Time Scales Analysis
Lecture 25

Properties of Partial Delta Derivatives

Svetlin G. Georgiev

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Example

Let $\Lambda^2 = \mathbb{N} \times \mathbb{N}$, $f(t) = t_1 t_2$, $t \in \Lambda^2$. Let $t_0 \in \Lambda^2$ be arbitrarily chosen. Here, $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{N}$ and

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

We will prove that f is completely delta differentiable at t_0. Let $\delta > 0$ be sufficiently small. For $t \in U_\delta(t^0)$, we have

Example

$$
\begin{aligned}
f(t^0) - f(t) &= t_1^0 t_2^0 - t_1 t_2 = t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 - t_2) \\
&\quad -t_2^0(t_1^0 - t_1) - t_1^0(t_2^0 - t_2) + t_1^0 t_2^0 - t_1 t_2 \\
&= t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 - t_2) - t_1^0 t_2^0 + t_1 t_2^0 - t_1^0 t_2^0 \\
&\quad +t_1^0 t_2 + t_1^0 t_2^0 - t_1 t_2 \\
&= t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 - t_2) - t_1^0 t_2^0 + t_1 t_2^0 + t_1^0 t_2 - t_1 t_2 \\
&= t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 - t_2) - t_2^0(t_1^0 - t_1) + t_2(t_1^0 - t_1) \\
&= t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 - t_2) + (t_2 - t_2^0)(t_1^0 - t_1).
\end{aligned}
$$

Example

Here,

$$a_1 = t_2^0, \quad a_2 = t_1^0, \quad \alpha_1 = t_2 - t_2^0, \quad \alpha_2 = 0.$$

We note that

$$\lim_{t \to t^0} \alpha_1 = 0.$$

For $t \in U_\delta(t^0)$, we have

$$\begin{aligned}
f_1^{\sigma_1}(t^0) - f(t) &= (t_1^0 + 1)t_2^0 - t_1 t_2 \\
&= t_2^0(t_1^0 + 1 - t_1) + t_1^0(t_2^0 - t_2) - t_2^0(t_1^0 + 1 - t_1) \\
&\quad -t_1^0(t_2^0 - t_2) + (t_1^0 + 1)t_2^0 - t_1 t_2 \\
&= t_2^0(t_1^0 + 1 - t_1) + t_1^0(t_2^0 - t_2) - t_1^0 t_2^0 - t_2^0 + t_1 t_2^0
\end{aligned}$$

Example

$$
\begin{aligned}
& -t_1^0 t_2^0 + t_1^0 t_2 + t_1^0 t_2^0 + t_2^0 - t_1 t_2 \\
= \quad & t_2^0(t_1^0 + 1 - t_1) + t_1^0(t_2^0 - t_2) - t_1^0 t_2^0 \\
& + t_1 t_2^0 + t_1^0 t_2 - t_1 t_2 \\
= \quad & t_2^0(t_1^0 + 1 - t_1) + t_1^0(t_2^0 - t_2) + (t_1 - t_1^0)(t_2^0 - t_2).
\end{aligned}
$$

Here,

$$A_{11} = t_2^0, \quad A_{21} = t_1^0, \quad \beta_{11} = 0, \quad \beta_{21} = t_1 - t_1^0.$$

We note that

$$\lim_{t \to t^0} \beta_{21} = 0.$$

Example

For $t \in U_\delta(t^0)$, we have

$$\begin{aligned}
f_2^{\sigma_2}(t^0) - f(t) &= t_1^0(t_2^0 + 1) - t_1 t_2 \\
&= t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 + 1 - t_2) - t_2^0(t_1^0 - t_1) \\
&\quad - t_1^0(t_2^0 + 1 - t_2) + t_1^0(t_2^0 + 1) - t_1 t_2 \\
&= t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 + 1 - t_2) - t_1^0 t_2^0 + t_1 t_2^0 - t_1^0 t_2^0 \\
&\quad - t_1^0 + t_1^0 t_2 + t_1^0 t_2^0 + t_1^0 - t_1 t_2 \\
&= t_2^0(t_1^0 - t_1) + t_1^0(t_2^0 + 1 - t_2) - t_1^0 t_2^0 \\
&\quad + t_1 t_2^0 + t_1^0 t_2 - t_1 t_2
\end{aligned}$$

