Time Scales Analysis
Lecture 24
Properties of Partial Delta Derivatives

Khaled Zennir

December 2, 2025

Example

Let $\Lambda^3 = \mathbb{N} \times \mathbb{N}_0 \times \mathbb{Z}$ and define $f : \Lambda^3 \to \mathbb{R}$ by

$$f(t) = t_1 t_2 t_3 + t_1^2 + t_2^2 + t_3^2, \quad t \in \Lambda^3.$$

We will find $f^{\Delta_1 \Delta_2 \Delta_3}_{t_1 t_2 t_3}(t)$ for $t \in \Lambda^{\kappa_1 \kappa_2 \kappa_3 3}_{123}$. Here, $\mathbb{T}_1 = \mathbb{N}$, $\mathbb{T}_2 = \mathbb{N}_0$, $\mathbb{T}_3 = \mathbb{Z}$, and

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2,$$

$$\sigma_3(t_3) = t_3 + 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

Example

$$\begin{aligned}
f_{t_1}^{\Delta_1}(t) &= t_2 t_3 + \sigma_1(t_1) + t_1 \\
&= t_2 t_3 + 2t_1 + 1, \quad t \in \Lambda_1^{\kappa_1 3}, \\
f_{t_1 t_2}^{\Delta_1 \Delta_2}(t) &= t_3, \quad t \in \Lambda_{12}^{\kappa_1 \kappa_2 3}, \\
f_{t_1 t_2 t_3}^{\Delta_1 \Delta_2 \Delta_3} &= 1, \quad t \in \Lambda_{123}^{\kappa_1 \kappa_2 \kappa_3 3}.
\end{aligned}$$

Example

Let $\Lambda^3 = 3^{\mathbb{N}} \times \mathbb{N}_0 \times \mathbb{Z}$ and define $f : \Lambda^3 \to \mathbb{R}$ by

$$f(t) = t_1^2 t_2^2 \sin(t_3), \quad t \in \Lambda^3.$$

Here, $\mathbb{T}_1 = 3^{\mathbb{N}}$, $\mathbb{T}_2 = \mathbb{N}_0$, $\mathbb{T}_3 = \mathbb{Z}$, and

$$\sigma_1(t_1) = 3t_1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2,$$

$$\sigma_3(t_3) = t_3 + 1, \quad t_3 \in \mathbb{T}_3.$$

Hence,

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t) &= (t_1+\sigma_1(t_1))t_2^2\sin(t_3) \\
&= 4t_1t_2^2\sin(t_3), \quad t\in\Lambda_1^{\kappa_1 3}, \\
f_{t_1t_2}^{\Delta_1\Delta_2}(t) &= 4t_1(t_2+\sigma_2(t_2))\sin(t_3) \\
&= 4t_1(2t_2+1)\sin(t_3), \quad t\in\Lambda_{12}^{\kappa_1\kappa_2 3},
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1 t_2 t_3}^{\Delta_1 \Delta_2 \Delta_3}(t) &= 4t_1(2t_2+1)\frac{\sin(t_3+1)-\sin(t_3)}{\sigma_3(t_3)-t_3} \\
&= 4t_1(2t_2+1)\left(\sin(t_3+1)-\sin(t_3)\right) \\
&= 8t_1(2t_2+1)\sin\left(\frac{1}{2}\right)\cos\left(t_3+\frac{1}{2}\right), \quad t \in \Lambda_{123}^{\kappa_1 \kappa_2 \kappa_3 3}.
\end{aligned}
$$

Theorem (Leibniz Formula)

Let $S_{ik}^{(m)}$ be the set consisting of all possible strings of length m, containing exactly k times σ_i and $m-k$ times Δ_i. If $f_{t_i^{m-k}}^{\alpha}$ exists for any $\alpha \in S_{ik}^{(m)}$ and $g_{t_i^k}^{\Delta_i^k}$ exists for any $k \in \{0, 1, \ldots, m\}$, then

$$(fg)_{t_i^m}^{\Delta_i^m} = \sum_{k=0}^{m} \left(\sum_{\alpha \in S_{ik}^{(m)}} f_{t_i^{m-k}}^{\alpha} \right) g_{t_i^k}^{\Delta_i^k} \tag{1}$$

holds for any $m \in \mathbb{N}$.

Proof.

