Time Scales Analysis
Lecture 23
Properties of Partial Delta Derivatives

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Theorem

Let $f, g : \Lambda^n \to \mathbb{R}$ be partial delta differentiable with respect to t_i at $t \in \Lambda_i^{\kappa_i n}$. Assume $g_i^{\sigma_i}(t)g(t) \neq 0$. Then $\frac{f}{g}$ is partial delta differentiable with respect to t_i at t and

$$\left(\frac{f}{g}\right)_{t_i}^{\Delta_i}(t) = \frac{f_{t_i}^{\Delta_i}(t)g(t) - f(t)g_{t_i}^{\Delta_i}(t)}{g_i^{\sigma_i}(t)g(t)}.$$

Proof.

Since $f, g : \Lambda^2 \to \mathbb{R}$ are partial delta differentiable with respect to t_i at t, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s_i \in (t_i - \delta, t_i + \delta)$, $s_i \in \mathbb{T}_i$, we have

$$\begin{aligned} |f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| &\leq \varepsilon^* |\sigma_i(t_i) - s_i|, \\ |g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| &\leq \varepsilon^* |\sigma_i(t_i) - s_i|, \\ |f(t)g(t_{s_i}) - f(t_{s_i})g(t)| &\leq \varepsilon^*, \end{aligned}$$

□

Proof.

where

$$0 < \varepsilon^* < \varepsilon \frac{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|}{1 + |g(t)||g(t_{s_i})| + |f(t_{s_i})||g(t)| + |g_{t_i}^{\Delta_i}(t)|}.$$

Hence,

$$\left| \left(\frac{f}{g}\right)_i^{\sigma_i}(t) - \left(\frac{f}{g}\right)(t_{s_i}) - \frac{f_{t_i}^{\Delta_i}(t)g(t) - f(t)g_{t_i}^{\Delta_i}(t)}{g_i^{\sigma_i}(t)g(t)}(\sigma_i(t) - s_i) \right|$$

$\square$

Proof.

$$
\begin{aligned}
&= \left| \frac{f_i^{\sigma_i}(t)}{g_i^{\sigma_i}(t)} - \frac{f(t_{s_i})}{g(t_{s_i})} - \frac{f_{t_i}^{\Delta_i}(t)g(t) - f(t)g_{t_i}^{\Delta_i}(t)}{g_i^{\sigma_i}(t)g(t)}(\sigma_i(t_i) - s_i) \right| \\
&= \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|} \Big| f_i^{\sigma_i}(t)g(t)g(t_{s_i}) - f(t_{s_i})g(t)g_i^{\sigma_i}(t) \\
&\quad - f_{t_i}^{\Delta_i}(t)g(t)g(t_{s_i})(\sigma_i(t_i) - s_i) + f(t)g(t_{s_i})g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i) \Big| \\
&= \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|} \Big| f_i^{\sigma_i}(t)g(t)g(t_{s_i}) - f(t_{s_i})g(t)g(t_{s_i})
\end{aligned}
$$

□

Proof.

$$+f(t_{s_i})g(t)g(t_{s_i})-f(t_{s_i})g(t)g_i^{\sigma_i}(t)$$

$$-f_{t_i}^{\Delta_i}(t)g(t)g(t_{s_i})(\sigma_i(t_i)-s_i)+f(t)g(t_{s_i})g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)\Big|$$

$$= \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|}\Big|g(t)g(t_{s_i})(f_i^{\sigma_i}(t)-f(t_{s_i})-f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i))$$

$$+f(t_{s_i})g(t)g(t_{s_i})-f(t_{s_i})g(t)g_i^{\sigma_i}(t)+f(t_{s_i})g(t)g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)$$

□

Proof.

$$-f(t_{s_i})g(t)g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)+f(t)g(t_{s_i})g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i)\Big|$$

$$= \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|}\Big|g(t)g(t_{s_i})(f_i^{\sigma_i}(t)-f(t_{s_i})-f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i))$$

$$-f(t_{s_i})g(t)(g_i^{\sigma_i}(t)-g(t_{s_i})-g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i)-s_i))$$

$$+g_{t_i}^{\Delta_i}(t)(f(t)g(t_{s_i})-f(t_{s_i})g(t))(\sigma_i(t_i)-s_i)\Big|$$

