Time Scales Analysis
Lecture 27
Delta Directional Derivative

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Let $\mathbb{T}$ be a time scale with forward jump operator σ and delta operator Δ. Let $x^0 \in \mathbb{T}$, $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ be a unit vector, and $t^0 = (t_1^0, t_2^0, \ldots, t_n^0)$ be a fixed point in $\mathbb{R}^n$. We set

$$\mathbb{T}_i = \{t_i = t_i^0 + (\xi - x^0)w_i : \xi \in \mathbb{T}\}, \quad i \in \{1, 2, \ldots, n\}.$$

We note that $\mathbb{T}_i$, $i \in \{1, 2, \ldots, n\}$, are time scales and $t_i^0 \in \mathbb{T}_i$, $i \in \{1, 2, \ldots, n\}$. Denote the forward jump operator of $\mathbb{T}_i$ by σ_i and the delta operator by Δ_i.

Definition

Let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n \to \mathbb{R}$ be given. The *directional delta derivative* of the function f at the point t^0 in the direction of the vector w is defined as the number

$$F^{\Delta}(x^0) = \frac{\partial f(t^0)}{\Delta w}.$$

Remark

*If f is σ_i-completely delta differentiable at t^0, then, using Theorem **??**, we have*

$$\begin{aligned} F^{\Delta}(x^0) &= f_{t_i}^{\Delta_i}(t^0)w_i \\ &\quad + f_{t_{i-1}}^{\Delta_{i-1}}(t_1^0, \ldots, t_{i-1}^0, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0)w_{i-1} \\ &\quad + \cdots \\ &\quad + f_{t_1}^{\Delta_1}(t_1^0, \sigma_2(t_2^0), \ldots, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0)w_1 \end{aligned}$$

Remark

$$+f^{\Delta_{i+1}}_{t_{i+1}}(\sigma_1(t_1^0), \ldots, \sigma_i(t_i^0), t_{i+1}^0, \ldots, t_n^0)w_{i+1}$$

$$+\cdots$$

$$+f^{\Delta_n}_{t_n}(\sigma_1(t_1^0), \ldots, \sigma_{n-1}(t_{n-1}^0), t_n^0)w_n.$$

Example

Let $\mathbb{T} = \mathbb{N}$, $x^0 = 1$, $w = (1, 0) \in \mathbb{R}^2$. We note that w is a unit vector in $\mathbb{R}^2$. Let $t^0 = (1, 1)$. We set

$$\begin{aligned}
\mathbb{T}_1 &= \{t_1 = 1 + (\xi - 1)1 : \xi \in \mathbb{T}\} = \mathbb{N}, \\
\mathbb{T}_2 &= \{t_2 = 1 + (\xi - 1)0 : \xi \in \mathbb{T}\} = \{1\}.
\end{aligned}$$

Example

We consider $f(t) = t_1^2 + t_2^2$, $t \in \mathbb{T}_1 \times \mathbb{T}_2$. We have that f is σ_1-completely delta differentiable in $\mathbb{T}_1 \times \mathbb{T}_2$ and

$$\begin{aligned} F^{\Delta}(1) &= \frac{\partial f}{\Delta w}(1,1) \\ &= f_{t_1}^{\Delta_1}(1,1)1 \\ &= (\sigma_1(t_1) + t_1)\Big|_{t_1=1} \\ &= (2t_1 + 1)\Big|_{t_1=1} = 3. \end{aligned}$$

Example

Let $\mathbb{T} = \mathbb{Z}$, $x^0 = 0$, $w = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right) \in \mathbb{R}^2$. We note that w is a unit vector in $\mathbb{R}^2$. Let $t^0 = (t_1^0, t_2^0) = (0, 0)$ be a fixed point in $\mathbb{R}^2$. We set

$$\begin{aligned} \mathbb{T}_1 &= \left\{ t_1 = \frac{2}{\sqrt{13}}\xi : \ \xi \in \mathbb{Z} \right\} = \frac{2}{\sqrt{13}}\mathbb{Z}, \\ \mathbb{T}_2 &= \left\{ t_2 = \frac{3}{\sqrt{13}}\xi : \ \xi \in \mathbb{Z} \right\} = \frac{3}{\sqrt{13}}\mathbb{Z} \end{aligned}$$

so that

Example

$$\sigma_1(t_1) = t_1 + \frac{2}{\sqrt{13}}, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + \frac{3}{\sqrt{13}}, \quad t_2 \in \mathbb{T}_2.$$

We consider

$$f(t) = t_1^3 + 3t_1^2 t_2 + t_2^2, \quad t = (t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