We will use induction. 1. Since

$$(fg)_{t_i}^{\Delta_i} = f_{t_i}^{\Delta_i} g + f_i^{\sigma_i} g_{t_i}^{\Delta_i},$$

(1) holds for $m = 1$.
2. We assume (1) holds for some $m \in \mathbb{N}$. We will prove

$$(fg)_{t_i^{m+1}}^{\Delta_i^{m+1}} = \sum_{k=0}^{m+1} \left(\sum_{\alpha \in S_{ik}^{(m+1)}} f_{t_i^{m-k+1}}^{\alpha} \right) g_{t_i^k}^{\Delta_i^k}. \tag{2}$$

Indeed, □

Proof.

$$
\begin{aligned}
(fg)^{\Delta_i^{m+1}}_{t_i^{m+1}} &= \left((fg)^{\Delta_i^m}_{t_i^m}\right)^{\Delta_i}_{t_i} \\
&= \left(\sum_{k=0}^{m}\left(\sum_{\alpha\in S_{ik}^{(m)}} f^{\alpha}_{t_i^{m-k}}\right) g^{\Delta_i^k}_{t_i^k}\right)^{\Delta_i}_{t_i} \\
&= \sum_{k=0}^{m}\left(\sum_{\alpha\in S_{ik}^{(m)}} f^{\alpha}_{t_i^{m-k}}\right)^{\Delta_i}_{t_i} g^{\Delta_i^k}_{t_i^k} + \sum_{k=0}^{m}\left(\sum_{\alpha\in S_{ik}^{(m)}} f^{\alpha}_{t_i^{m-k}}\right)^{\sigma_i} g^{\Delta_i^{k+1}}_{t_i^{k+1}}
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
&= \sum_{k=0}^{m}\left(\sum_{\alpha\in S_{ik}^{(m)}} f_{t_i^{m-k}}^{\alpha}\right)_{t_i}^{\Delta_i} g_{t_i^k}^{\Delta_i^k} + \sum_{k=1}^{m+1}\left(\sum_{\alpha\in S_{ik-1}^{(m)}} f_{t_i^{m-k+1}}^{\alpha\sigma_i}\right) g_{t_i^k}^{\Delta_i^k} \\
&= \sum_{\alpha\in S_{i0}^{(m)}} f_{t_i^{m+1}}^{\alpha\Delta_i} g + \sum_{k=1}^{m}\left(\sum_{\alpha\in S_{ik}^{(m)}} f_{t_i^{m-k+1}}^{\alpha\Delta_i}\right) g_{t_i^k}^{\Delta_i^k} \\
&\quad + \sum_{k=1}^{m}\left(\sum_{\alpha\in S_{ik-1}^{(m)}} f_{t_i^{m-k+1}}^{\alpha\sigma_i}\right) g_{t_i^k}^{\Delta_i^k} + \sum_{\alpha\in S_{im}^{(m)}} f_{t_i^0}^{\alpha\sigma_i} g_{t_i^{m+1}}^{\Delta_i^{m+1}}
\end{aligned}
$$

$\square$

Proof.

$$= \sum_{\alpha \in S_{i0}^{(m)}} f_{t_i^{m+1}}^{\alpha\Delta_i} g + \sum_{\alpha \in S_{im}^{(m)}} f_{t_i^0}^{\alpha\sigma_i} g_{t_i^{m+1}}^{\Delta_i^{m+1}}$$

$$+ \sum_{k=1}^{m} \left(\sum_{\alpha \in S_{ik-1}^{(m)}} f_{t_i^{m-k}}^{\alpha\sigma_i} + \sum_{\alpha \in S_{ik-1}^{(m)}} f_{t_i^{m+1-k}}^{\alpha\Delta_i} \right) g_{t_i^k}^{\Delta_i^k}$$

$\square$

Proof.

$$
\begin{aligned}
= & \left(\sum_{\alpha\in S^{(m+1)}_{im+1}} f^{\alpha}\right) g^{\Delta_i^{m+1}}_{t_i^{m+1}} + \left(\sum_{\alpha\in S^{(m+1)}_{i0}} f^{\alpha}_{t_i^{m+1}}\right) g \\
& + \sum_{k=1}^{m}\left(\sum_{\alpha\in S^{(m+1)}_{ik}} f^{\alpha}_{t_i^{m+1-k}}\right) g^{\Delta_i^{k}}_{t_i^{k}},
\end{aligned}
$$

i.e., (2) holds. By the principle of mathematical induction, (1) holds for any $m \in \mathbb{N}$. □

Remark

We note that (1) *holds for* $m = 0$ *with the convention that*

$$\sum_{\alpha \in \emptyset} f^{\alpha}_{t_i^k} = f.$$

Definition

We put

$$\Lambda^n_\kappa = \mathbb{T}_{1\kappa} \times \mathbb{T}_{2\kappa} \times \ldots \times \mathbb{T}_{n\kappa},$$

$$\Lambda^n_{i\kappa_i} = \mathbb{T}_1 \times \ldots \times \mathbb{T}_{i-1} \times \mathbb{T}_{i\kappa} \times \mathbb{T}_{i+1} \times \ldots \times \mathbb{T}_n, \quad i = 1, 2, \ldots, n,$$

$$\Lambda^n_{i_1 i_2 \ldots i_l \kappa_{i_1} \kappa_{i_2} \ldots \kappa_{i_l}} = \ldots \times \mathbb{T}_{i_1\kappa} \times \ldots \times \mathbb{T}_{i_2\kappa} \times \ldots \times \mathbb{T}_{i_l\kappa} \times \ldots,$$

where $1 \leq i_1 < i_2 < \ldots < i_l \leq n$, $i_m \in \mathbb{N}$, $m = 1, 2, \ldots, l$.