Example

Here,

$$A_{12} = t_2^0, \quad A_{22} = t_1^0, \quad \beta_{12} = t_2 - t_2^0, \quad \beta_{22} = 0.$$

We note that

$$\lim_{t \to t^0} \beta_{12} = 0.$$

Theorem

Every function defined on $\mathbb{Z}^n$ is completely delta differentiable at every point.

Proof.

We may use

$$A_{ii} = f_{t_i}^{\Delta_i}(t^0), \quad A_{li} = 0, \quad l \neq i, \quad \beta_{li} = 0, \quad l, i \in \{1, 2, \dots, n\}.$$

□

Theorem

Let $f : \Lambda^n \to \mathbb{R}$ be completely delta differentiable at $t \in \Lambda^{\kappa n}$. Then $f_{t_i}^{\Delta_i}(t)$ exists and

$$A_{ii} = f_{t_i}^{\Delta_i}(t) \quad \text{for all} \quad i \in \{1, 2, \ldots, n\}. \tag{1}$$

Proof.

From (**??**), we get, for $i \in \{1, 2, \ldots, n\}$,

$$\lim_{s_i \to t_i} \frac{f_i^{\sigma_i}(t) - f(t_{s_i})}{\sigma_i(t_i) - s_i} = A_{ii},$$

which implies (1). □

Example

Let $\mathbb{T} = [0,2] \cup \{4\}$, where $[0,2]$ is the real number interval. Let $f : \mathbb{T} \to \mathbb{R}$ be defined by $f(t) = t^3$, $t \in [0,2]$, $f(4) = 2$. Then

$$
\begin{aligned}
f^{\Delta}(t) &= 3t^2, \quad t \in [0,2), \\
f^{\Delta}(2-) &= 12, \\
f^{\Delta}(2) &= \frac{f(\sigma(2)) - f(2)}{\sigma(2) - 2} \\
&= \frac{f(4) - f(2)}{4 - 2} = \frac{2 - 8}{2} = -3.
\end{aligned}
$$

Therefore, the function f is not completely delta differentiable at the point $t = 2$.

Example

Let $\mathbb{T} = [2,3] \cup \{5\}$, where $[2,3]$ is the real number interval. Let $f : \mathbb{T} \to \mathbb{R}$ be defined by

$$f(t) = t^2 + t + 1, \quad t \in [2,3] \quad \text{and} \quad f(5) = c,$$

where $c \in \mathbb{R}$. Then

$$\begin{aligned} f^{\Delta}(t) &= 2t + 1, \quad t \in [2,3), \\ f^{\Delta}(3-) &= 7, \end{aligned}$$

Example

$$
\begin{aligned}
f^{\Delta}(3) &= \frac{f(\sigma(3)) - f(3)}{\sigma(3) - 3} \\
&= \frac{f(5) - f(3)}{5 - 3} \\
&= \frac{c - (9 + 3 + 1)}{2} \\
&= \frac{c - 13}{2}.
\end{aligned}
$$

Example

Hence, $f^{\Delta}(3-) = f^{\Delta}(3)$ if

$$\frac{c-13}{2} = 7, \quad \text{i.e.,} \quad c = 27.$$

Consequently, the function f is completely delta differentiable at $t = 3$ if $c = 27$. If $c \neq 27$, then the function f is not completely delta differentiable at $t = 3$.

