Time Scales Analysis
Lecture 22
Properties of Partial Delta Derivatives

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Theorem

Let $t \in \Lambda_i^{\kappa_i n}$ *and* $t_i = \sigma_i(t_i)$. *Then* f *is partial delta differentiable with respect to* t_i *at* t *if and only if the limit*

$$\lim_{s_i \to t_i} \frac{f(t) - f(t_{s_i})}{t_i - s_i}$$

exists as a finite number. In this case,

$$f_{t_i}^{\Delta_i}(t) = \lim_{s_i \to t_i} \frac{f(t) - f(t_{s_i})}{t_i - s_i}. \tag{1}$$

Proof.

Let f be partial delta differentiable with respect to t_i at t. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s_i \in (t - \delta_i, t + \delta_i)$, $s_i \in \mathbb{T}_i$, we have (**??**). Hence, using that $\sigma_i(t_i) = t_i$, we get

$$\left| \frac{f(t) - f(t_{s_i})}{t_i - s_i} - f_{t_i}^{\Delta_i}(t) \right| \leq \varepsilon.$$

Because $\varepsilon > 0$ was arbitrarily chosen, we conclude (1). $\square$

Proof.

Suppose (1) holds. Then, for $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s_i \in (t_i - \delta, t_i + \delta)$, $s_i \in \mathbb{T}_i$, we have

$$|f(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(t_i - s_i)| \leq \varepsilon |t_i - s_i|.$$

Hence, using $t_i = \sigma_i(t_i)$, we get the definition for partial delta derivatives. The proof is complete. $\square$

Example

Let $\Lambda^2 = \left(\frac{1}{2}\right)^{\mathbb{N}} \times \mathbb{N}$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1^2 t_2, \quad t = (t_1, t_2) \in \Lambda^2.$$

We will find $f_{t_1}^{\Delta_1}\left(\frac{1}{2}, t_2\right)$, $t_2 \in \mathbb{N}$. We note that

$$\mathbb{T}_1 = \left(\frac{1}{2}\right)^{\mathbb{N}}, \quad \mathbb{T}_2 = \mathbb{N}.$$

Hence, $\sigma_2\left(\frac{1}{2}\right) = \frac{1}{2}$. Thus,

$$\lim_{s_1 \to \frac{1}{2}} \frac{f\left(\frac{1}{2}, t_2\right) - f(s_1, t_2)}{\frac{1}{2} - s_1} = \lim_{s_1 \to \frac{1}{2}} \frac{\frac{1}{4} t_2 - s_1^2 t_2}{\frac{1}{2} - s_1}$$

Example

$$
\begin{aligned}
&= \lim_{s_1 \to \frac{1}{2}} \frac{\left(\frac{1}{2} - s_1\right)\left(\frac{1}{2} + s_1\right) t_2}{\frac{1}{2} - s_1} \\
&= \lim_{s_1 \to \frac{1}{2}} \left(\frac{1}{2} + s_1\right) t_2 \\
&= t_2.
\end{aligned}
$$

Consequently, $f_{t_1}^{\Delta_1}\left(\frac{1}{2}, t_2\right) = t_2$.

Example

Let $\Lambda^2 = \mathbb{Z} \times (-2\mathbb{N})$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1 t_2 + t_2^2, \quad t = (t_1, t_2) \in \Lambda^2.$$

We will find $f_{t_2}^{\Delta_2}(t_1, -2)$, $t_1 \in \mathbb{Z}$. We note that $\mathbb{T}_1 = \mathbb{Z}$, $\mathbb{T}_2 = -2\mathbb{N}$. Hence, $\sigma_2(-2) = -2$ and

$$\begin{aligned}
\lim_{t_2 \to -2} \frac{f(t_1, -2) - f(t_1, t_2)}{-2 - t_2} &= \lim_{t_2 \to -2} \frac{-2t_1 + 4 - t_1 t_2 - t_2^2}{-2 - t_2} \\
&= \lim_{t_2 \to -2} \frac{(2 + t_2)t_1 + (t_2 - 2)(t_2 + 2)}{2 + t_2} \\
&= \lim_{t_2 \to -2} (t_1 + t_2 - 2) \\
&= t_1 - 4.
\end{aligned}$$

Consequently, $f_{t_2}^{\Delta_2}(t_1, -2) = t_1 - 4$.

