3 - t_2 t_3(\sigma_1(t_1) - s_1)| \\ = \; & 0 \\ \leq \; & \varepsilon_1 |\sigma_1(t_1) - s_1|. \end{aligned}$$

Example

Let $f(t) = t_2^2 + t_1 t_3$, $t = (t_1, t_2, t_3) \in \Lambda^3$. We will prove that

$$f_{t_2}^{\Delta_2}(t) = \sigma_2(t_2) + t_2.$$

Indeed, for every $\varepsilon_2 > 0$, there exists

$$\delta_2 > 0, \quad \delta_2 \leq \varepsilon_2,$$

such that for every

$$s_2 \in (t - \delta_2, t + \delta_2), \quad s_2 \in \mathbb{T}_2,$$

we have

$$|s_2 - t_2| \leq \delta_2 \leq \varepsilon_2$$

and

Example

$$
\begin{aligned}
&|f(t_1,\sigma_2(t_2),t_3)-f(t_1,s_2,t_3)-f_{t_2}^{\Delta_2}(t)(\sigma_2(t_2)-s_2)| \\
&= |(\sigma_2(t_2))^2+t_1t_3-s_2^2-t_1t_3-(\sigma_2(t_2)+t_2)(\sigma_2(t_2)-s_2)| \\
&= |(\sigma_2(t_2))^2-s_2^2-(\sigma_2(t_2)+t_2)(\sigma_2(t_2)-s_2)| \\
&= |(\sigma_2(t_2)-s_2)(\sigma_2(t_2)+s_2)-(\sigma_2(t_2)+t_2)(\sigma_2(t_2)-s_2)| \\
&= |(\sigma_2(t_2)-s_2)(\sigma_2(t_2)+s_2-\sigma_2(t_2)-t_2)| \\
&= |t_2-s_2||\sigma_2(t_2)-s_2| \le \varepsilon_2|\sigma_2(t_2)-s_2|.
\end{aligned}
$$

Example

Let $f(t) = t_1 t_2 \sqrt{t_3}$, $t = (t_1, t_2, t_3) \in \Lambda^3$. We will prove that

$$f_{t_3}^{\Delta_3}(t) = \frac{t_1 t_2}{\sqrt{\sigma_3(t_3)} + \sqrt{t_3}}.$$

Indeed, for every $\varepsilon_3 > 0$, there exists $\delta_3 > 0$, $\delta_3 \leq \varepsilon_3 \frac{\sigma_3(t_3)\sqrt{t_3}}{1+|t_1 t_2|}$, such that for every $s_3 \in (t_3 - \delta_3, t_3 + \delta_3)$, $s_3 \in \mathbb{T}_3$, we have

$$|t_3 - s_3| \leq \delta_3$$

and

Example

$$
\begin{aligned}
&\left| f(t_1, t_2, \sigma_3(t_3)) - f(t_1, t_2, s_3) - f_{t_3}^{\Delta_3}(t)(\sigma_3(t_3) - s_3) \right| \\
&= \left| t_1 t_2 \sqrt{\sigma_3(t_3)} - t_1 t_2 \sqrt{s_3} - \frac{t_1 t_2}{(\sqrt{\sigma_3(t_3)} + \sqrt{t_3})}(\sigma_3(t_3) - s_3) \right| \\
&= \left| t_1 t_2 \frac{(\sqrt{\sigma_3(t_3)} - \sqrt{s_3})(\sqrt{\sigma_3(t_3)} + \sqrt{s_3})}{\sqrt{\sigma_3(t_3)} + \sqrt{s_3}} - \frac{t_1 t_2}{\sqrt{\sigma_3(t_3)} + \sqrt{t_3}}(\sigma_3(t_3) - s_3) \right. \\
&= |\sigma_3(t_3) - s_3| |t_1 t_2| \left| \frac{1}{\sqrt{\sigma_3(t_3)} + \sqrt{s_3}} - \frac{1}{\sqrt{\sigma_3(t_3)} + \sqrt{t_3}} \right|
\end{aligned}
$$