Example

Let $\mathbb{T}_1 = [0,1] \cup \{3\}$, $\mathbb{T}_2 = [0,1]$, where $[0,1]$ is the real number interval. Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be defined by $f(t) = t_1^2 + t_2^2$ for $(t_1, t_2) \in [0,1] \times [0,1]$ and $f(3, t_2) = a + bt_2$, $t_2 \in [0,1]$, where a and b are real constants. We have

$$f_{t_1}^{\Delta_1}(t) \quad = \quad 2t_1, \quad (t_1, t_2) \in [0,1) \times [0,1],$$

Example

$$
\begin{aligned}
f_{t_2}^{\Delta_2}(t) &= 2t_2, \quad (t_1, t_2) \in [0,1) \times [0,1], \\
f_{t_1}^{\Delta_1}(1-, t_2) &= 2, \\
f_{t_2}^{\Delta_2}(1-, t_2) &= 2t_2, \quad t_2 \in [0,1], \\
f_{t_1}^{\Delta_1}(1, t_2) &= \frac{f(\sigma_1(1), t_2) - f(1, t_2)}{\sigma_1(1) - 1} \\
&= \frac{f(3, t_2) - f(1, t_2)}{3 - 1} \\
&= \frac{a + bt_2 - 1 - t_2^2}{2},
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(1,0) &= \frac{a-1}{2}, \\
f_{t_2}^{\Delta_2}(1,0) &= f_{t_2}^{\Delta_2}(1-,0) \\
&= 0.
\end{aligned}
$$

Therefore, f is completely delta differentiable at $(1,0)$ if

$$\frac{a-1}{2} = 2, \quad \text{i.e.,} \quad a = 5.$$

Definition

For $i \in \{1, 2, \dots, n\}$, we say that a function

$$f : \Lambda^n \to \mathbb{R}$$

is *σ_i-completely delta differentiable* at $t^0 \in \Lambda^{\kappa n}$ if it is completely delta differentiable at that point in the sense of the conditions (**??**) and (**??**) and moreover, along with the numbers A_{ii}, there exist numbers B_{li} independent of $t \in \Lambda^n$ (but in general, dependent on t^0), $l \neq i$, $l \in \{1, 2, \dots, n\}$, such that

Definition

$$f^\sigma(t^0) - f(t) = A_{ii}(\sigma_i(t_i^0) - t_i) + \sum_{l=1, l\neq i}^{n} B_{li}(\sigma_l(t_l^0) - t_l) + \sum_{l=1}^{n} \gamma_l(\sigma_l(t_l^0) - t_l) \tag{2}$$

for all $t \in V^{\sigma_i}(t^0)$, where $V^{\sigma_i}(t^0)$ is a union of some neighbourhoods of t^0 and

$$(t_1^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0),$$

and $\gamma_i = \gamma_i(t^0, t)$, $\gamma_l = \gamma_l(t^0, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$, $l \neq i$, are equal to zero for $t = t^0$ and

Definition

$$\lim_{t\to t^0}\gamma_i(t^0,t)=0,\quad \lim_{t_l\to t_l^0}\gamma_l(t^0,t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)=0,\quad l\neq i,$$

where $l\in\{1,2,\ldots,n\}$.

Remark

In fact, in (2), we have

$$B_{li}=f_{t_l}^{\Delta_l}(\sigma_1(t_1^0),\ldots,\sigma_{l-1}(t_{l-1}^0),t_l^0,\sigma_{l+1}(t_{l+1}^0),\ldots,\sigma_n(t_n^0)),\quad l\neq i,$$

where $l\in\{1,2,\ldots,n\}$.

Example

Let $\mathbb{T}_1 = [0,1] \times \{2\}$, $\mathbb{T}_2 = [0,1]$, where $[0,1]$ is the real interval. Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be defined by

$$f(t_1, t_2) = t_1^2 + t_2^2 \quad \text{for} \quad (t_1, t_2) \in [0,1] \times [0,1]$$