Example

Let $\Lambda^2 = \mathbb{Z} \times \left(\frac{1}{3}\right)^{\mathbb{N}_0}$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1 + t_2 + t_2^2, \quad t = (t_1, t_2) \in \Lambda^2.$$

We will find $f_{t_2}^{\Delta_2}(t_1, 1)$, $t_1 \in \mathbb{Z}$. We note that

$$\mathbb{T}_1 = \mathbb{Z}, \quad \mathbb{T}_2 = \left(\frac{1}{3}\right)^{\mathbb{N}_0}.$$

Hence, $\sigma_2(1) = 1$ and

$$\lim_{s_2 \to 1} \frac{f(t_1, 1) - f(t_1, s_2)}{1 - s_2} = \lim_{s_2 \to 1} \frac{t_1 + 2 - t_1 - s_2 - s_2^2}{1 - s_2}$$

Example

$$
\begin{aligned}
&= \lim_{s_2 \to 1} \frac{(1-s_2)+(1-s_2)(1+s_2)}{1-s_2} \\
&= \lim_{s_2 \to 1} (2+s_2) = 3.
\end{aligned}
$$

Consequently, $f_{t_2}^{\Delta_2}(t_1, 1) = 3$.

Theorem

Let $t \in \Lambda_i^{\kappa_i n}$*. Suppose* $f : \Lambda^n \to \mathbb{R}$ *is a function that is partial delta differentiable with respect to* t_i *at* t*. If* $\alpha \in \mathbb{R}$*, then* αf *is partial delta differentiable with respect to* t_i *at* t *and*

$$(\alpha f)_{t_i}^{\Delta_i}(t) = \alpha f_{t_i}^{\Delta_i}(t).$$

Proof.

Without loss of generality, we may assume $\alpha \neq 0$. Since f is partial delta differentiable with respect to t_i at t, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s_i \in (t_i - \delta, t_i + \delta)$, $s_i \in \mathbb{T}_i$, we have

$$|f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \leq \frac{\varepsilon}{|\alpha|}|\sigma_i(t_i) - s_i|.$$

Hence, □

Proof.

$$
\begin{aligned}
&\left|(\alpha f)_i^{\sigma_i}(t) - (\alpha f)(t_{s_i}) - \alpha f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)\right| \\
&= |\alpha|\left|f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)\right| \\
&\leq |\alpha|\frac{\varepsilon}{|\alpha|}|\sigma_i(t_i) - s_i| \\
&= \varepsilon|\sigma_i(t_i) - s_i|,
\end{aligned}
$$

which completes the proof. □

Theorem

Let $t \in \Lambda_i^{\kappa_i n}$. *Assume* $f, g : \Lambda^n \to \mathbb{R}$ *are partial delta differentiable with respect to* t_i *at* t. *Then* $f + g$ *is partial delta differentiable with respect to* t_i *at* t *and*

$$(f+g)_{t_i}^{\Delta_i}(t) = f_{t_i}^{\Delta_i}(t) + g_{t_i}^{\Delta_i}(t).$$

Proof.

Since f and g are partial delta differentiable with respect to t_i at t, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $s_i \in (t_i - \delta, t_i + \delta)$, $s_i \in \mathbb{T}_i$, we have

$$|f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \leq \frac{\varepsilon}{2}|\sigma_i(t_i) - s_i|$$

and

$$|g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \leq \frac{\varepsilon}{2}|\sigma_i(t_i) - s_i|,$$

whereupon □

Proof.