Remark

If $(i_1, i_2, \ldots, i_l) = (1, 2, \ldots, n)$, *then*

$$\Lambda^n_{i_1 i_2 \ldots i_l \kappa_1 \kappa_2 \ldots \kappa_l} = \Lambda^n_\kappa.$$

Definition

Assume that $f : \Lambda^n \to \mathbb{R}$ is a function and let $t \in \Lambda_{i\kappa_i n}$. We define

$$\frac{\partial f(t_1, t_2, \ldots, t_n)}{\nabla_i t_i} = \frac{\partial f(t)}{\nabla_i t_i} = \frac{\partial f}{\nabla_i t_i}(t) = f_{t_i}^{\nabla_i}(t)$$

to be the number, provided it exists, with the property such that for any $\varepsilon_i > 0$, there exists a neighbourhood

$$U_i = (t_i - \delta_i, t_i + \delta_i) \cap \mathbb{T}_i$$

for some $\delta_i > 0$ such that

$$\Big| f(t_1, \ldots, t_{i-1}, \rho_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n) - f_{t_i}^{\nabla_i}(t)(\rho_i(t_i) - s_i)\Big| \leq \varepsilon_i |\rho_i(t_i) - s_i| \quad (3)$$

for all $s_i \in U_i$.

Definition

We call $f_{t_i}^{\nabla_i}(t)$ the *partial nabla derivative* of f with respect to t_i at t. We say that f is *partial nabla differentiable* with respect to t_i in $\Lambda^n_{i\kappa_i}$ if $f_{t_i}^{\nabla_i}(t)$ exists for all $t \in \Lambda^n_{i\kappa_i}$. The function $f_{t_i}^{\nabla_i} : \Lambda^n_{i\kappa_i} \to \mathbb{R}$ is said to be the *partial nabla derivative* with respect to t_i of f in $\Lambda^n_{i\kappa_i}$.

Theorem

The partial nabla derivative is well defined.

Proof.

Let $t \in \Lambda^n_{i\kappa_i}$ for some $i \in \{1, 2, \ldots, n\}$. We assume that the partial nabla derivative $f_{t_i}^{\nabla_i}(t)$ exists and

$$g_1(t) = f_{t_i}^{\nabla_i}(t), \quad g_2(t) = f_{t_i}^{\nabla_i}(t).$$

Let $\varepsilon_i > 0$ be arbitrarily chosen. Then there exists $\delta_i > 0$ such that for every

$$s_i \in (t_i - \delta_i, t_i + \delta_i) \cap \mathbb{T}_i,$$

we have □

Proof.

$$\Big| f(t_1, t_2, \ldots, t_{i-1}, \rho_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)$$

$$- g_1(t)(\rho_i(t_i) - s_i)\Big| \leq \frac{\varepsilon_i}{2}|\rho_i(t_i) - s_i| \quad (4)$$

and

$$\Big| f(t_1, t_2, \ldots, t_{i-1}, \rho_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)$$

$$- g_2(t)(\rho_i(t_i) - s_i)\Big| \leq \frac{\varepsilon_i}{2}|\rho_i(t_i) - s_i|. \quad (5)$$

$\square$

Proof.

From (4) and (5), we obtain

$$|g_1(t) - g_2(t)| = \Bigg| g_1(t) - \frac{f(t_1, t_2, \dots, t_{i-1}, \rho_i(t_i), t_{i+1}, \dots, t_n)}{\rho_i(t_i)_- s_i}$$

$$+ \frac{f(t_1, t_2, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n)}{\rho_i(t_i) - s_i} + \frac{f(t_1, t_2, \dots, t_{i-1}, \rho_i(t_i), t_{i+1}, \dots, t_n)}{\rho_i(t_i) - s_i}$$

$$- \frac{f(t_1, t_2, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n)}{\rho_i(t_i) - s_i} - g_2(t) \Bigg|$$

□

Proof.