and

$$f(2, t_2) = 3 + \sqrt{t_2} \quad \text{for} \quad t_2 \in [0,1].$$

Thus,

$$\begin{aligned} f_{t_1}^{\Delta_1}(t_1, t_2) &= \sigma_1(t_1) + t_1 \\ &= 2t_1, \quad t_1 \in [0,1), \quad t_2 \in [0,1], \end{aligned}$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(1-, t_2) &= 2, \quad t_2 \in [0, 1], \\
f_{t_1}^{\Delta_1}(1-, 0) &= 2, \\
f_{t_1}^{\Delta_1}(1, t_2) &= \frac{f(\sigma_1(1), t_2) - f(1, t_2)}{\sigma_1(1) - 1} \\
&= \frac{f(2, t_2) - f(1, t_2)}{2 - 1} \\
&= 3 + \sqrt{t_2} - 1 - t_2^2 \\
&= 2 + \sqrt{t_2} - t_2^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(1,0) &= 2, \\
f_{t_2}^{\Delta_2}(t_1,t_2) &= \sigma_2(t_2)+t_2 \\
&= 2t_2, \\
f_{t_2}^{\Delta_2}(t_1,0) &= 0.
\end{aligned}
$$

Therefore, f is completely delta differentiable. Also,

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t_1, \sigma_2(t_2)) &= 2t_1, \quad t_1 \in [0,1), \\
f_{t_1}^{\Delta_1}(1-, \sigma_2(t_2)) &= 2, \\
f_{t_1}^{\Delta_1}(1, \sigma_2(t_2)) &= \frac{f(\sigma_1(1), \sigma_2(t_2)) - f(1, \sigma_2(t_2))}{\sigma_1(1) - 1} \\
&= \frac{f(2, \sigma_2(t_2)) - f(1, \sigma_2(t_2))}{2 - 1} \\
&= 3 + \sqrt{\sigma_2(t_2)} - 1 - \sigma_2^2(t_2) \\
&= 2 + \sqrt{t_2} - t_2^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(1,0) &= 2,\\
f_{t_2}^{\Delta_2}(\sigma_1(1),0) &= \lim_{s\to 0}\frac{f(\sigma_1(1),s)-f(\sigma_1(1),0)}{s-0}\\
&= \lim_{s\to 0}\frac{f(2,s)-f(2,0)}{s}\\
&= \lim_{s\to 0}\frac{3+\sqrt{s}-3}{s}\\
&= \infty,
\end{aligned}
$$

and thus, while f is σ_1-completely delta differentiable, it is not σ_2-completely delta differentiable.

Example

Now assume that

$$f(2, t_2) = 3 + t_2 \quad \text{for} \quad t_2 \in [0, 1]$$

so that

$$\begin{aligned} f_{t_1}^{\Delta_1}(1, t_2) &= 2 + t_2 - t_2^2, \\ f_{t_1}^{\Delta_1}(1, 0) &= 2, \\ f_{t_2}^{\Delta_2}(t_1, t_2) &= 2t_2, \end{aligned}$$

Example

$$
\begin{aligned}
f_{t_2}^{\Delta_2}(1,0) &= 0, \\
f_{t_1}^{\Delta_1}(1,\sigma_2(t_2)) &= 2t_1, \\
f_{t_2}^{\Delta_2}(1-,0) &= 2, \\
f_{t_1}^{\Delta_1}(1,\sigma_2(t_2)) &= 2+t_2-t_2^2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(1,0) &= 2, \\
f_{t_2}^{\Delta_2}(\sigma_1(1),0) &= \lim_{s\to 0} \frac{f(\sigma_1(1),s) - f(\sigma_1(1),0)}{s} \\
&= \lim_{s\to 0} \frac{f(2,s) - f(2,0)}{s} \\
&= \lim_{s\to 0} \frac{3+s-3}{s} = 1.
\end{aligned}
$$

Therefore, f is both σ_1-completely delta differentiable and σ_2-completely delta differentiable.

