Time Scales Analysis
Lecture 26
Mean Value Theorems

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Theorem (Mean Value Theorem)

Let $f : \Lambda^n \to \mathbb{R}$, $x, y \in \Lambda^n$, $\alpha = \min_{i\in\{1,\dots,n\}} x_i$, $\beta = \max_{i\in\{1,\dots,n\}} y_i$. *Assume* f *is continuous on*

$$[\alpha, \beta] \times [\alpha, \beta] \times \cdots \times [\alpha, \beta] \subset \Lambda^n$$

and $f_{x_i}^{\Delta_i}(x)$ *exists for all* $x \in \Lambda_i^{\kappa_i n}$. *Then there exist numbers* $\xi_i, \eta_i \in [\alpha, \beta)$, $i \in \{1, 2, \dots, n\}$, *such that*

Theorem

$$
\begin{aligned}
&f_{x_1}^{\Delta_1}(\xi_1, x_2, \ldots, x_n)(x_1 - y_1) + f_{x_2}^{\Delta_2}(y_1, \xi_2, \ldots, x_n)(x_2 - y_2) \\
&\quad + \cdots + f_{x_n}^{\Delta_n}(y_1, y_2, \ldots, \xi_n)(x_n - y_n) \leq f(x) - f(y) \\
&\leq f_{x_1}^{\Delta_1}(\eta_1, x_2, \ldots, x_n)(x_1 - y_1) + f_{x_2}^{\Delta_2}(y_1, \eta_2, \ldots, x_n)(x_2 - y_2) \\
&\quad + \cdots + f_{x_n}^{\Delta_n}(y_1, y_2, \ldots, \eta_n)(x_n - y_n).
\end{aligned} \tag{1}
$$

Proof.

We have

$$\begin{aligned} f(x)-f(y) =& f(x_1,x_2,\dots,x_n) - f(y_1,x_2,\dots,x_n) + f(y_1,x_2,\dots,x_n) \\ & - f(y_1,y_2,x_3,\dots,x_n) + f(y_1,y_2,x_3,\dots,x_n) \\ & - \dots + f(y_1,y_2,\dots,y_{n-1},x_n) - f(y_1,y_2,\dots,y_{n-1},y_n). \end{aligned} \tag{2}$$

Using the mean value theorem, Theorem **??**, there exist numbers $\xi_i, \eta_i \in [\alpha, \beta)$, $i \in \{1, 2, \dots, n\}$, such that □

Proof.

$$
\begin{aligned}
f_{x_1}^{\Delta_1}(\xi_1, x_2, \ldots, x_n)(x_1 - y_1) &\leq f(x_1, x_2, \ldots, x_n) - f(y_1, x_2, \ldots, x_n) \\
&\leq f_{x_1}^{\Delta_1}(\eta_1, x_2, \ldots, x_n)(x_1 - y_1), \\
f_{x_2}^{\Delta_2}(y_1, \xi_2, \ldots, x_n)(x_2 - y_2) &\leq f(y_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, x_n)
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
& \leq f_{x_2}^{\Delta_2}(y_1, \eta_2, \dots, x_n)(x_2 - y_2), \\
& \vdots \\
f_{x_n}^{\Delta_n}(y_1, y_2, \dots, x_n)(x_n - y_n) & \leq f(y_1, y_2, \dots, x_n) - f(y_1, y_2, \dots, y_n) \\
& \leq f_{x_n}^{\Delta_n}(y_1, y_2, \dots, \eta_n)(x_n - y_n).
\end{aligned}
$$

Thus, using (2), we get (1). □

Corollary

Suppose $f : \Lambda^n \to \mathbb{R}$ *is a continuous function and* $f_{x_i}^{\Delta_i}(x)$ *exists for every* $x \in \Lambda_i^{\kappa_i n}$, $i \in \{1, 2, \ldots, n\}$. *If these derivatives are identically zero, then* f *is a constant function on* Λ^n.

Theorem

Suppose the function $f : \Lambda^n \to \mathbb{R}$ is continuous and has first-order partial delta derivatives $f_{t_1}^{\Delta_i}(t)$ in some neighbourhood $U_\delta(t^0)$ of the point $t^0 \in \Lambda^{\kappa n}$. If these derivatives are continuous at the point t^0, then f is completely delta differentiable at t^0.