$$
\begin{aligned}
&\leq \left| g_1(t) - \frac{f(t_1, t_2, \ldots, t_{i-1}, \rho_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, s_i, t_{i+}}{\rho_i(t_i) - s_i} \right. \\
&\quad + \left| g_2(t) - \frac{f(t_1, t_2, \ldots, t_{i-1}, \rho_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, s_i, t}{\rho_i(t_i) - s_i} \right. \\
&\leq \frac{\varepsilon_i}{2} + \frac{\varepsilon_i}{2} \\
&= \varepsilon_i .
\end{aligned}
$$

Because $\varepsilon_i > 0$ was arbitrarily chosen, we conclude that $g_1(t) = g_2(t)$. □

Example

Let

$$\Lambda^2 = \mathbb{N} \times \mathbb{N}_0, \quad f(t) = t_1^2 t_2, \quad t = (t_1, t_2) \in \Lambda^2.$$

Here, $\mathbb{T}_1 = \mathbb{N}$, $\mathbb{T}_2 = \mathbb{N}_0$. We will prove that

$$f_{t_1}^{\nabla_1}(t) = \begin{cases} (2t_1 - 1)t_2 & \text{if} \quad t_1 \in \mathbb{T}_1, \quad t_1 \geq 2, \quad t_2 \in \mathbb{T}_2, \\ 2t_2 & \text{if} \quad t_1 = 1, \quad t_2 \in \mathbb{T}_2. \end{cases}$$

We have $\rho_1(t_1) = t_1 - 1$ for $t_1 \in \mathbb{T}_1$, $t_1 \geq 2$, $\rho_1(1) = 1$. Let $\varepsilon > 0$ be arbitrarily chosen. Then, for every $s_1 \in (t_1 - \varepsilon^*, t_1 + \varepsilon^*)$, $s_1 \in \mathbb{T}_1$, we have, for $\varepsilon^* \leq \frac{\varepsilon}{1+t_2}$,

$$|t_1 - s_1| \leq \varepsilon^*.$$

Hence, for $t = (t_1, t_2) \in \Lambda^2_{1\kappa_1}$, $t_1 \geq 2$, we get

Example

$$
\begin{aligned}
f(\rho_1(t_1), t_2) &= \rho_1^2(t_1)t_2 \\
&= (t_1-1)^2 t_2,
\end{aligned}
$$

$$
\begin{aligned}
&|f(\rho_1(t_1), t_2) - f(s_1, t_2) - (2t_1-1)t_2(\rho_1(t_1)-s_1)| \\
&= |(t_1-1)^2 t_2 - s_1^2 t_2 - (2t_1-1)t_2(t_1-1-s_1)| \\
&= |(t_1-1-s_1)(t_1+s_1-1)t_2 - (2t_1-1)t_2(t_1-1-s_1)| \\
&= |s_1-t_1||t_2||t_1-s_1-1| \\
&\leq |s_1-t_1|(1+t_2)|t_1-s_1-1|
\end{aligned}
$$

Example

For $t \in \Lambda^2_{1\kappa_1}$, $t_1 = 1$, we have

$$f(\rho_1(1), t_2) = \rho_1^2(1) t_2 = t_2$$

and

$$\begin{aligned}
|f(\rho_1(1), t_2) - f(s_1, t_2) - 2t_2(\rho_1(1) - s_1)| &= |f(1, t_2) - f(s_1, t_2) - 2t_2(1 - \\
&= |t_2 - s_1^2 t_2 - 2t_2(1 - s_1)| \\
&= |1 - s_1| t_2 |1 + s_1 - 2| \\
&\leq (1 + t_2)(1 - s_1)^2 \\
&\leq \varepsilon^*(1 + t_2)|1 - s_1|
\end{aligned}$$

The proofs of the following theorems repeat the main steps of the proofs of the corresponding theorems for partial delta derivatives. Thus, these proofs are omitted and left to the reader.

Theorem

Let $f : \Lambda^n \to \mathbb{R}$ be a function and $t \in \Lambda^n_{i\kappa_i}$. If f is nabla differentiable with respect to t_i at t, then

$$\lim_{s_i \to t_i} f(t_{s_i}) = f(t).$$

Theorem

Let $f : \Lambda^n \to \mathbb{R}$, $t \in \Lambda^n_{i\kappa_i}$, and

$$\lim_{s_i \to t_i} f(t_{s_i}) = f(t).$$

If $\rho_i(t_i) < t_i$, then f is nabla differentiable with respect to t_i at t and

$$f^{\nabla_i}_{t_i}(t) = \frac{f^{\rho_i}_i(t) - f(t)}{\rho_i(t_i) - t_i}.$$

Theorem

Let $t \in \Lambda^n_{i\kappa_i}$ and $t_i = \rho_i(t_i)$. Then f is partial nabla differentiable with respect to t_i at t if and only if the limit

$$\lim_{s_i \to t_i} \frac{f(t) - f(t_{s_i})}{t_i - s_i}$$

exists as a finite number. In this case,

$$f^{\nabla_i}_{t_i}(t) = \lim_{s_i \to t_i} \frac{f(t) - f(t_{s_i})}{t_i - s_i}.$$