Example

Let $\mathbb{T}_1 = [1,2] \cup \{3\}$, $\mathbb{T}_2 = [1,2]$, where $[1,2]$ is the real interval. Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be defined by

$$f(t_1, t_2) = t_1 + t_2, \quad (t_1, t_2) \in [1,2] \times [1,2]$$

and

$$f(3, t_2) = a + t_2, \quad t_2 \in [1,2],$$

where a is a real constant. We will find a constant a such that the function f is σ_1-completely delta differentiable at the point $(2,1)$. We have

$$f_{t_1}^{\Delta_1}(t_1, t_2) \;\; = \;\; 1, \quad (t_1, t_2) \in [1,2) \times [1,2],$$

Example

$$
\begin{aligned}
f_{t_2}^{\Delta_2}(t_1, t_2) &= 1, \quad (t_1, t_2) \in [1,2] \times [1,2], \\
f_{t_1}^{\Delta_1}(2, t_2) &= \frac{f(3, t_2) - f(2, t_2)}{1} \\
&= \frac{a + t_2 - 2 - t_2}{1} \\
&= a - 2, \\
f_{t_1}^{\Delta_1}(2, 1) &= a - 2,
\end{aligned}
$$

Example

$$f_{t_2}^{\Delta_2}(2, t_2) = 1, \quad t_2 \in [1, 2],$$

$$f_{t_1}^{\Delta_1}(2-, t_2) = 1, \quad t_2 \in [1, 2].$$

Hence, f is σ_1-completely delta differentiable iff $a - 2 = 1$, i.e., $a = 3$.

Example

Let $\mathbb{T}_1 = [0, 2] \cup \{4\}$, $\mathbb{T}_2 = [0, 2]$, where $[0, 2]$ is the real interval. Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be defined by

$$f(t_1, t_2) = 2t_1 + 3t_2, \quad (t_1, t_2) \in [0, 2] \times [0, 2]$$

and

$$f(4, t_2) = a + \sqrt{t_2}, \quad t_2 \in [0, 2],$$

where a is a real constant. We will find a constant a so that the function f is completely delta differentiable, σ_1-completely delta differentiable, and σ_2-completely delta differentiable at the point $(2, 2)$. We have

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t_1, t_2) &= 2, \quad (t_1, t_2) \in [0, 2) \times [0, 2], \\
f_{t_1}^{\Delta_1}(2-, t_2) &= 2, \quad t_2 \in [0, 2], \\
f_{t_2}^{\Delta_1}(t_1, t_2) &= 3, \quad (t_1, t_2) \in [0, 2] \times [0, 2],
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(2, t_2) &= \frac{f(\sigma_1(2), t_2) - f(2, t_2)}{\sigma_1(2) - 2} \\
&= \frac{f(4, t_2) - f(2, t_2)}{4 - 2} \\
&= \frac{a + \sqrt{t_2} - (4 + 3t_2)}{2} \\
&= \frac{a-4}{2} + \frac{1}{2}\sqrt{t_2} - \frac{3}{2} t_2, \\
f_{t_1}^{\Delta_1}(2, 2) &= \frac{a-4}{2} + \frac{\sqrt{2}}{2} - 3 \\
&= \frac{a}{2} + \frac{\sqrt{2}}{2} - 5,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(2-, \sigma_2(2)) &= 2, \\
f_{t_2}^{\Delta_2}(\sigma_1(2), 2) &= \lim_{s \to 2} \frac{f(\sigma_1(2), s) - f(\sigma_1(2), 2)}{s-2} \\
&= \lim_{s \to 2} \frac{f(4, s) - f(4, 2)}{s-2} \\
&= \lim_{s \to 2} \frac{a + \sqrt{s} - a - \sqrt{2}}{s-2} \\
&= \lim_{s \to 2} \frac{1}{\sqrt{s} + \sqrt{2}} \\
&= \frac{\sqrt{2}}{4}.
\end{aligned}
$$

Example

Therefore, the function f is completely delta differentiable, σ_1-completely delta differentiable, and σ_2-completely delta differentiable at the point $(2,2)$ iff

$$\frac{a}{2} + \frac{\sqrt{2}}{2} - 5 = 2, \quad \text{i.e.,} \quad 4 = a + \sqrt{2} - 10, \quad \text{i.e.,} \quad a = 14 - \sqrt{2}.$$