Proof.

We have

$$\begin{aligned} f(t^0) - f(t) \;&=\; f(t^0) - f(t_1, t_2^0, \ldots, t_n^0) + f(t_1, t_2^0, \ldots, t_n^0) - \cdots \\ &\quad + f(t_1, t_2, \ldots, t_{n-1}, t_n^0) - f(t_1, t_2, \ldots, t_n). \end{aligned}$$

$\square$

Proof.

By the one-variable case, we have

$$f(t^0) - f(t_1, t_2^0, \ldots, t_n^0) = f_{t_1}^{\Delta_1}(t^0)(t_1^0 - t_1) + \alpha_1(t_1^0 - t_1)$$

for $(t_1, t_2^0, \ldots, t_n^0) \in U_\delta(t^0)$, where

$$\alpha_1 = \alpha_1(t^0, t_1), \quad \alpha_1 \to 0 \quad \text{as} \quad (t_1, t_2^0, \ldots, t_n^0) \to (t_1^0, t_2^0, \ldots, t_n^0).$$

Further, applying the one-variable mean value result, we get □

Proof.

$$
\begin{aligned}
f_{t_2}^{\Delta_2}(t_1,\xi_1,\dots,t_n^0)(t_2^0-t_2) \leq & f(t_1,t_2^0,\dots,t_n^0)-f(t_1,t_2,\dots,t_n^0) \\
\leq & f_{t_2}^{\Delta_2}(t_1,\xi_2,\dots,t_n^0)(t_2^0-t_2),
\end{aligned} \tag{3}
$$

where $\xi_1,\xi_2 \in [\alpha,\beta)$ and $\alpha=\min\{t_2^0,t_2\}$, $\beta=\max\{t_2^0,t_2\}$. □

Proof.

Since $\xi_1, \xi_2 \to t_2^0$ as $t_2 \to t_2^0$ and $f_{t_2}^{\Delta_2}(\cdot)$ is continuous at t^0, we get

$$\lim_{t \to t^0} f_{t_2}^{\Delta_2}(t_1, \xi_1, \dots, t_n^0) = \lim_{t \to t^0} f_{t_2}^{\Delta_2}(t_1, \xi_2, \dots, t_n^0) = f_{t_2}^{\Delta_2}(t^0).$$

Thus, using (3), we find

$$f(t_1, t_2^0, \dots, t_n^0) - f(t_1, t_2, \dots, t_n^0) = f_{t_2}^{\Delta_2}(t^0)(t_2^0 - t_2) + \alpha_2(t_2^0 - t_2),$$

where $\alpha_2 = \alpha_2(t^0, t_1, t_2)$, $\alpha_2 \to 0$ as $t \to t^0$, and so on,

$$f(t_1, t_2, \dots, t_n^0) - f(t_1, t_2, \dots, t_n) = f_{t_n}^{\Delta_n}(t^0)(t_n^0 - t_n) + \alpha_n(t_n^0 - t_n),$$

where $\alpha_n = \alpha_n(t^0, t_1, t_2, \dots, t_n)$, $\alpha_n \to 0$ as $t \to t^0$. $\square$

Proof.

Consequently,

$$f(t^0) - f(t) = \sum_{i=1}^{n} f_{t_i}^{\Delta_i}(t^0)(t_i^0 - t_i) + \sum_{i=1}^{n} \alpha_i (t_i^0 - t_i). \tag{4}$$

Now, we consider the difference

$$f(t_1^0, t_2^0, \dots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \dots, t_n^0) - f(t_1, t_2, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n)$$

$$= f(t_1^0, t_2^0, \dots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \dots, t_n^0) - f(t_1^0, t_2^0, \dots, t_{i-1}^0, t_i^0, t_{i+1}^0, \dots, t_n^0)$$

$$+ f(t_1^0, t_2^0, \dots, t_{i-1}^0, t_i^0, t_{i+1}^0, \dots, t_n^0) - f(t_1, t_2, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n). \tag{5}$$

□

Proof.