$$
\begin{aligned}
&|(f+g)_i^{\sigma_i}(t) - (f+g)(t_{s_i}) - (f_{t_i}^{\Delta_i}(t) + g_{t_i}^{\Delta_i}(t))(\sigma_i(t_i) - s_i)| \\
&= \; |f_i^{\sigma_i}(t) + g_i^{\sigma_i}(t) - f(t_{s_i}) - g(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - \\
&\leq \; |f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| + |g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(\sigma_i(t_i) - \\
&\leq \; \frac{\varepsilon}{2}|\sigma_i(t_i) - s_i| + \frac{\varepsilon}{2}|\sigma_i(t_i) - s_i| \\
&= \; \varepsilon|\sigma_i(t_i) - s_i|,
\end{aligned}
$$

which completes the proof. $\square$

Theorem

Let $t \in \Lambda_i^{\kappa_i n}$. Assume $f, g : \Lambda^n \to \mathbb{R}$ are partial delta differentiable with respect to t_i at t. Then fg is partial delta differentiable with respect to t_i at t and

$$(fg)_{t_i}^{\Delta_i}(t) = f_{t_i}^{\Delta_i}(t)g(t) + f_i^{\sigma_i}(t)g_{t_i}^{\Delta_i}(t) = f(t)g_{t_i}^{\Delta_i}(t) + f_{t_i}^{\Delta_i}(t)g_i^{\sigma_i}(t).$$

Proof.

Since f and g are partial delta differentiable at t, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $s_i \in (t_i - \delta, t_i + \delta)$, $s_i \in \mathbb{T}_i$, we have

$$\begin{aligned}
|f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| &\leq \varepsilon^* |\sigma_i(t_i) - s_i|, \\
|g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| &\leq \varepsilon^* |\sigma_i(t_i) - s_i|, \\
|f(t_{s_i}) - f(t)| &\leq \varepsilon^*, \\
|g(t_{s_i}) - g(t)| &\leq \varepsilon^*,
\end{aligned}$$

□

Proof.

where

$$\varepsilon^* < \min\left\{ \frac{\varepsilon}{1 + |f_i^{\sigma_i}(t)| + |g(t_{s_i})| + |f_{t_i}^{\Delta_i}(t)|}, \frac{\varepsilon}{1 + |g_i^{\sigma_i}(t)| + |f(t_{s_i})| + |g_{t_i}^{\Delta_i}(t)|} \right.$$

Hence,

$$|(fg)_i^{\sigma_i}(t) - (fg)(t_{s_i}) - (f_{t_i}^{\Delta_i}(t)g(t) + f_i^{\sigma_i}(t)g_{t_i}^{\Delta_i}(t))(\sigma_i(t_i) - s_i)|$$

$$= \quad |f_i^{\sigma_i}(t)g_i^{\sigma_i}(t) - f(t_{s_i})g(t_{s_i}) - (f_{t_i}^{\Delta_i}(t)g(t) + f_i^{\sigma_i}(t)g_{t_i}^{\Delta_i}(t))(\sigma_i(t_i) - s_i)|$$

□

Proof.

$$
\begin{aligned}
&= |f_i^{\sigma_i}(t)(g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)) \\
&\quad + f_i^{\sigma_i}(t)g(t_{s_i}) - f(t_{s_i})g(t_{s_i}) - f_{t_i}^{\Delta_i}(t)g(t)(\sigma_i(t_i) - s_i)| \\
&= |f_i^{\sigma_i}(t)(g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)) \\
&\quad + g(t_{s_i})(f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i))
\end{aligned}
$$

$\square$

Proof.

$$+(g(t_{s_i}) - g(t))f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)|$$

$$\leq \quad \varepsilon^*|\sigma_i(t_i) - s_i||f_i^{\sigma_i}(t)| + \varepsilon^*|g(t_{s_i})||\sigma_i(t_i) - s_i|$$

$$+|g(t_{s_i}) - g(t)||f_{t_i}^{\Delta_i}(t)||\sigma_i(t_i) - s_i|$$

$$\leq \quad \varepsilon^*|\sigma_i(t_i) - s_i||f_i^{\sigma_i}(t)| + \varepsilon^*|g(t_{s_i})||\sigma_i(t_i) - s_i| + \varepsilon^*|f_{t_i}^{\Delta_i}(t)||\sigma_i(t_i) - s_i|$$

$\square$

Proof.