$\square$

Proof.

$$
\begin{aligned}
\leq\ & \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|}\Big(|g(t)||g(t_{s_i})||f_i^{\sigma_i}(t) - f(t_{s_i}) - f_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \\
& + |f(t_{s_i})||g(t)||g_i^{\sigma_i}(t) - g(t_{s_i}) - g_{t_i}^{\Delta_i}(t)(\sigma_i(t_i) - s_i)| \\
& + |g_{t_i}^{\Delta_i}(t)||(f(t)g(t_{s_i}) - f(t_{s_i})g(t)||\sigma_i(t_i) - s_i|\Big) \\
\leq\ & \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|}(\varepsilon^*|g(t)||g(t_{s_i})||\sigma_i(t_i) - s_i|
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
& +\varepsilon^*|f(t_{s_i})||g(t)||\sigma_i(t_i)-s_i|+\varepsilon^*|g_{t_i}^{\Delta_i}(t)||\sigma_i(t_i)-s_i|) \\
\leq \;& \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|}\varepsilon^*(|g(t)||g(t_{s_i})||\sigma_i(t_i)-s_i| \\
& +|f(t_{s_i})||g(t)||\sigma_i(t_i)-s_i|+|g_{t_i}^{\Delta_i}(t)||\sigma_i(t_i)-s_i|) \\
\leq \;& \frac{1}{|g_i^{\sigma_i}(t)||g(t_{s_i})||g(t)|}\varepsilon^*(1+|g(t)||g(t_{s_i})| \\
& +|f(t_{s_i})||g(t)|+|g_{t_i}^{\Delta_i}(t)|)|\sigma_i(t_i)-s_i| \\
\leq \;& \varepsilon|\sigma_i(t_i)-s_i|.
\end{aligned}
$$

This completes the proof. □

Example

Let $\Lambda^2 = \mathbb{N} \times \mathbb{N}$ and define $h : \Lambda^2 \to \mathbb{R}$ by

$$h(t) = \frac{t_1^2 + 2t_1 t_2 + t_2^3}{t_1 + t_2}, \quad t = (t_1, t_2) \in \Lambda^2.$$

We will find $h_{t_1}^{\Delta_1}(t)$ for $t \in \Lambda_1^{\kappa_1 2}$ and $h_{t_2}^{\Delta_2}(t)$ for $t_2 \in \Lambda_2^{\kappa_2 2}$. Let

$$f(t) = t_1^2 + 2t_1 t_2 + t_2^3, \quad g(t) = t_1 + t_2, \quad t = (t_1, t_2) \in \Lambda^2.$$

Here,

$$\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{N}$$

and

Example

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

We have

$$\begin{aligned} f_{t_1}^{\Delta_1}(t) &= \sigma_1(t_1) + t_1 + 2t_2 \\ &= 2t_1 + 2t_2 + 1, \quad t \in \Lambda_1^{\kappa_1 2}, \\ f_{t_2}^{\Delta_2}(t) &= 2t_1 + (\sigma_2(t_2))^2 + t_2\sigma_2(t_2) + t_2^2 \\ &= 2t_1 + (t_2+1)^2 + t_2(t_2+1) + t_2^2 \\ &= 2t_1 + t_2^2 + 2t_2 + 1 + t_2^2 + t_2 + t_2^2 \\ &= 2t_1 + 3t_2^2 + 3t_2 + 1, \quad t \in \Lambda_2^{\kappa_2 2}, \end{aligned}$$

Example

$$
\begin{aligned}
g_{t_1}^{\Delta_1}(t) &= 1, \quad t \in \Lambda_1^{\kappa_1 2}, \\
g_{t_2}^{\Delta_2}(t) &= 1, \quad t \in \Lambda_2^{\kappa_2 2}, \\
g_1^{\sigma_1}(t) &= g_2^{\sigma_2}(t) \\
&= t_1 + t_2 + 1, \quad t \in \Lambda^2.
\end{aligned}
$$