Example

$$
\begin{aligned}
&= |\sigma_3(t_3) - s_3||t_1 t_2| \frac{|\sqrt{t_3} - \sqrt{s_3}|}{(\sqrt{\sigma_3(t_3)} + \sqrt{t_3})(\sqrt{\sigma_3(t_3)} + \sqrt{s_3})} \\
&= |\sigma_3(t_3) - s_3||t_1 t_2| \frac{|t_3 - s_3|}{(\sqrt{\sigma_3(t_3)} + \sqrt{t_3})(\sqrt{\sigma_3(t_3)} + \sqrt{s_3})(\sqrt{t_3} + \sqrt{s_3})} \\
&\leq |\sigma_3(t_3) - s_3||t_1 t_2| \frac{|t_3 - s_3|}{\sigma_3(t_3)\sqrt{t_3}} \\
&\leq \delta_3 |\sigma_3(t_3) - s_3||t_1 t_2| \frac{1}{\sigma_3(t_3)\sqrt{t_3}} \\
&\leq \delta_3 |\sigma_3(t_3) - s_3|(1 + |t_1 t_2|) \frac{1}{\sigma_3(t_3)\sqrt{t_3}} \\
&\leq \varepsilon_3 |\sigma_3(t_3) - s_3|.
\end{aligned}
$$

Remark

For $t \in \Lambda^n$ and $s_i \in \mathbb{T}_i$, we write

$$t_{s_i} = (t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n), \quad i \in \{1, 2, \ldots, n\}.$$

Then we can rewrite (1) *in the form*

$$|f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \le \varepsilon_i |\sigma_i(t_i) - s_i|. \tag{4}$$

Theorem

Let $f : \Lambda^n \to \mathbb{R}$ be a function and $t \in \Lambda_i^{\kappa_i n}$. If f is delta differentiable with respect to t_i at t, then

$$\lim_{s_i \to t_i} f(t_{s_i}) = f(t). \tag{5}$$

Proof.

Because f is delta differentiable with respect to t_i at t, we have that for every $\varepsilon > 0$, there exists $\delta > 0$, $\delta < \{1, \varepsilon^*\}$, such that for every $s_i \in (t_i - \delta, t_i + \delta)$, $s_i \in \mathbb{T}_i$, we have

$$|f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \leq \varepsilon^* |\sigma_i(t_i) - s_i|$$

and $\square$

Proof.

$$|f_i^{\sigma_i}(t) - f(t) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - t_i)| \leq \varepsilon^* |\sigma_i(t_i) - t_i|,$$

for

$$\varepsilon^* = \frac{\varepsilon}{1 + 2\mu_i(t_i) + |f_{t_i}^{\Delta_i}(t)|}.$$

Hence, for every $s_i \in (t_i - \delta, t_i + \delta)$, $s_i \in \mathbb{T}_i$, we have

$$\begin{aligned} |f(t) - f(t_{s_i})| &= \Big| f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i) \\ &\quad - (f_i^{\sigma_i}(t) - f(t) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - t_i)) + f_{t_i}^{\Delta_i}(t)(t_i - s_i) \Big| \\ &\leq \quad |f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \end{aligned}$$

□

Proof.

$$
\begin{aligned}
& +|f_i^{\sigma_i}(t) - f(t) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - t_i)| + |f_{t_i}^{\Delta_i}(t)||t_i - s_i| \\
\leq \quad & \varepsilon^* |\sigma_i(t_i) - s_i| + \varepsilon^* |\sigma_i(t_i) - t_i| + |f_{t_i}^{\Delta_i}(t)||t_i - s_i| \\
\leq \quad & \varepsilon^* |\sigma_i(t_i) - s_i| + \varepsilon^* \mu_i(t_i) + \varepsilon^* |f_{t_i}^{\Delta_i}(t)| \\
= \quad & \varepsilon^* \left(\mu_i(t_i) + |\sigma_i(t_i) - s_i| + |f_{t_i}^{\Delta_i}(t)| \right) \\
= \quad & \varepsilon^* \left(\mu_i(t_i) + |\sigma_i(t_i) - t_i + t_i - s_i| + |f_{t_i}^{\Delta_i}(t)| \right)
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
&\le \varepsilon^* \left(\mu_i(t_i) + \mu_i(t_i) + |t_i - s_i| + |f_{t_i}^{\Delta_i}(t)|\right) \\
&\le \varepsilon^* \left(1 + 2\mu_i(t_i) + |f_{t_i}^{\Delta_i}(t)|\right) \\
&= \varepsilon.
\end{aligned}
$$

Because $\varepsilon > 0$ was arbitrarily chosen, we obtain (5). □

Theorem

Let $f : \Lambda^n \to \mathbb{R}$, $t \in \Lambda_i^{\kappa_i n}$, *and*

$$\lim_{s_i \to t_i} f(t_{s_i}) = f(t). \tag{6}$$

If $t_i < \sigma_i(t_i)$, *then* f *is delta differentiable with respect to* t_i *at* t *and*

$$f_{t_i}^{\Delta_i}(t) = \frac{f_i^{\sigma_i}(t) - f(t)}{\mu_i(t_i)}. \tag{7}$$

Proof.