Theorem

Let $t \in \Lambda^n_{i\kappa_i}$. Suppose $f : \Lambda^n \to \mathbb{R}$ is a function that is partial nabla differentiable with respect to t_i at t. If $\alpha \in \mathbb{R}$, then αf is partial nabla differentiable with respect to t_i at t and

$$(\alpha f)_{t_i}^{\nabla_i}(t) = \alpha f_{t_i}^{\nabla_i}(t).$$

Theorem

Let $t \in \Lambda^n_{i\kappa_i}$. Suppose $f, g : \Lambda^n \to \mathbb{R}$ are partial nabla differentiable with respect to t_i at t. Then fg is partial nabla differentiable with respect to t_i at t and

$$(fg)_{t_i}^{\nabla_i}(t) = f_{t_i}^{\nabla_i}(t)g(t) + f_i^{\rho_i}(t)g_{t_i}^{\nabla_i}(t) = f(t)g_{t_i}^{\nabla_i}(t) + f_{t_i}^{\nabla_i}(t)g_i^{\rho_i}(t).$$

Theorem

Let $f, g : \Lambda^n \to \mathbb{R}$ be partial nabla differentiable with respect to t_i at $t \in \Lambda^n_{i\kappa_i}$. Assume $g_i^{\rho_i}(t)g(t) \neq 0$. Then $\frac{f}{g}$ is partial nabla differentiable with respect to t_i at t and

$$\left(\frac{f}{g}\right)_{t_i}^{\nabla_i}(t) = \frac{f_{t_i}^{\nabla_i}(t)g(t) - f(t)g_{t_i}^{\nabla_i}(t)}{g_i^{\rho_i}(t)g(t)}.$$

Theorem (Leibniz Formula)

Let $Q_{ik}^{(m)}$ be the set consisting of all possible strings of length m, containing exactly k times ρ_i and $m-k$ times ∇_i. If $f_{t_i^{m-k}}^{\beta}$ exists for any $\beta \in Q_{ik}^{(m)}$ and $g_{t_i^k}^{\nabla_i^k}$ exists for any $k \in \{0, 1, \ldots, m\}$, then

$$(fg)_{t_i^m}^{\nabla_i^m} = \sum_{k=0}^{m} \left(\sum_{\beta \in Q_{ik}^{(m)}} f_{t_i^{m-k}}^{\beta} \right) g_{t_i^k}^{\nabla_i^k}$$

holds for any $m \in \mathbb{N}$.

We can define higher-order nabla derivatives and also mixed derivatives obtained by combining both delta and nabla differentiations such as, for instance, $f^{\nabla_i \Delta_j}_{t_i t_j}$ or $f^{\nabla_i \nabla_j \Delta_l}_{t_i t_j t_l}$.

Example

Let $\Lambda^2 = \mathbb{N} \times \mathbb{Z}$, $f(t) = t_1^2 t_2 + t_1 t_2^2 + t_2^2$, $t \in \Lambda^2$. Here, $\mathbb{T}_1 = \mathbb{N}$, $\mathbb{T}_2 = \mathbb{Z}$, and

$$
\begin{aligned}
\sigma_1(t_1) &= t_1 + 1, \quad t_1 \in \mathbb{T}_1, \\
\sigma_2(t_2) &= t_2 + 1, \quad t_2 \in \mathbb{T}_2, \\
\rho_1(t_1)
\end{aligned}
$$

Example

$$= \begin{cases} t_1 - 1 & \text{if} \quad t_1 \in \mathbb{T}_1, \quad t_1 \geq 2, \\ 1 & \text{if} \quad t_1 = 1, \end{cases}$$

$$\rho_2(t_2) = t_2 - 1, \quad t_2 \in \mathbb{T}_2.$$

Hence,

$$\begin{aligned} f_{t_1}^{\Delta_1}(t) &= (\sigma_1(t_1) + t_1)t_2 + t_2^2 \\ &= (t_1 + 1 + t_1)t_2 + t_2^2 \end{aligned}$$

Example

$$
\begin{aligned}
&= 2t_1 t_2 + t_2^2 + t_2, \quad t \in \Lambda_1^{\kappa_1 2}, \\
f_{t_1 t_2}^{\Delta_1 \nabla_2}(t) &= \left(f_{t_1}^{\Delta_1}\right)_{t_2}^{\nabla_2}(t) \\
&= 2t_1 + \rho_2(t_2) + t_2 + 1 \\
&= 2t_1 + t_2 - 1 + t_2 + 1 \\
&= 2t_1 + 2t_2, \quad t \in \Lambda_{12\kappa_2}^{\kappa_1 2}.
\end{aligned}
$$