Hence, for $t \in \Lambda_1^{\kappa_1 2}$, we have

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t) &= \frac{f_{t_1}^{\Delta_1}(t)g(t) - f(t)g_{t_1}^{\Delta_1}(t)}{g_1^{\sigma_1}(t)g(t)} \\
&= \frac{(2t_1 + 2t_2 + 1)(t_1 + t_2) - (t_1^2 + 2t_1t_2 + t_2^3)}{(t_1 + t_2 + 1)(t_1 + t_2)} \\
&= \frac{2t_1^2 + 2t_1t_2 + 2t_1t_2 + 2t_2^2 + t_1 + t_2 - t_1^2 - 2t_1t_2 - t_2^3}{(t_1 + t_2 + 1)(t_1 + t_2)} \\
&= \frac{t_1^2 + 2t_1t_2 + 2t_2^2 - t_2^3 + t_1 + t_2}{(t_1 + t_2 + 1)(t_1 + t_2)}.
\end{aligned}
$$

Example

For $t \in \Lambda_2^{\kappa_2 2}$, we have

$$
\begin{aligned}
h_{t_2}^{\Delta_2}(t) &= \frac{f_{t_2}^{\Delta_2}(t)g(t) - f(t)g_{t_2}^{\Delta_2}(t)}{g_2^{\sigma_2}(t)g(t)} \\
&= \frac{(2t_1 + 3t_2^2 + 3t_2 + 1)(t_1 + t_2) - (t_1^2 + 2t_1 t_2 + t_2^3)}{(t_1 + t_2 + 1)(t_1 + t_2)} \\
&= \frac{2t_1^2 + 2t_1 t_2 + 3t_1 t_2^2 + 3t_2^3 + 3t_1 t_2 + 3t_2^2 + t_1 + t_2 - t_1^2 - 2t_1 t_2 -}{(t_1 + t_2 + 1)(t_1 + t_2)} \\
&= \frac{t_1^2 + 3t_1 t_2 + 3t_1 t_2^2 + 2t_2^3 + 3t_2^2 + t_1 + t_2}{(t_1 + t_2 + 1)(t_1 + t_2)}.
\end{aligned}
$$

Example

Let $\Lambda^2 = 2^{\mathbb{N}} \times \mathbb{N}_0$ and define $h : \Lambda^2 \to \mathbb{R}$ by

$$h(t) = \frac{t_1 - t_2}{t_1^2 + t_2}, \quad t \in \Lambda^2.$$

We will find $h_{t_1}^{\Delta_1}(t)$ for $t \in \Lambda_1^{\kappa_1 2}$ and $h_{t_2}^{\Delta_2}(t)$ for $t \in \Lambda_2^{\kappa_2 2}$. Let

$$f(t) = t_1 - t_2, \quad g(t) = t_1^2 + t_2, \quad t \in \Lambda^2.$$

Here,

$$\mathbb{T}_1 = 2^{\mathbb{N}}, \quad \mathbb{T}_2 = \mathbb{N}.$$

$$\sigma_1(t_1) = 2t_1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2,$$

$$f_{t_1}^{\Delta_1}(t) = 1, \quad g_{t_1}^{\Delta_1}(t) = \sigma_1(t_1) + t_1 = 2t_1 + t_1 = 3t_1, \quad t \in \Lambda_1^{\kappa_1 2},$$

Example

$$f_{t_2}^{\Delta_2}(t) = -1, \quad g_{t_2}^{\Delta_2}(t) = 1, \quad t \in \Lambda_2^{\kappa_2 2},$$

$$g_1^{\sigma_1}(t) = 4t_1^2 + t_2, \quad g_2^{\sigma_2}(t) = t_1^2 + t_2 + 1, \quad t \in \Lambda^2.$$

Hence, for $t \in \Lambda_1^{\kappa_1 2}$, we have

$$\begin{aligned} h_{t_1}^{\Delta_1}(t) &= \frac{f_{t_1}^{\Delta_1}(t)g(t) - f(t)g_{t_1}^{\Delta_1}(t)}{g_1^{\sigma_1}(t)g(t)} \\ &= \frac{(t_1^2 + t_2) - (t_1 - t_2)3t_1}{(4t_1^2 + t_2)(t_1^2 + t_2)} \end{aligned}$$