Using the definition of the partial delta derivative, we get

$$f(t_1^0, t_2^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0) - f(t_1^0, t_2^0, \ldots, t_{i-1}^0, t_i^0, t_{i+1}^0, \ldots, t_n^0)$$

$$= f_{t_i}^{\Delta_i}(t^0)(\sigma_i(t_i^0) - t_i^0) + \beta_{ii}(\sigma_i(t_i^0) - t_i^0). \quad (6)$$

Substituting (4) and (6) in (5), we obtain □

Proof.

$$f(t_1^0, t_2^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0) - f(t)$$

$$= \quad f_{t_i}^{\Delta_i}(t^0)(\sigma_i(t_i^0) - t_i) + \beta_{ii}(\sigma_i(t_i^0) - t_i)$$

$$+ \sum_{i=1}^{n} f_{t_i}^{\Delta_i}(t^0)(t_i^0 - t_i) + \sum_{i=1}^{n} \alpha_i(t_i^0 - t_i).$$

Therefore, the function f is completely delta differentiable at t^0. □

Theorem

Assume $f : \Lambda^n \to \mathbb{R}$ *is a continuous function that has partial derivatives* $f_{t_i}^{\Delta_i}(t)$, $i \in \{1, 2, \ldots, n\}$, *in a union of some neighbourhoods of the points* t^0 *and*

$$\left(\sigma_1(t_1^0), \ldots, \sigma_i(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0)\right).$$

If these derivatives are continuous at the point t^0 *and, moreover,*

$$f_{t_i}^{\Delta_i}(\sigma_1(t_1^0), \ldots, \sigma_{i-1}(t_{i-1}^0), t_i, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0))$$

is continuous at $t_i = t_i^0$, *then* f *is* σ_i*-completely delta differentiable at* t^0.

Proof.

From Theorem 4, it follows that the function f is completely delta differentiable at t^0. Now, we consider the difference

$$f(\sigma_1(t_1^0), \sigma_2(t_2^0), \ldots, \sigma_n(t_n^0)) - f(t^0)$$

$$= f(t_1^0, t_2^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0) - f(t_1^0, t_2^0, \ldots, t_{i-1}^0, t_i^0, t_{i+1}^0, \ldots, t_n^0)$$

$$- f(t_1^0, t_2^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0) + f(\sigma_1(t_1^0), \sigma_2(t_2^0), \ldots, \sigma_n(t_n^0)). \tag{7}$$

□

Proof.

Using the definition of the partial delta derivative, we have

$$f(t_1^0, t_2^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0) - f(t_1^0, t_2^0, \ldots, t_{i-1}^0, t_i^0, t_{i+1}^0, \ldots, t_n^0)$$

$$= f_{t_i}^{\Delta_i}(t^0)(\sigma_i(t_i^0) - t_i^0). \quad (8)$$

Also,

$$f(\sigma_1(t_1^0), \sigma_2(t_2^0), \ldots, \sigma_n(t_n^0)) - f(t_1^0, t_2^0, \ldots$$

□

Proof.

$$
\begin{aligned}
= \quad & f(\sigma_1(t_1^0), \sigma_2(t_2^0), \ldots, \sigma_{i-1}(t_{i-1}^0), \sigma_i(t_i^0), \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0)) \\
& -f(\sigma_1(t_1^0), \sigma_2(t_2^0) \ldots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0)) \\
& +f(\sigma_1(t_1^0), \sigma_2(t_2^0), \ldots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0)) \\
& -f(t_1^0, t_2^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0)
\end{aligned}
$$

□

Proof.

$$= f_{t_i}^{\Delta_i}(\sigma_1(t_1^0), \sigma_2(t_2^0), \ldots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0))(\sigma_i(t_i^0) -$$

$$+f(\sigma_1(t_1^0), \ldots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0))$$

$$-f(t_1^0, \sigma_2(t_2^0), \ldots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0))$$

$$+f(t_1^0, \sigma_2(t_2^0), \ldots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0))$$

$$- \cdots$$

$\square$

Proof.