Definition

We say that a function $f : \Lambda^n \to \mathbb{R}$ is *completely delta differentiable* at a point $t^0 \in \Lambda^{\kappa n}$ if there exist numbers a_i and A_{ij}, $i, j \in \{1, 2, \dots, n\}$ independent of $t \in \Lambda^n$ but, in general, dependent on t^0, such that for all $t \in U_\delta(t_0)$,

$$f(t^0) - f(t) = \sum_{i=1}^{n} a_i(t_i^0 - t_i) + \sum_{i=1}^{n} \alpha_i(t_i^0 - t_i) \tag{6}$$

and, for each $i \in \{1, 2, \dots, n\}$ and all $t \in U_\delta(t^0)$,

Definition

$$
\begin{aligned}
f_i^{\sigma_i}(t^0) - f(t) = & A_{ii}(\sigma_i(t_i^0) - t_i) + \sum_{l=1, l\neq i}^{n} A_{li}(t_l^0 - t_l) \\
& + \beta_{ii}(\sigma_i(t_i^0) - t_i) + \sum_{l=1, l\neq i}^{n} \beta_{li}(t_l^0 - t_l),
\end{aligned}
\tag{7}
$$

where $\delta > 0$ is a sufficiently small real number, $U_\delta(t^0)$ is the δ-neighbourhood of t^0 in Λ^n, $\alpha_i = \alpha_i(t^0, t)$, $\beta_{li} = \beta_{li}(t^0, t)$ are defined in $U_\delta(t^0)$, $l, i \in \{1, 2, \dots, n\}$, such that

$$
\lim_{t \to t^0} \alpha_i(t^0, t) = \lim_{t \to t^0} \beta_{ij}(t^0, t) = 0 \quad \text{for all} \quad i, j \in \{1, 2, \dots, n\}.
$$

Remark

If $\Lambda^n = \mathbb{R}^n$, then Definition 24 coincides with the classical total differentiability of functions of n real variables.

Example

Let $\Lambda^2 = 2^{\mathbb{N}} \times \mathbb{N}$, $f(t) = t_1^2 + t_2^2$, $t \in \Lambda^2$. Let $t^0 \in \Lambda^{\kappa 2}$ be arbitrarily chosen. We will prove that f is completely delta differentiable at t^0. Here, $\mathbb{T}_1 = 2^{\mathbb{N}}$, $\mathbb{T}_2 = \mathbb{N}$, and

$$\sigma_1(t_1) = 2t_1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

Let $\delta > 0$ be sufficiently small. For every $t \in U_\delta(t^0)$, we have

Example

$$
\begin{aligned}
f(t^0) - f(t) &= t_1^{02} + t_2^{02} - t_1^2 - t_2^2 \\
&= t_1^{02} + t_2^{02} - t_1^2 - t_2^2 + 2t_1^0(t_1^0 - t_1) - 2t_1^0(t_1^0 - t_1) \\
&\quad +2t_2^0(t_2^0 - t_2) - 2t_2^0(t_2^0 - t_2) \\
&= 2t_1^0(t_1^0 - t_1) + 2t_2^0(t_2^0 - t_2) + (t_1^{02} - 2t_1^{02} + 2t_1^0 t_1 - t_1^2) \\
&\quad +(t_2^{02} - t_2^2 - 2t_2^{02} + 2t_2^0 t_2) \\
&= 2t_1^0(t_1^0 - t_1) + 2t_2^0(t_2^0 - t_2) - (t_1^0 - t_1)^2 - (t_2^0 - t_2)^2.
\end{aligned}
$$

Therefore,

Example

$$\alpha_1 = -(t_1^0 - t_1), \quad \alpha_2 = -(t_2^0 - t_2), \quad a_1 = 2t_1^0, \quad a_2 = 2t_2^0.$$

We note that

$$\lim_{t \to t^0} \alpha_1 = \lim_{t \to t^0} \alpha_2 = 0.$$

For $t \in U_\delta(t^0)$, we have

$$f_1^{\sigma_1}(t^0) - f(t) \;=\; \sigma_1^2(t_1^0) + t_2^{02} - t_1^2 - t_2^2$$