Example

$$
\begin{aligned}
&= \frac{t_1^2 + t_2 - 3t_1^2 + 3t_1 t_2}{(4t_1^2 + t_2)(t_1^2 + t_2)} \\
&= \frac{-2t_1^2 + 3t_1 t_2 + t_2}{(4t_1^2 + t_2)(t_1^2 + t_2)}.
\end{aligned}
$$

For $t \in \Lambda_2^{\kappa_2 2}$, we get

$$
\begin{aligned}
h_{t_2}^{\Delta_2}(t) &= \frac{f_{t_2}^{\Delta_2}(t) g(t) - f(t) g_{t_2}^{\Delta_2}(t)}{g_2^{\sigma_2}(t) g(t)} \\
&= \frac{-(t_1^2 + t_2) - (t_1 - t_2)}{(t_1^2 + t_2 + 1)(t_1^2 + t_2)} \\
&= -\frac{t_1^2 + t_1}{(t_1^2 + t_2 + 1)(t_1^2 + t_2)}.
\end{aligned}
$$

Example

Let $\Lambda^2 = \mathbb{Z} \times \mathbb{N}_0$ and define $h : \Lambda^2 \to \mathbb{R}$ by

$$h(t) = \frac{t_1 + 2t_2 + 3}{t_1 - t_2 + 1}, \quad t \in \Lambda^2.$$

We will find

$$h_{t_1}^{\Delta_1}(t), \quad t \in \Lambda_1^{\kappa_1 2}, \quad (t_1 - t_2 + 1)(t_1 - t_2 + 2) \neq 0$$

and

$$h_{t_2}^{\Delta_2}(t), \quad t \in \Lambda_2^{\kappa_2 2}, \quad (t_1 - t_2 + 1)(t_1 - t_2) \neq 0.$$

Here,

$$\mathbb{T}_1 = \mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{N}_0$$

and

Example

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

Let

$$f(t) = t_1 + 2t_2 + 3, \quad g(t) = t_1 - t_2 + 1, \quad t \in \Lambda^2.$$

We have

$$f_{t_1}^{\Delta_1}(t) = g_{t_1}^{\Delta_1}(t) = 1, \quad t \in \Lambda_1^{\kappa_1 2},$$

$$f_{t_2}^{\Delta_2} = 2, \quad g_{t_2}^{\Delta_2} = -1, \quad t \in \Lambda_2^{\kappa_2 2},$$

$$g_1^{\sigma_1}(t) = t_1 - t_2 + 2, \quad g_2^{\sigma_2}(t) = t_1 - t_2, \quad t \in \Lambda^2.$$

Example

For $t \in \Lambda_1^{\kappa_1 2}$ with $(t_1 - t_2 + 1)(t_1 - t_2 + 2) \neq 0$, we have

$$\begin{aligned} h_{t_1}^{\Delta_1}(t) &= \frac{f_{t_1}^{\Delta_1}(t)g(t) - f(t)g_{t_1}^{\Delta_1}(t)}{g_1^{\sigma_1}(t)g(t)} \\ &= \frac{t_1 - t_2 + 1 - (t_1 + 2t_2 + 3)}{(t_1 - t_2 + 2)(t_1 - t_2)} \\ &= \frac{-3t_2 - 2}{(t_1 - t_2 + 2)(t_1 - t_2)}. \end{aligned}$$

For $t \in \Lambda_2^{\kappa_2 2}$ with $(t_1 - t_2 + 1)(t_1 - t_2) \neq 0$, we have

Example

$$
\begin{aligned}
h_{t_2}^{\Delta_2}(t) &= \frac{f_{t_2}^{\Delta_2}(t)g(t) - f(t)g_{t_2}^{\Delta_2}(t)}{g_2^{\sigma_2}(t)g(t)} \\
&= \frac{2(t_1 - t_2 + 1) + (t_1 + 2t_2 + 3)}{(t_1 - t_2)(t_1 - t_2 + 1)} \\
&= \frac{3t_1 + 5}{(t_1 - t_2)(t_1 - t_2 + 1)}.
\end{aligned}
$$