$$+f(t_1^0, t_2^0, \ldots, t_{i-1}^0, t_i^0, t_{i+1}^0, \ldots, t_n^0)$$

$$-f(t_1^0, t_2^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0)$$

$$= \quad f_{t_1}^{\Delta_1}(\sigma_1(t_1^0), \sigma_2(t_2^0), \ldots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \ldots, \sigma_n(t_n^0))(\sigma_i(t_i^0) -$$

$\square$

Proof.

$$+f_{t_1}^{\Delta_1}(t_1^0, \sigma_2(t_2^0), \dots, \sigma_{i-1}(t_{i-1}^0), t_i^0, \sigma_{i+1}(t_{i+1}^0), \dots, \sigma_n(t_n^0))(\sigma_1(t_1^0) - t_1^0)$$

$$+\cdots$$

$$+f_{t_n}^{\Delta_n}(t^0)(\sigma_n(t_n^0) - t_n^0) - f_{t_i}^{\Delta_i}(t^0)(\sigma_i(t_i^0) - t_i^0).$$

Hence, using (7) and (8), we conclude that f is σ_i-completely delta differentiable at t^0. $\square$

Theorem

Suppose the function $f : \Lambda^n \to \mathbb{R}$ *has mixed partial derivatives*

$$f^{\Delta_i \Delta_j}_{t_i t_j}(t) \quad and \quad f^{\Delta_j \Delta_i}_{t_j t_i}(t)$$

in some neighbourhood of the point $t^0 \in \Lambda^{\kappa_i \kappa_j n}_{ij}$. *If these derivatives are continuous at the point* t^0, *then*

$$f^{\Delta_i \Delta_j}_{t_i t_j}(t^0) = f^{\Delta_j \Delta_i}_{t_j t_i}(t^0).$$

Here, $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$.

Proof.

For convenience, we suppose that $i < j$. Let

$$\begin{aligned}\Phi(t) &= f(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n) \\ &\quad - f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n) \\ &\quad - f(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n) + f(t)\end{aligned}$$

and □

Proof.

$$\phi(t_i) = f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n) - f(t).$$

Then

$$\begin{aligned}\phi(\sigma_i(t_i^0)) &= f(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n) \\ &\quad - f(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_n).\end{aligned}$$

Therefore, □

Proof.

$$\Phi(t) = \phi(\sigma_i(t_i^0)) - \phi(t_i).$$

Hence, using the mean value theorem, there exist points $\xi_i^1, \xi_i^2 \in [\alpha_i, \beta_i)$, where

$$\alpha_i = \min\{t_i, \sigma_i(t_i^0)\}, \quad \beta_i = \max\{t_i, \sigma_i(t_i^0)\},$$

such that

$$\phi^{\Delta}(\xi_i^1)(\sigma_i(t_i^0) - t_i) \leq \phi(\sigma_i(t_i^0)) - \phi(t_i) \leq \phi^{\Delta}(\xi_i^2)(\sigma_i(t_i^0) - t_i),$$

i.e., $\square$

Proof.

$$
\begin{aligned}
&\Big(f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^1,t_{i+1},\ldots,t_{j-1},\sigma_j(t_j^0),t_{j+1},\ldots,t_n)\\
&\quad -f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^1,t_{i+1},\ldots,t_n)\Big)(\sigma_i(t_i^0)-t_i)\\
&\le \Phi(t)\\
&\le \Big(f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^2,t_{i+1},\ldots,t_{j-1},\sigma_j(t_j^0),t_{j+1},\ldots,t_n)\\
&\quad -f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^2,t_{i+1},\ldots,t_n)\Big)(\sigma_i(t_i^0)-t_i),
\end{aligned}
$$

whereupon □

Proof.

$$f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^1,t_{i+1},\ldots,t_{j-1},\sigma_j(t_j^0),t_{j+1},\ldots,t_n)$$

$$-f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^1,t_{i+1},\ldots,t_n)$$

$$\leq \frac{\Phi(t)}{\sigma_i(t_i^0)-t_i}$$

$$\leq f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^2,t_{i+1},\ldots,t_{j-1},\sigma_j(t_j^0),t_{j+1},\ldots,t_n)$$

$$-f_{t_i}^{\Delta_i}(t_1,\ldots,t_{i-1},\xi_i^2,t_{i+1},\ldots,t_n)$$

if $\sigma_i(t_i^0) > t_i$, and $\square$

Proof.