Example

$$
\begin{aligned}
&= 4t_1^{02} + t_2^{02} - t_1^2 - t_2^2 \\
&= (2t_1^0 - t_1)(2t_1^0 + t_1) + (t_2^0 - t_2)(t_2^0 + t_2) \\
&= (2t_1^0 - t_1)(3t_1^0 - t_1^0 + t_1) + (t_2^0 - t_2)(2t_2^0 - t_2^0 + t_2) \\
&= 3t_1^0(2t_1^0 - t_1) + 2t_2^0(t_2^0 - t_2) - (2t_1^0 - t_1)(t_1^0 - t_1) - (t_2^0 - t_2)^2.
\end{aligned}
$$

Here,

$$A_{11} = 3t_1^0, \quad A_{21} = 2t_2^0, \quad \beta_{11} = -(t_1^0 - t_1), \quad \beta_{21} = -(t_2^0 - t_2).$$

We note that

Example

$$\lim_{t \to t^0} \beta_{11} = \lim_{t \to t^0} \beta_{21} = 0.$$

For all $t \in U_\delta(t^0)$, we have

$$\begin{aligned}
f_2^{\sigma_2}(t^0) - f(t) &= t_1^{02} + (t_2^0 + 1)^2 - t_1^2 - t_2^2 \\
&= (t_1^0 - t_1)(t_1^0 + t_1) + (t_2^0 + 1 - t_2)(t_2^0 + 1 + t_2) \\
&= (t_1^0 - t_1)(2t_1^0 + t_1 - t_1^0) + (t_2^0 + 1 - t_2)(2t_2^0 + 1 + t_2 - t_2^0 \\
&= 2t_1^0(t_1^0 - t_1) + (2t_2^0 + 1)(t_2^0 + 1 - t_2) \\
&\quad -(t_1^0 - t_1)^2 - (t_2^0 + 1 - t_2)(t_2^0 - t_2).
\end{aligned}$$

Example

Thus,

$$A_{12} = 2t_1^0, \quad A_{22} = 2t_2^0 + 1, \quad \beta_{12} = -(t_1^0 - t_1), \quad \beta_{22} = -(t_2^0 - t_2).$$

We note that

$$\lim_{t \to t^0} \beta_{12} = \lim_{t \to t^0} \beta_{22} = 0.$$

Example

Let $\Lambda^2 = \mathbb{Z} \times 3^{\mathbb{N}}$, $f(t) = t_1 t_2^2$, $t \in \Lambda^2$. Let $t^0 \in \Lambda^2$ be arbitrarily chosen. We will prove that the function f is completely delta differentiable at t_0.

Example

Here, $\mathbb{T}_1 = \mathbb{Z}$, $\mathbb{T}_2 = 3^{\mathbb{N}}$, and

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = 3t_2, \quad t_2 \in \mathbb{T}_2.$$

Let $\delta > 0$ be arbitrarily chosen. For $t \in U_\delta(t^0)$, we have

$$\begin{aligned} f(t^0) - f(t) &= t_1^0 t_2^{02} - t_1 t_2^2 \\ &= t_1^0 t_2^{02} - t_1 t_2^{02} + t_1 t_2^{02} - t_1 t_2^2 \end{aligned}$$

Example

$$
\begin{aligned}
&= (t_1^0 - t_1)t_2^{02} + t_1(t_2^0 - t_2)(t_2^0 + t_2) \\
&= (t_1^0 - t_1)t_2^{02} + 2t_1 t_2^0(t_2^0 - t_2) - t_1(t_2^0 - t_2)^2 \\
&= (t_1^0 - t_1)t_2^{02} + 2t_1^0 t_2^0(t_2^0 - t_2) - 2t_1^0 t_2^0(t_2^0 - t_2) \\
&\quad +2t_1 t_2^0(t_2^0 - t_2) - t_1(t_2^0 - t_2)^2 \\
&= t_2^{02}(t_1^0 - t_1) + 2t_1^0 t_2^0(t_2^0 - t_2) - 2t_2^0(t_2^0 - t_2)(t_1^0 - t_1) \\
&\quad -t_1(t_2^0 - t_2)^2.
\end{aligned}
$$

Example

Here,

$$a_1 = t_2^{02}, \quad a_2 = 2t_1^0 t_2^0, \quad \alpha_1 = -2t_2^0(t_2^0 - t_2), \quad \alpha_2 = -t_1(t_2^0 - t_2).$$

We note that

$$\lim_{t \to t^0} \alpha_1 = \lim_{t \to t^0} \alpha_2 = 0.$$

For $t \in U_\delta(t^0)$, we have

$$\begin{aligned} f_1^{\sigma_1}(t^0) - f(t) &= (t_1^0 + 1)t_2^{02} - t_1 t_2^2 \\ &= t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0(t_2^0 - t_2) - t_2^{02}(t_1^0 + 1 - t_1) \end{aligned}$$