Definition

For a function $f : \Lambda^n \to \mathbb{R}$, we shall talk about the *second-order partial delta derivative* with respect to t_i and t_j, $i, j \in \{1, 2, \ldots, n\}$, $f_{t_i t_j}^{\Delta_i \Delta_j}$, provided $f_{t_i}^{\Delta_i}$ is partial delta differentiable with respect to t_j on $\Lambda_{ij}^{\kappa_i \kappa_j n} = \left(\Lambda_i^{\kappa_i n}\right)_j^{\kappa_j n}$ with partial delta derivative

$$f_{t_i t_j}^{\Delta_i \Delta_j} = \left(f_{t_i}^{\Delta_i}\right)_{t_j}^{\Delta_j} : \Lambda_{ij}^{\kappa_i \kappa_j n} \to \mathbb{R}.$$

For $i = j$, we will write

$$f_{t_i t_i}^{\Delta_i \Delta_i} = f_{t_i^2}^{\Delta_i^2}.$$

Definition

Similarly, we define *higher-order partial delta derivatives*

$$f^{\Delta_i \Delta_j \ldots \Delta_l}_{t_i t_j \ldots t_l} : \Lambda^{\kappa_i \kappa_j \ldots \kappa_l n}_{ij \ldots l} \to \mathbb{R}.$$

For $t \in \Lambda^n$, we define

$$\sigma^2(t) = \sigma(\sigma(t)) = (\sigma_1(\sigma_1(t_1)), \sigma_2(\sigma_2(t_2)), \ldots, \sigma_n(\sigma_n(t_n)))$$

and

$$\rho^2(t) = \rho(\rho(t)) = (\rho_1(\rho_1(t_1)), \rho_2(\rho_2(t_2)), \ldots, \rho_n(\rho_n(t_n))),$$

and $\sigma^m(t)$ and $\rho^m(t)$ for $m \in \mathbb{N}$ are defined accordingly.

Definition

Finally, we put

$$\rho^0(t) = \sigma^0(t) = t, \quad f_{t_i}^{\Delta_i^0} = f, \quad \Lambda_i^{\kappa_i^0 n} = \Lambda^n.$$

Example

Let $\Lambda^3 = \mathbb{N} \times \mathbb{N}_0 \times \mathbb{Z}$. Here

$$\mathbb{T}_1 = \mathbb{N}, \quad \mathbb{T}_2 = \mathbb{N}_0, \quad \mathbb{T}_3 = \mathbb{Z}$$

and

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2,$$

$$\sigma_3(t_3) = t_3 + 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

Example

$$\begin{aligned}
\sigma^2(t) &= (\sigma_1(\sigma_1(t_1)), \sigma_2(\sigma_2(t_2)), \sigma_3(\sigma_3(t_3))) \\
&= (\sigma_1(t_1) + 1, \sigma_2(t_2) + 1, \sigma_3(t_3) + 1) \\
&= (t_1 + 1 + 1, t_2 + 1 + 1, t_3 + 1 + 1) \\
&= (t_1 + 2, t_2 + 2, t_3 + 2).
\end{aligned}$$

Example

Let $\Lambda^2 = 2^{\mathbb{N}} \times \mathbb{N}_0$. Here,

$$\mathbb{T}_1 = 2^{\mathbb{N}}, \quad \mathbb{T}_2 = \mathbb{N}_0.$$

Then

$$\rho_1(t_1) = \frac{t_1}{2}, \quad t_1 \neq 2, \quad t_1 \in \mathbb{T}_1, \quad \rho_1(2) = 2,$$

$$\rho_2(t_2) = t_2 - 1, \quad t_2 \in \mathbb{T}_2, \quad t_2 \neq 0, \quad \rho_2(0) = 0.$$