$$
\begin{aligned}
& f_{t_i}^{\Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^2, t_{i+1}, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n) \\
& \qquad -f_{t_i}^{\Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^2, t_{i+1}, \ldots, t_n) \\
& \quad \leq \frac{\Phi(t)}{\sigma_i(t_i^0) - t_i} \\
& \quad \leq f_{t_i}^{\Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^1, t_{i+1}, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n) \\
& \qquad -f_{t_i}^{\Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^2, t_{i+1}, \ldots, t_n)
\end{aligned}
$$

if $\sigma_i(t_i^0) < t_i$. $\square$

Proof.

Without loss of generality, we assume that $\sigma_i(t_i^0) > t_i$. Again, we apply the mean value theorem and find that there exist $\xi_j^1, \xi_j^2 \in [\alpha_j, \beta_j)$, where

$$\alpha_j = \min\{\sigma_j(t_j^0), t_j\}, \quad \beta_j = \max\{\sigma_j(t_j^0), t_j\},$$

such that

$$f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^1, t_{i+1}, \ldots, t_{j-1}, \xi_j^1, t_{j+1}, \ldots, t_n)(\sigma_j(t_j^0) - t_j)$$

$$\leq \frac{\Phi(t)}{\sigma_i(t_i^0) - t_i}$$

$$\leq f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^2, t_{i+1}, \ldots, t_{j-1}, \xi_j^2, t_{j+1}, \ldots, t_n)(\sigma_j(t_j^0) - t_j),$$

from where □

Proof.

$$f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \dots, t_{i-1}, \xi_i^1, t_{i+1}, \dots, t_{j-1}, \xi_j^1, t_{j+1}, \dots, t_n)$$

$$\leq \frac{\Phi(t)}{(\sigma_j(t_j^0) - t_j)(\sigma_i(t_i^0) - t_i)}$$

$$\leq f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \dots, t_{i-1}, \xi_i^2, t_{i+1}, \dots, t_{j-1}, \xi_j^2, t_{j+1}, \dots, t_n)$$

if $\sigma_j(t_j^0) > t_j$, and

$$f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \dots, t_{i-1}, \xi_i^2, t_{i+1}, \dots, t_{j-1}, \xi_j^2, t_{j+1}, \dots, t_n)$$

□

Proof.

$$\leq \frac{\Phi(t)}{(\sigma_j(t_j^0) - t_j)(\sigma_i(t_i^0) - t_i)}$$

$$\leq f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^1, t_{i+1}, \ldots, t_{j-1}, \xi_j^1, t_{j+1}, \ldots, t_n)$$

if $\sigma_j(t_j^0) < t_j$. Without loss of generality, we assume that $\sigma_j(t_j^0) > t_j$. Since $f_{t_j t_i}^{\Delta_j \Delta_i}$ is continuous at t^0, we get

$$\lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^1, t_{i+1}, \ldots, t_{j-1}, \xi_j^1, t_{j+1}, \ldots, t_n) = f_{t_j t_i}^{\Delta_j \Delta_i}(t^0$$

$\square$

Proof.

$$\leq \lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} \frac{\Phi(t)}{(\sigma_j(t_j^0) - t_j)(\sigma_i(t_i^0) - t_i)}$$

$$\leq \lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} f_{t_j t_i}^{\Delta_j \Delta_i}(t_1, \ldots, t_{i-1}, \xi_i^2, t_{i+1}, \ldots, t_{j-1}, \xi_j^2, t_{j+1}, \ldots, t_n)$$

$$= f_{t_j t_i}^{\Delta_j \Delta_i}(t^0),$$

i.e.,

$$f_{t_j t_i}^{\Delta_j \Delta_i}(t^0) = \lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} \frac{\Phi(t)}{(\sigma_j(t_i^0) - t_j)(\sigma_i(t_i^0) - t_i)}. \tag{9}$$

Proof.