Example

$$
\begin{aligned}
& -2t_1^0 t_2^0(t_2^0 - t_2) + (t_1^0 + 1)t_2^{02} - t_1 t_2^2 \\
= \quad & t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0(t_2^0 - t_2) - t_1^0 t_2^{02} \\
& -t_2^{02} + t_1 t_2^{02} - 2t_1^0 t_2^{02} + 2t_1^0 t_2^0 t_2 + t_1^0 t_2^{02} + t_2^{02} - t_1 t_2^2 \\
= \quad & t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0(t_2^0 - t_2) + t_1(t_2^0 - t_2)(t_2^0 + t_2) \\
& -2t_1^0 t_2^0(t_2^0 - t_2)
\end{aligned}
$$

Example

$$
\begin{aligned}
&= t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0 (t_2^0 - t_2) + (t_1 t_2^0 + t_1 t_2 \\
&\quad -2t_1^0 t_2^0)(t_2^0 - t_2) \\
&= t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0 (t_2^0 - t_2) + (t_1 t_2^0 - t_1^0 t_2^0 \\
&\quad -t_1^0 t_2^0 + t_1 t_2)(t_2^0 - t_2) \\
&= t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0 (t_2^0 - t_2)
\end{aligned}
$$

Example

$$
\begin{aligned}
& -(t_1^0 - t_1)t_2^0(t_2^0 - t_2) + (t_1 t_2 - t_1^0 t_2 + t_1^0 t_2 - t_1^0 t_2^0)(t_2^0 - t_2) \\
= \;& t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0(t_2^0 - t_2) - (t_1^0 - t_1)t_2^0(t_2^0 - t_2) \\
& +(t_1 - t_1^0)t_2(t_2^0 - t_2) - t_1^0(t_2^0 - t_2)^2 \\
= \;& t_2^{02}(t_1^0 + 1 - t_1) + 2t_1^0 t_2^0(t_2^0 - t_2) - (t_2 + t_2^0)(t_2^0 - t_2)(t_1^0 - t_1) \\
& -t_1^0(t_2^0 - t_2)^2.
\end{aligned}
$$

Example

Here,

$$A_{11} = t_2^{02}, \quad A_{21} = 2t_1^0 t_2^0, \quad \beta_{11} = -(t_2^0 + t_2)(t_2^0 - t_2), \quad \beta_{21} = -t_1^0(t_2^0 - t_2).$$

We note that

$$\lim_{t \to t^0} \beta_{11} = \lim_{t \to t^0} \beta_{21} = 0.$$

For $t \in U_\delta(t^0)$, we have

$$\begin{aligned} f_2^{\sigma_2}(t^0) - f(t) &= t_1^0(3t_2^0)^2 - t_1 t_2^2 \\ &= 9t_1^0 t_2^{02} - t_1 t_2^2 \end{aligned}$$

Example

$$
\begin{aligned}
&= t_2^{02}(t_1^0 - t_1) + 4t_1^0 t_2^0(3t_2^0 - t_2) - t_2^{02}(t_1^0 - t_1) \\
&\quad -4t_1^0 t_2^0(3t_2^0 - t_2) + 9t_1^0 t_2^{02} - t_1 t_2^2 \\
&= t_2^{02}(t_1^0 - t_1) + 4t_1^0 t_2^0(3t_2^0 - t_2) - t_2^{02}(t_1^0 - t_1) \\
&\quad -4t_1^0 t_2^0(3t_2^0 - t_2) + 9t_1^0 t_2^{02} - t_1^0 t_2^2 + t_1^0 t_2^2 - t_1 t_2^2 \\
&= t_2^{02}(t_1^0 - t_1) + 4t_1^0 t_2^0(3t_2^0 - t_2) - t_2^{02}(t_1^0 - t_1) \\
&\quad -4t_1^0 t_2^0(3t_2^0 - t_2) + t_1^0(3t_2^0 - t_2)(3t_2^0 + t_2) + (t_1^0 - t_1)t_2^2
\end{aligned}
$$

Example

$$= \quad t_2^{02}(t_1^0 - t_1) + 4t_1^0 t_2^0(3t_2^0 - t_2)$$

$$+(t_2^2 - t_2^{02})(t_1^0 - t_1) + (t_1^0 t_2 - t_1^0 t_2^0)(3t_2^0 - t_2).$$

Here,

$$A_{12} = t_2^{02}, \quad A_{22} = 4t_1^0 t_2^0, \quad \beta_{12} = t_2^2 - t_2^{02}, \quad \beta_{22} = t_1^0 t_2 - t_1^0 t_2^0.$$

We note that

$$\lim_{t \to t^0} \beta_{12} = \lim_{t \to t^0} \beta_{22} = 0.$$