Therefore,

$$\rho(t) = \begin{cases} \left(\frac{t_1}{2}, t_2 - 1\right) & \text{if} \quad t_1 \geq 4, \quad t_2 \geq 1, \\ (2, t_2 - 1) & \text{if} \quad t_1 = 2, \quad t_2 \geq 1, \\ \left(\frac{t_1}{2}, 0\right) & \text{if} \quad t_1 \geq 4, \quad t_2 = 0, \end{cases}$$

Example

1. Let $t_1 \geq 4$, $t_2 \geq 1$. Then

$$\begin{aligned} \rho^2(t) &= (\rho_1(\rho_1(t_1)), \rho_2(\rho_2(t_2))) \\ &= \left(\rho_1\left(\frac{t_1}{2}\right), \rho_2(t_2-1)\right) \\ &= \begin{cases} \left(\frac{t_1}{4}, t_2-2\right) & \text{if} \quad t_1 \geq 8, \quad t_2 \geq 2, \\ \left(\frac{t_1}{4}, 0\right) & \text{if} \quad t_1 \geq 8, \quad t_2 = 1, \\ (2, t_1-2) & \text{if} \quad t_1 = 4, \quad t_2 \geq 2, \\ (2, 0) & \text{if} \quad t_1 = 4, \quad t_2 = 1. \end{cases} \end{aligned}$$

Example

2. Let $t_1 = 2$, $t_2 \geq 1$. Then

$$\begin{aligned} \rho^2(t) &= (\rho_1(2), \rho_2(t_2 - 1)) \\ &= \begin{cases} (2, t_2 - 2) & \text{if} \quad t_2 \geq 2, \\ (2, 0) & \text{if} \quad t_2 = 1. \end{cases} \end{aligned}$$

Example

2. Let $t_1 \geq 4$, $t_2 = 0$. Then

$$\begin{aligned} \rho^2(t) &= \left(\rho_1\left(\frac{t_1}{2}\right), \rho_2(0)\right) \\ &= \begin{cases} \left(\frac{t_1}{4}, 0\right) & \text{if} \quad t_1 \geq 8, \\ (2, 0) & \text{if} \quad t_1 = 4. \end{cases} \end{aligned}$$

3. $\rho^2(2, 0) = (\rho_1(2), \ \rho_2(0)) = (2, 0)$.

Example

Let $\Lambda^2 = \mathbb{Z} \times \mathbb{N}$. We will find $(\sigma^2(t)) \cdot (\rho^3(t))$, $t \in \Lambda^2$, where $\cdot$ is the inner product in $\mathbb{R}^2$. Here,

$$\mathbb{T}_1 = \mathbb{Z}, \quad \mathbb{T}_2 = \mathbb{N}$$

and

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

Hence,

Example

$$
\begin{aligned}
\sigma(t) &= (\sigma_1(t_1), \sigma_2(t_2)) \\
&= (t_1 + 1, t_2 + 1), \\
\sigma^2(t) &= (\sigma_1(\sigma_1(t_1)), \sigma_2(\sigma_2(t_2))) \\
&= (\sigma_1(t_1 + 1), \sigma_2(t_2 + 1)) \\
&= (t_1 + 2, t_2 + 2), \quad t \in \Lambda^2.
\end{aligned}
$$

Also,

Example

$$
\begin{aligned}
\rho(t) &= (\rho_1(t_1), \rho_2(t_2)) \\
&= \begin{cases} (t_1 - 1, t_2 - 1) & \text{if} \quad t_2 \geq 2, \\ (t_1 - 1, 1) & \text{if} \quad t_2 = 1, \end{cases} \\
\rho^2(t) &= (\rho_1(\rho_1(t_1)), \rho_2(\rho_2(t_2))) \\
&= \begin{cases} (\rho_1(t_1 - 1), \rho_2(t_2 - 1)) & \text{if} \quad t_2 \geq 2, \\ (\rho_1(t_1 - 1), \rho_2(1)) & \text{if} \quad t_2 = 1 \end{cases}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \begin{cases} (t_1 - 2, t_2 - 2) & \text{if} \quad t_2 \geq 3, \\ \\ (t_1 - 2, 1) & \text{if} \quad t_2 \in \{1, 2\}, \end{cases} \\
\rho^3(t) &= (\rho_1(\rho_1^2(t_1)), \rho_2(\rho_2^2(t_2))) \\
&= \begin{cases} (\rho_1(t_1 - 2), \rho_2(t_2 - 2) & \text{if} \quad t_2 \geq 3, \\ \\ (\rho_1(t_1 - 2), \rho_2(1)) & \text{if} \quad t_2 \in \{1, 2\} \end{cases}
\end{aligned}
$$