Let

$$\psi(t_j) = f(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n) - f(t).$$

Thus,

$$\begin{aligned} \psi(\sigma_j(t_j^0)) &= f(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n) \\ &\quad -f(t_1, \ldots, t_{j-1}, \sigma_j(t_j^0), t_{j+1}, \ldots, t_n). \end{aligned}$$

Hence, $\square$

Proof.

$$\Phi(t) = \psi(\sigma_j(t_j^0)) - \psi(t_j).$$

From the mean value theorem, it follows that there exist $\eta_j^1, \eta_j^2 \in [\alpha_j, \beta_j)$ such that

$$\psi^\Delta(\eta_j^1)(\sigma_j(t_j^0) - t_j) \leq \psi(\sigma_j(t_j^0)) - \psi(t_j) \leq \psi^\Delta(\eta_j^2)(\sigma_j(t_j^0) - t_j),$$

i.e.,

$$\Big(f_{t_j}^{\Delta_j}(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, \eta_j^1, t_{j+1}, \ldots, t_n)$$

$$-f_{t_j}^{\Delta_j}(t_1, \ldots, t_{j-1}, \eta_j^1, t_{j+1}, \ldots, t_n)\Big)(\sigma_j(t_j^0) - t_j)$$

□

Proof.

$$\leq \Phi(t)$$

$$\leq \Big(f_{t_j}^{\Delta_j}(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, \eta_j^2, t_{j+1}, \ldots, t_n)$$

$$-f_{t_j}^{\Delta_j}(t_1, \ldots, t_{j-1}, \eta_j^2, t_{j+1}, \ldots, t_n)\Big)(\sigma_j(t_j^0) - t_j),$$

whereupon

$$f_{t_j}^{\Delta_j}(t_1, \ldots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \ldots, t_{j-1}, \eta_j^1, t_{j+1}, \ldots, t_n)$$

$\square$

Proof.

$$-f_{t_j}^{\Delta_j}(t_1, \dots, t_{j-1}, \eta_j^1, t_{j+1}, \dots, t_n)$$

$$\leq \quad \frac{\Phi(t)}{(\sigma_j(t_j^0) - t_j)}$$

$$\leq \quad \Big(f_{t_j}^{\Delta_j}(t_1, \dots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \dots, t_{j-1}, \eta_j^2, t_{j+1}, \dots, t_n)$$

$$-f_{t_j}^{\Delta_j}(t_1, \dots, t_{j-1}, \eta_j^2, t_{j+1}, \dots, t_n)\Big)$$

if $\sigma_j(t_j^0) > t_j$, and

$$f_{t_j}^{\Delta_j}(t_1, \dots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \dots, t_{j-1}, \eta_j^2, t_{j+1}, \dots, t_n)$$

$\square$

Proof.

$$
\begin{aligned}
& -f_{t_j}^{\Delta_j}(t_1, \dots, t_{j-1}, \eta_j^2, t_{j+1}, \dots, t_n) \\
\leq\ & \frac{\Phi(t)}{(\sigma_j(t_j^0) - t_j)} \\
\leq\ & \Big(f_{t_j}^{\Delta_j}(t_1, \dots, t_{i-1}, \sigma_i(t_i^0), t_{i+1}, \dots, t_{j-1}, \eta_j^1, t_{j+1}, \dots, t_n) \\
& -f_{t_j}^{\Delta_j}(t_1, \dots, t_{j-1}, \eta_j^1, t_{j+1}, \dots, t_n) \Big)
\end{aligned}
$$

if $\sigma_j(t_j^0) < t_j$. Without loss of generality, we suppose that $\sigma_j(t_j^0) > t_j$. Again, we apply the mean value theorem and get □

Proof.