$$\begin{aligned} &= \varepsilon^*(|f_i^{\sigma_i}(t)| + |g(t_{s_i})| + |f_{t_i}^{\Delta_i}(t)|)|\sigma_i(t_i) - s_i| \\ &\leq \varepsilon^*(1 + |f_i^{\sigma_i}(t)| + |g(t_{s_i})| + |f_{t_i}^{\Delta_i}(t)|)|\sigma_i(t_i) - s_i| \\ &\leq \varepsilon|\sigma_i(t_i) - s_i| \end{aligned}$$

and

$$|(fg)_i^{\sigma_i}(t) - (fg)(t_{s_i}) - (f(t)g_{t_i}^{\Delta_i}(t) + f_{t_i}^{\Delta_i}(t)g_i^{\sigma_i}(t))(\sigma_i(t_i) - s_i)|$$

$$= |f_i^{\sigma_i}(t)g_i^{\sigma_i}(t) - f(t_{s_i})g(t_{s_i}) - (f(t)g_{t_1}^{\Delta_i}(t) + f_{t_i}^{\Delta_i}(t)g_i^{\sigma_i}(t))(\sigma_i(t_i) - s_i)|$$

□

Proof.

$$
\begin{aligned}
&= |g_i^{\sigma_i}(t)(f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(\sigma_i(t_i) - s_i)) \\
&\quad + g_i^{\sigma_i}(t)f(t_{s_i}) - f(t_{s_i})g(t_{s_i}) - f(t)g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \\
&= |g_i^{\sigma_i}(t)(f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)) \\
&\quad + f(t_{s_i})(g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i))
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
& +f(t_{s_i})g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)-f(t)g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)| \\
\leq\ & |g_i^{\sigma_i}(t)||f_i^{\sigma_i}(t)-f(t_{s_i})-f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)| \\
& +|f(t_{s_i})||g_i^{\sigma_i}(t)-g(t_{s_i})-g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)| \\
& +|g_{t_i}^{\Delta_i}(t)||f(t_{s_i})-f(t)||\sigma_i(t_i)-s_i|
\end{aligned}
$$

$\square$

Proof.

$$\begin{aligned}
&\leq \varepsilon^*|g_i^{\sigma_i}(t)||\sigma_i(t_i)-s_i|+\varepsilon^*|f(t_{s_i})||\sigma_i(t_i)-s_i|+\varepsilon^*|g_{t_i}^{\Delta_i}(t)||\sigma_i(t_i)-s_i| \\
&\leq \varepsilon^*(1+|g_i^{\sigma_i}(t)|+|f(t_{s_i})|+|g_{t_i}^{\Delta_i}(t)|)|\sigma_i(t_i)-s_i| \\
&\leq \varepsilon|\sigma_i(t_i)-s_i|.
\end{aligned}$$

This completes the proof. □

Example

Let $\Lambda^2 = \mathbb{N} \times \mathbb{N}_0^2$ and define $h : \Lambda^2 \to \mathbb{R}$ by

$$h(t) = (t_1^2 + 2t_1)(t_1^3 + t_2), \quad t \in \Lambda^2.$$

Here, $\mathbb{T}_1 = \mathbb{N}$ and $\mathbb{T}_2 = \mathbb{N}_0^2$. Then

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = (\sqrt{t_2} + 1)^2, \quad t_2 \in \mathbb{T}_2.$$

We will find $h_{t_1}^{\Delta_1}(t)$, $t \in \Lambda_1^{\kappa_1 2}$. Let

$$f(t) = t_1^2 + 2t_1, \quad g(t) = t_1^3 + t_2, \quad t \in \Lambda^2.$$

Hence, $h(t) = f(t)g(t)$, $t \in \Lambda^2$. Then, for $t \in \Lambda_1^{\kappa_1 2}$, we have