Example

$$= \begin{cases} (t_1 - 3, t_2 - 3) & \text{if} \quad t_2 \geq 4, \\ \\ (t_1 - 3, 1) & \text{if} \quad t_2 \in \{1, 2, 3\}, \quad t \in \Lambda^2. \end{cases}$$

Hence, for $t \in \Lambda^2$, we have

Example

$$\sigma^2(t)\cdot\rho^3(t) = \begin{cases} (t_1+2, t_2+2)\cdot(t_1-3, t_2-3) & \text{if} \quad t_2 \geq 4, \\ (t_1+2, t_2+2)\cdot(t_1-3, 1) & \text{if} \quad t_2 \in \{1,2,3\} \end{cases}$$

$$= \begin{cases} (t_1+2, t_2+2)\cdot(t_1-3, t_2-3) & \text{if} \quad t_2 \geq 4, \\ (t_1+2, 3)\cdot(t_1-3, 1) & \text{if} \quad t_2 = 1, \\ (t_1+2, 4)\cdot(t_1-3, 1) & \text{if} \quad t_2 = 2, \\ (t_1+2, 5)\cdot(t_1-3, 1) & \text{if} \quad t_2 = 3, \end{cases}$$

Example

$$= \begin{cases} (t_1+2)(t_1-3)+(t_2+2)(t_2-3) & \text{if} \quad t_2 \geq 4, \\ (t_1+2)(t_1-3)+3 & \text{if} \quad t_2 = 1, \\ (t_1+2)(t_1-3)+4 & \text{if} \quad t_2 = 2, \\ (t_1+2)(t_1-3)+5 & \text{if} \quad t_2 = 3 \end{cases}$$

Example

$$= \begin{cases} t_1^2 + t_2^2 - t_1 - t_2 - 12 & \text{if} \quad t_2 \geq 4, \\ t_1^2 - t_1 - 3 & \text{if} \quad t_2 = 1, \\ t_1^2 - t_1 - 2 & \text{if} \quad t_2 = 2, \\ t_1^2 - t_1 - 1 & \text{if} \quad t_2 = 3. \end{cases}$$

Example

Let $\Lambda^2 = 2^{\mathbb{N}} \times \mathbb{N}_0$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1^2 t_2 + t_1 t_2^2, \quad t \in \Lambda^2.$$

Here, $\mathbb{T}_1 = 2^{\mathbb{N}}$, $\mathbb{T}_2 = \mathbb{N}_0$, and

$$\sigma_1(t_1) = 2t_1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

Thus,

$$\begin{aligned} f_{t_1}^{\Delta_1}(t) &= t_2(\sigma_1(t_1) + t_1) + t_2^2 \\ &= t_2(2t_1 + t_1) + t_2^2 \\ &= 3t_1 t_2 + t_2^2, \quad t \in \Lambda_1^{\kappa_1 2}, \end{aligned}$$

Example

$$
\begin{aligned}
f_{t_2}^{\Delta_2}(t) &= t_1^2 + t_1(\sigma_2(t_2) + t_2) \\
&= t_1^2 + t_1(2t_2 + 1) \\
&= t_1^2 + 2t_1t_2 + t_1, \quad t \in \Lambda_2^{\kappa_2 2}, \\
f_{t_1t_2}^{\Delta_1\Delta_2}(t) &= 3t_1 + t_2 + \sigma_2(t_2) \\
&= 3t_1 + 2t_2 + 1, \\
f_{t_2t_1}^{\Delta_2\Delta_1}(t) &= \sigma_1(t_1) + t_1 + 2t_2 + 1 \\
&= 3t_1 + 2t_2 + 1, \quad t \in \Lambda_{12}^{\kappa_1\kappa_2 2}.
\end{aligned}
$$