$\eta_i^1, \eta_i^2 \in [\alpha_i, \beta_i)$ such that

$$f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^1, t_{i+1}, \ldots, t_{j-1}, \eta_j^1, t_{j+1}, \ldots, t_n)(\sigma_i(t_i^0) - t_i)$$

$$\begin{aligned} &\leq \frac{\Phi(t)}{(\sigma_j(t_j^0) - t_j)} \\ &\leq f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^2, t_{i+1}, \ldots, t_{j-1}, \eta_j^2, t_{j+1}, \ldots, t_n)(\sigma_i(t_i^0) - t_i), \end{aligned}$$

from where

$$f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^1, t_{i+1}, \ldots, t_{j-1}, \eta_j^1, t_{j+1}, \ldots, t_n)$$

$\square$

Proof.

$$\leq \frac{\Phi(t)}{(\sigma_i(t_i^0) - t_i)(\sigma_j(t_j^0) - t_j)}$$

$$\leq f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^2, t_{i+1}, \ldots, t_{j-1}, \eta_j^2, t_{j+1}, \ldots, t_n)$$

if $\sigma_i(t_i^0) > t_i$, and

$$f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^2, t_{i+1}, \ldots, t_{j-1}, \eta_j^2, t_{j+1}, \ldots, t_n)$$

$$\leq \frac{\Phi(t)}{(\sigma_i(t_i^0) - t_i)(\sigma_j(t_j^0) - t_j)}$$

$$\leq f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^1, t_{i+1}, \ldots, t_{j-1}, \eta_j^1, t_{j+1}, \ldots, t_n)$$

if $\sigma_i(t_i^0) < t_i$. □

Proof.

Without loss of generality, we assume that $\sigma_i(t_i^0) > t_i$. Then

$$\lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^1, t_{i+1}, \ldots, t_{j-1}, \eta_j^1, t_{j+1}, \ldots, t_n) = f_{t_i t_j}^{\Delta_i \Delta_j}(t^0$$

$$\leq \lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} \frac{\Phi(t)}{(\sigma_i(t_i^0) - t_i)(\sigma_j(t_j^0) - t_j)}$$

$$\leq \lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} f_{t_i t_j}^{\Delta_i \Delta_j}(t_1, \ldots, t_{i-1}, \eta_i^2, t_{i+1}, \ldots, t_{j-1}, \eta_j^2, t_{j+1}, \ldots, t_n) = f_{t_i}^{\Delta}$$

i.e., □

Proof.

$$f_{t_i t_j}^{\Delta_i \Delta_j}(t^0) = \lim_{\substack{t \to t^0 \\ t_i \neq \sigma_i(t_i^0) \\ t_j \neq \sigma_j(t_j^0)}} \frac{\Phi(t)}{(\sigma_i(t_i^0) - t_i)(\sigma_j(t_j^0) - t_j)}.$$

From the last equality and from (9), we find

$$f_{t_i t_j}^{\Delta_i \Delta_j}(t^0) = f_{t_j t_i}^{\Delta_j \Delta_i}(t^0),$$

completing the proof. □

Example

Let $\Lambda^2 = \mathbb{N} \times \mathbb{Z}$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = t_1^2 t_2 + t_1 t_2^2, \quad t \in \Lambda^2.$$

Here,

$$\mathbb{T}_1 = \mathbb{N}, \quad \mathbb{T}_2 = \mathbb{Z}$$

and

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

Hence,

$$f_{t_1}^{\Delta_1}(t) \;\; = \;\; (\sigma_1(t_1) + t_1)t_2 + t_2^2$$

Example

$$
\begin{aligned}
&= (2t_1 + 1)t_2 + t_2^2, \\
f_{t_1 t_2}^{\Delta_1 \Delta_2}(t) &= 2t_1 + 1 + \sigma_2(t_2) + t_2 \\
&= 2t_1 + 2t_2 + 2, \\
f_{t_2}^{\Delta_2}(t) &= t_1^2 + (\sigma_2(t_2) + t_2)t_1 \\
&= t_1^2 + (2t_2 + 1)t_1,
\end{aligned}
$$

Example

$$\begin{aligned} f_{t_2 t_1}^{\Delta_2 \Delta_1}(t) &= \sigma_1(t_1) + t_1 + 2t_2 + 1 \\ &= 2t_1 + 2t_2 + 2, \quad t \in \Lambda^2. \end{aligned}$$

Example

Consequently,

$$f_{t_1 t_2}^{\Delta_1 \Delta_2}(t) = f_{t_2 t_1}^{\Delta_2 \Delta_1}(t), \quad t \in \Lambda^2.$$