Example

$$
\begin{aligned}
h_{t_1}^{\Delta_1}(t) &= f_{t_1}^{\Delta_1}(t)g(t) + f_1^{\sigma_1}(t)g_{t_1}^{\Delta_1}(t) \\
&= (\sigma_1(t_1) + t_1 + 2)g(t) + ((\sigma_1(t_1))^2 + 2\sigma_1(t_1))((\sigma_1(t_1))^2 + t_1\sigma_1(t \\
&= (t_1 + 1 + t_1 + 2)(t_1^3 + t_2) + ((t_1 + 1)^2 + 2t_1 + 2)((t_1 + 1)^2 + t_1 \\
&= (2t_1 + 3)(t_1^3 + t_2) + (t_1^2 + 4t_1 + 3)(3t_1^2 + 3t_1 + 1) \\
&= 2t_1^4 + 2t_1t_2 + 3t_1^3 + 3t_2 + 3t_1^4 + 3t_1^3 + t_1^2 + 12t_1^3 + 12t_1^2 + 4t_1 + \\
&= 5t_1^4 + 18t_1^3 + 22t_1^2 + 13t_1 + 2t_1t_2 + 3t_2 + 3.
\end{aligned}
$$

Example

Let $\Lambda^2 = 2^{\mathbb{N}} \times \mathbb{N}$ and define $h : \Lambda^2 \to \mathbb{R}$ by

$$h(t) = (t_1^3 + t_2)(t_1^2 + t_1 t_2 + t_2^2), \quad t \in \Lambda^2.$$

We will find $h_{t_1}^{\Delta_1}(t)$ for $t \in \Lambda_1^{\kappa_1 2}$ and $h_{t_2}^{\Delta_2}(t)$ for $t \in \Lambda_2^{\kappa_2 2}$. Let

$$f(t) = t_1^3 + t_2, \quad g(t) = t_1^2 + t_1 t_2 + t_2^2, \quad t = (t_1, t_2) \in \Lambda^2.$$

Here,

$$\mathbb{T}_1 = 2^{\mathbb{N}}, \quad \mathbb{T}_2 = \mathbb{N}.$$

Then

$$\sigma_1(t_1) = 2t_1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

Then, for $t \in \Lambda_1^{\kappa_1 2}$, we have

Example

$$
\begin{aligned}
h_{t_1}^{\Delta_1}(t) &= f_{t_1}^{\Delta_1}(t)g(t) + f_1^{\sigma_1}(t)g_{t_1}^{\Delta_1}(t) \\
&= ((\sigma_1(t_1))^2 + t_1\sigma_1(t_1) + t_1^2)(t_1^2 + t_1t_2 + t_2^2) \\
&\quad +((\sigma_1(t_1))^3 + t_2)(\sigma_1(t_1) + t_1 + t_2) \\
&= (4t_1^2 + 2t_1^2 + t_1^2)(t_1^2 + t_1t_2 + t_2^2) \\
&\quad +(8t_1^3 + t_2)(2t_1 + t_1 + t_2)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= 7t_1^2(t_1^2 + t_1 t_2 + t_2^2) \\
&\quad +(8t_1^3 + t_2)(3t_1 + t_2) \\
&= 7t_1^4 + 7t_1^3 t_2 + 7t_1^2 t_2^2 + 24t_1^4 \\
&\quad +8t_1^3 t_2 + 3t_1 t_2 + t_2^2 \\
&= 31t_1^4 + 15t_1^3 t_2 + 7t_1^2 t_2^2 + 3t_1 t_2 + t_2^2.
\end{aligned}
$$

For $t \in \Lambda_2^{\kappa_2 2}$, we have

Example

$$
\begin{aligned}
h_{t_2}^{\Delta_2}(t) &= f_{t_2}^{\Delta_2}(t)g(t) + f_2^{\sigma_2}(t)g_{t_2}^{\Delta_2}(t) \\
&= g(t) + (t_1^3 + \sigma_2(t_2))(t_1 + \sigma_2(t_2) + t_2) \\
&= t_1^2 + t_1 t_2 + t_2^2 + (t_1^3 + t_2 + 1)(t_1 + t_2 + 1 + t_2) \\
&= t_1^2 + t_1 t_2 + t_2^2 + (t_1^3 + t_2 + 1)(t_1 + 2t_2 + 1) \\
&= t_1^2 + t_1 t_2 + t_2^2 + t_1^4 + 2t_1^3 t_2 + t_1^3 + t_1 t_2 + 2t_2^2 + t_2 + t_1 + 2t_2 + 1 \\
&= t_1^4 + t_1^3(2t_2 + 1) + t_1^2 + t_1(2t_2 + 1) + 3t_2^2 + 3t_2 + 1.
\end{aligned}
$$