When $s_i \to t_i$, using (1) and (6), we get (7). □

Example

Let $\Lambda^2 = \mathbb{N} \times 2^{\mathbb{N}}$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1^2 t_2 - 2t_1, \quad t = (t_1, t_2) \in \Lambda^2.$$

Then

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = 2t_2, \quad t_2 \in 2^{\mathbb{N}}.$$

Hence,

$$t_1 < \sigma_1(t_1), \quad t_1 \in \mathbb{N}, \quad t_2 < \sigma_2(t_2), \quad t_2 \in 2^{\mathbb{N}}.$$

Therefore,

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t) &= \frac{f_1^{\sigma_1}(t) - f(t)}{\sigma_1(t_1) - t_1} \\
&= \frac{(\sigma_1(t_1))^2 t_2 - 2\sigma_1(t_1) - t_1^2 t_2 + 2t_1}{t_1 + 1 - t_1} \\
&= (t_1+1)^2 t_2 - 2(t_1+1) - t_1^2 t_2 + 2t_1 \\
&= t_1^2 t_2 + 2t_1 t_2 + t_2 - 2t_1 - 2 - t_1^2 t_2 + 2t_1 \\
&= 2t_1 t_2 + t_2 - 2, \quad (t_1, t_2) \in \Lambda_1^{\kappa_1 2},
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_2}^{\Delta_2}(t) &= \frac{f_2^{\sigma_2}(t) - f(t)}{\sigma_2(t_2) - t_2} \\
&= \frac{t_1^2\sigma_2(t_2) - 2t_1 - (t_1^2 t_2 - 2t_1)}{2t_2 - t_2} \\
&= \frac{2t_1^2 t_2 - 2t_1 - t_1^2 t_2 + 2t_1}{t_2} \\
&= t_1^2, \quad (t_1, t_2) \in \Lambda_2^{\kappa_2 2}.
\end{aligned}
$$

Example

Let $\Lambda^2 = 3^{\mathbb{N}} \times \mathbb{N}_0^2$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1^2 + 2t_1 t_2 + t_2, \quad t = (t_1, t_2) \in \Lambda^2.$$

We will find $f_{t_1}^{\Delta_1}$ and $f_{t_2}^{\Delta_2}$. Here,

$$\sigma_1(t_1) = 3t_1, \quad t_1 \in 3^{\mathbb{N}}, \quad \sigma_2(t_2) = \left(\sqrt{t_2} + 1\right)^2, \quad t_2 \in \mathbb{N}_0^2.$$

Then

$$\sigma_1(t_1) > t_1 \quad \text{for all} \quad t_1 \in 3^{\mathbb{N}}, \quad \sigma_2(t_2) > t_2 \quad \text{for all} \quad t_2 \in \mathbb{N}_0^2.$$

Hence,

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t) &= \frac{f(\sigma_1(t_1), t_2) - f(t_1, t_2)}{\sigma_1(t_1) - t_1} \\
&= \frac{(\sigma_1(t_1))^2 + 2\sigma_1(t_1)t_2 + t_2 - t_1^2 - 2t_1 t_2 - t_2}{\sigma_1(t_1) - t_1} \\
&= \frac{(\sigma_1(t_1) - t_1)(\sigma_1(t_1) + t_1) + 2(\sigma_1(t_1) - t_1)t_2}{\sigma_1(t_1) - t_1} \\
&= \sigma_1(t_1) + t_1 + 2t_2
\end{aligned}
$$