Example

Let $\Lambda^2 = 2^{\mathbb{N}} \times \mathbb{N}$ and define $f : \Lambda^2 \to \mathbb{R}$ by

$$f(t) = (\log t_2)t_1 + \sin t_1, \quad t \in \Lambda^2.$$

Here,

$$\mathbb{T}_1 = 2^{\mathbb{N}}, \quad \mathbb{T}_2 = \mathbb{N}$$

and

$$\sigma_1(t_1) = 2t_1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

Hence,

$$f_{t_1}^{\Delta_1}(t) = \log t_2 + \frac{\sin(\sigma_1(t_1)) - \sin t_1}{\sigma_1(t_1) - t_1}$$

Example

$$
\begin{aligned}
&= \log t_2 + \frac{\sin(2t_1) - \sin(t_1)}{2t_1 - t_1} \\
&= \log t_2 + 2\frac{\sin\frac{t_1}{2}\cos\frac{3t_1}{2}}{t_1}, \\
f_{t_1 t_2}^{\Delta_1\Delta_2}(t) &= \frac{\log(\sigma_2(t_2)) - \log(t_2)}{\sigma_2(t_2) - t_2} \\
&= \frac{\log(t_2+1) - \log(t_2)}{t_2 + 1 - t_2}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \log \frac{t_2+1}{t_2}, \\
f_{t_2}^{\Delta_2}(t) &= \frac{\log(\sigma_2(t_2)) - \log t_2}{\sigma_2(t_2) - t_2} t_1 \\
&= \frac{\log(t_2+1) - \log(t_2)}{t_2+1-t_2} t_1 \\
&= \log \frac{t_2+1}{t_2} t_1,
\end{aligned}
$$

Example

$$f_{t_2 t_1}^{\Delta_2 \Delta_1}(t) \quad = \quad \log \frac{t_2 + 1}{t_2}, \quad t \in \Lambda^2.$$

Consequently,

$$f_{t_1 t_2}^{\Delta_1 \Delta_2}(t) = f_{t_2 t_1}^{\Delta_2 \Delta_1}(t), \quad t \in \Lambda^2.$$

Example

Let

$$\mathbb{T}_1 = [0,2] \times \{4\}, \quad \mathbb{T}_2 = [0,2],$$

where $[0,2]$ is real number interval. Define $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ by

$$f(t) = t_1 + t_1 t_2, \quad t \in [0,2] \times [0,2]$$

and

$$f(4, t_2) = t_2^2, \quad t_2 \in [0,2].$$

This yields

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(2, t_2) &= \frac{f(\sigma_1(2), t_2) - f(2, t_2)}{\sigma_1(2) - 2} \\
&= \frac{f(4, t_2) - f(2, t_2)}{2}
\end{aligned}
$$

Example

$$
\begin{aligned}
&= \frac{t_2^2 - 2 - 2t_2}{2} \\
&= \frac{1}{2}(t_2^2 - 2t_2) - 1, \\
f_{t_1 t_2}^{\Delta_1 \Delta_2}(2, t_2) &= \frac{1}{2}(\sigma_2(t_2) + t_2 - 2) \\
&= \frac{1}{2}(2t_2 - 2) \\
&= t_2 - 1, \\
f_{t_2}^{\Delta_2}(2, t_2) &= 2,
\end{aligned}
$$

Example

$$f_{t_2}^{\Delta_2}(4, t_2) = 2t_2,$$

$$f_{t_2 t_1}^{\Delta_2 \Delta_1}(2, t_2) = \frac{f_{t_2}^{\Delta_2}(4, t_2) - f_{t_2}^{\Delta_2}(2, t_2)}{2}$$

$$= \frac{2t_2 - 2}{2}$$

Example

$$= \quad t_2 - 1, \quad t_2 \in [0, 2].$$

Therefore,

$$f_{t_1 t_2}^{\Delta_1 \Delta_2}(2, t_2) = f_{t_2 t_1}^{\Delta_2 \Delta_1}(2, t_2), \quad t_2 \in [0, 2].$$