Example

Let $\Lambda^3 = \mathbb{N} \times \mathbb{Z} \times \mathbb{R}$ and define $h : \Lambda^3 \to \mathbb{R}$ by

$$h(t) = t_1^2 t_2 t_3 (3t_1 t_2 + t_1^2 + t_3^2), \quad t \in \Lambda^3.$$

We will find $h_{t_1}^{\Delta_1}(t)$ for $t \in \Lambda_1^{\kappa_1 3}$, $h_{t_2}^{\Delta_2}(t)$ for $t \in \Lambda_2^{\kappa_2 3}$, and $h_{t_3}^{\Delta_3}(t)$ for $t \in \Lambda_3^{\kappa_3 3}$. Let

$$f(t) = t_1^2 t_2 t_3, \quad g(t) = 3t_1 t_2 + t_1^2 + t_3^2, \quad t \in \Lambda^3.$$

Here,

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1 = \mathbb{N}, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2 = \mathbb{Z},$$

$$\sigma_3(t_3) = t_3, \quad t_3 \in \mathbb{T}_3 = \mathbb{R}.$$

Example

For $t \in \Lambda_1^{\kappa_1 3}$, we have

$$
\begin{aligned}
h_{t_1}^{\Delta_1}(t) &= f_{t_1}^{\Delta_1}(t)g(t) + f_1^{\sigma_1}(t)g_{t_1}^{\Delta_1}(t) \\
&= (\sigma_1(t_1)+t_1)t_2t_3(3t_1t_2+t_1^2+t_3^2) + (\sigma_1(t_1))^2t_2t_3(3t_2+\sigma_1(t_1)+ \\
&= (2t_1+1)t_2t_3(3t_1t_2+t_1^2+t_3^2) + (t_1+1)^2t_2t_3(3t_2+2t_1+1) \\
&= (2t_1+1)(3t_1t_2^2t_3+t_1^2t_2t_3+t_2t_3^3) + (t_1^2+2t_1+1)(3t_2^2t_3+2t_1t_2
\end{aligned}
$$

Example

$$
\begin{aligned}
= \;& 6t_1^2t_2^2t_3 + 2t_1^3t_2t_3 + 2t_1t_2t_3^3 + 3t_1t_2^2t_3 + t_1^2t_2t_3 + t_2t_3^3 + 3t_1^2t_2^2t_3 \\
& +2t_1^3t_2t_3 + t_1^2t_2t_3 + 6t_1t_2^2t_3 + 4t_1^2t_2t_3 + 2t_1t_2t_3 + 3t_2^2t_3 + 2t_1t_2t_3 + t_2t \\
= \;& 4t_1^3t_2t_3 + 9t_1^2t_2^2t_3 + 6t_1^2t_2t_3 + 9t_1t_2^2t_3 + 2t_1t_2t_3^3 \\
& +4t_1t_2t_3 + 3t_2^2t_3 + t_2t_3 + t_2t_3^3.
\end{aligned}
$$

For $t \in \Lambda_2^{\kappa_2 3}$, we have

Example

$$
\begin{aligned}
h_{t_2}^{\Delta_2}(t) &= f_{t_2}^{\Delta_2}(t)(g(t) + f_2^{\sigma_2}(t) g_{t_2}^{\Delta_2}(t) \\
&= t_1^2 t_3 (3t_1 t_2 + t_1^2 + t_3^2) + 3t_1^2 \sigma_2(t_2) t_3 t_1 \\
&= t_1^2 t_3 (3t_1 t_2 + t_1^2 + t_3^2) + 3t_1^2 (t_2 + 1) t_3 t_1 \\
&= 3t_1^3 t_2 t_3 + t_1^4 t_3 + t_1^2 t_3^3 + 3t_1^3 t_2 t_3 + 3t_1^3 t_3 \\
&= 6t_1^3 t_2 t_3 + 3t_1^3 t_3 + t_1^2 t_3^3 + t_1^4 t_3.
\end{aligned}
$$