Example

$$
\begin{aligned}
&= 3t_1 + t_1 + 2t_2 \\
&= 4t_1 + 2t_2, \quad (t_1, t_2) \in \Lambda_1^{\kappa_1 2}, \\
f_{t_2}^{\Delta_2}(t) &= \frac{f(t_1, \sigma_2(t_2)) - f(t_1, t_2)}{\sigma_2(t_2) - t_2} \\
&= \frac{t_1^2 + 2t_1\sigma_2(t_2) + \sigma_2(t_2) - t_1^2 - 2t_1 t_2 - t_2}{\sigma_2(t_2) - t_2} \\
&= \frac{2t_1(\sigma_2(t_2) - t_2) + \sigma_2(t_2) - t_2}{\sigma_2(t_2) - t_2} \\
&= 2t_1 + 1, \quad (t_1, t_2) \in \Lambda^2, \quad (t_1, t_2) \in \Lambda_2^{\kappa_2 2}.
\end{aligned}
$$

Example

Let $\Lambda^2 = \mathbb{N} \times 4^{\mathbb{N}}$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1^3 + t_2^3 + t_1^2 t_2^2 + t_2^4, \quad (t_1, t_2) \in \Lambda^2.$$

We will find $f_{t_1}^{\Delta_1}$ and $f_{t_2}^{\Delta_2}$. We have

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{N}, \quad \sigma_2(t_2) = 4t_2, \quad t_2 \in 4^{\mathbb{N}}.$$

Therefore,

$$\sigma_1(t_1) > t_1 \quad \text{for all} \quad t_1 \in \mathbb{N}, \quad \sigma_2(t_2) > t_2 \quad \text{for all} \quad t_2 \in 4^{\mathbb{N}}.$$

Hence,

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t) &= \frac{f(\sigma_1(t_1), t_2) - f(t_1, t_2)}{\sigma_1(t_1) - t_1} \\
&= \frac{(\sigma_1(t_1))^3 + t_2^3 + (\sigma_1(t_1))^2 t_2^2 + t_2^4 - t_1^3 - t_2^3 - t_1^2 t_2^2 - t_2^4}{\sigma_1(t_1) - t_1} \\
&= \frac{(\sigma_1(t_1) - t_1)((\sigma_1(t_1))^2 + t_1\sigma_1(t_1) + t_1^2) + t_2^2(\sigma_1(t_1) - t_1)(\sigma_1(t_1}{\sigma_1(t_1) - t_1} \\
&= (\sigma_1(t_1))^2 + \sigma_1(t_1)t_1 + t_1^2 + t_2^2(\sigma_1(t_1) + t_1) \\
&= (t_1 + 1)^2 + t_1(t_1 + 1) + t_1^2 + t_2^2(2t_1 + 1)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= t_1^2 + 2t_1 + 1 + t_1^2 + t_1 + t_1^2 + 2t_1 t_2^2 + t_2^2 \\
&= 3t_1^2 + 3t_1 + 2t_1 t_2^2 + t_2^2 + 1, \quad (t_1, t_2) \in \Lambda_1^{\kappa_1 2}, \\
f_{t_2}^{\Delta_2}(t) &= \frac{f(t_1, \sigma_2(t_2)) - f(t_1, t_2)}{\sigma_2(t_2) - t_2} \\
&= \frac{t_1^3 + (\sigma_2(t_2))^3 + t_1^2 \sigma_2^2(t_2) + (\sigma_2(t_2))^4 - t_1^3 - t_2^3 - t_1^2 t_2^2 - t_2^4}{\sigma_2(t_2) - t_2} \\
&= \frac{(\sigma_2(t_2) - t_2)((\sigma_2(t_2))^2 + t_2 \sigma_2(t_2) + t_2^2) + t_1^2(\sigma_2(t_2) - t_2)(\sigma_2(t_2}{\sigma_2(t_2) - t_2}
\end{aligned}
$$

Example

$$
\begin{aligned}
& +\frac{(\sigma_2(t_2)-t_2)((\sigma_2(t_2))^3+t_2(\sigma_2(t_2))^2+t_2^2\sigma_2(t_2)+t_2^3)}{\sigma_2(t_2)-t_2} \\
= \;& (\sigma_2(t_2))^2+t_2\sigma_2(t_2)+t_2^2+t_1^2(\sigma_2(t_2)+t_2)+(\sigma_2(t_2))^3 \\
& +t_2(\sigma_2(t_2))^2+t_2^2\sigma_2(t_2)+t_2^3 \\
= \;& 16t_2^2+4t_2^2+t_2^2+5t_1^2t_2+64t_2^3+16t_2^3+4t_2^3+t_2^3 \\
= \;& 5t_1^2t_2+21t_2^2+85t_2^3, \quad (t_1,t_2)\in\Lambda_2^{\kappa_2 2}.
\end{aligned}
$$