For $t \in \Lambda_3^{\kappa_3 3}$, we have

Example

$$
\begin{aligned}
h_{t_3}^{\Delta_3}(t) &= f_{t_3}^{\Delta_3}(t)g(t) + f_3^{\sigma_3}(t)g_{t_3}^{\Delta_3}(t) \\
&= t_1^2 t_2(3t_1 t_2 + t_1^2 + t_3^2) + t_1^2 t_2 \sigma_3(t_3)(\sigma_3(t_3) + t_3) \\
&= t_1^2 t_2(3t_1 t_2 + t_1^2 + t_3^2) + 2t_1^2 t_2 t_3 t_3 \\
&= 3t_1^3 t_2^2 + t_1^4 t_2 + t_1^2 t_2 t_3^2 + 2t_1^2 t_2 t_3^2 \\
&= t_1^4 t_2 + 3t_1^3 t_2^2 + 3t_1^2 t_2 t_3^2.
\end{aligned}
$$

Example

Let $f : \Lambda^n \to \mathbb{R}$ be partial delta differentiable with respect to t_i at $t \in \Lambda_i^{\kappa_i n}$. Then

$$
\begin{aligned}
\left(f^2\right)_{t_i}^{\Delta_i} &= (ff)_{t_i}^{\Delta_i} \\
&= f_{t_i}^{\Delta_i} f + f_i^{\sigma_i} f_{t_i}^{\Delta_i} \\
&= f_{t_i}^{\Delta_i}\left(f_i^{\sigma_i} + f\right), \\
(f^3)_{t_i}^{\Delta_i} &= (ff^2)_{t_i}^{\Delta_i}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= f_{t_i}^{\Delta_i} f^2 + f_i^{\sigma_i} (f^2)_{t_i}^{\Delta_i} \\
&= f_{t_i}^{\Delta_i} f^2 + f_i^{\sigma_i} f_{t_i}^{\Delta_i} (f_i^{\sigma_i} + f) \\
&= f_{t_i}^{\Delta_i} (f^2 + f f_i^{\sigma_i} + (f_i^{\sigma_i})^2).
\end{aligned}
$$

We assume that

$$(f^n)_{t_i}^{\Delta_i} = f_{t_i}^{\Delta_i} \sum_{k=0}^{n-1} f^k (f_i^{\sigma_i})^{n-1-k}$$

for some $n \in \mathbb{N}$.

Example

We now prove that

$$(f^{n+1})_{t_i}^{\Delta_i} = f_{t_i}^{\Delta_i} \sum_{k=0}^{n} f^k (f_i^{\sigma_i})^{n-k}.$$

Indeed,

$$\begin{aligned}
(f^{n+1})_{t_i}^{\Delta_i} &= (ff^n)_{t_i}^{\Delta_i} \\
&= f_{t_i}^{\Delta_i} f^n + f_i^{\sigma_i} (f^n)_{t_i}^{\Delta_i} \\
&= f_{t_i}^{\Delta_i} f^n + f_{t_i}^{\Delta_i} (f^{n-1} + f^{n-2} f_i^{\sigma_i} + \cdots
\end{aligned}$$

Example

$$
\begin{aligned}
& +f(f_i^{\sigma_i})^{n-2} + (f_i^{\sigma_i})^{n-1})f_i^{\sigma_i} \\
= \quad & f_{t_i}^{\Delta_i}\left(f^n + f^{n-1}f_i^{\sigma_i} + f^{n-2}(f_i^{\sigma_i})^2 + \cdots + (f_i^{\sigma_i})^n\right) \\
= \quad & f_{t_i}^{\Delta_i}\sum_{k=0}^{n} f^k(f_i^{\sigma_i})^{n-k}.
\end{aligned}
$$