We note that f is σ_1-completely delta differentiable in $\mathbb{T}_1 \times \mathbb{T}_2$. Hence,

$$\frac{\partial f(0,0)}{\Delta w} = f_{t_1}^{\Delta_1}(0,0)\frac{2}{\sqrt{13}} + f_{t_2}^{\Delta_2}(\sigma_1(0),0)\frac{3}{\sqrt{13}},$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(t) &= (\sigma_1(t_1))^2 + t_1\sigma_1(t_1) + t_1^2 \\
&\quad +3(\sigma_1(t_1) + t_1)t_2 \\
&= \left(t_1 + \frac{2}{\sqrt{13}}\right)^2 + t_1\left(t_1 + \frac{2}{\sqrt{13}}\right) + t_1^2 \\
&\quad +3\left(2t_1 + \frac{2}{\sqrt{13}}\right)t_2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(0) &= \left(\frac{2}{\sqrt{13}}\right)^2 \\
&= i\frac{4}{13}, \\
f_{t_2}^{\Delta_2}(t) &= 3t_1^2 + \sigma_2(t_2) + t_2 \\
&= 3t_1^2 + 2t_2 + \frac{3}{\sqrt{13}},
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_2}^{\Delta_2}(\sigma_1(0),0) &= f_{t_2}^{\Delta_2}\left(\frac{2}{\sqrt{13}},0\right)\\
&= 3\left(\frac{2}{\sqrt{13}}\right)^2+\frac{3}{\sqrt{13}}\\
&= \frac{12}{13}+\frac{3}{\sqrt{13}}.
\end{aligned}
$$

Hence,

Example

$$
\begin{aligned}
\frac{\partial f(0,0)}{\Delta w} &= \frac{4}{13}\cdot\frac{2}{\sqrt{13}} + \frac{3}{\sqrt{13}}\left(\frac{12}{13} + \frac{3}{\sqrt{13}}\right) \\
&= \frac{8}{13\sqrt{13}} + \frac{36}{13\sqrt{13}} + \frac{9}{13} \\
&= \frac{44}{13\sqrt{13}} + \frac{9}{13}.
\end{aligned}
$$

Example

Let $\mathbb{T} = \mathbb{Z}$, $x^0 = 0$, $w = \left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \in \mathbb{R}^2$. We note that w is a unit vector in $\mathbb{R}^2$. Let $t^0 = (0,0)$. We set

$$\begin{aligned}
\mathbb{T}_1 &= \left\{ t_1 = -\frac{\xi}{\sqrt{3}} : \xi \in \mathbb{T} \right\} = -\frac{1}{\sqrt{3}}\mathbb{Z}, \\
\mathbb{T}_2 &= \left\{ t_2 = \frac{2\xi}{\sqrt{3}} : \xi \in \mathbb{T} \right\} = \frac{2}{\sqrt{3}}\mathbb{Z}.
\end{aligned}$$

Example

We consider

$$f(t) = t_1^2 + 2t_1 t_2, \quad t \in \mathbb{T}_1 \times \mathbb{T}_2.$$

We have that f is σ_1-completely delta differentiable in $\mathbb{T}_1 \times \mathbb{T}_2$ and

$$\sigma_1(t_1) = t_1 + \frac{1}{\sqrt{3}}, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + \frac{2}{\sqrt{3}}, \quad t_2 \in \mathbb{T}_2.$$

Hence,

Example

$$
\begin{aligned}
\frac{\partial f(0,0)}{\Delta w} &= f_{t_1}^{\Delta_1}(0,0)\left(-\frac{1}{\sqrt{3}}\right) + f_{t_2}^{\Delta_2}(\sigma_1(0),0)\frac{2}{\sqrt{3}}, \\
f_{t_1}^{\Delta_1}(t) &= \sigma_1(t_1) + t_1 + 2t_2 \\
&= 2t_1 + \frac{1}{\sqrt{3}} + 2t_2,
\end{aligned}
$$

Example

$$
\begin{aligned}
f_{t_1}^{\Delta_1}(0,0) &= \frac{1}{\sqrt{3}}, \\
f_{t_2}^{\Delta_2}(t) &= 2t_1, \\
f_{t_2}^{\Delta_2}(\sigma_1(0),0) &= f_{t_2}^{\Delta_2}\left(\frac{1}{\sqrt{3}},0\right) \\
&= \frac{2}{\sqrt{3}},
\end{aligned}
$$

Example

$$\begin{aligned}\frac{\partial f(0,0)}{\Delta w} &= \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\\ &= -\frac{1}{3}+\frac{4}{3}\\ &= 1.\end{aligned}$$

Let $\mathbb{T}_i$, $1 \leq i \leq n-1$, be time scales. We consider the equation

$$f(t_1, t_2, \ldots, t_{n-1}, x) = 0 \tag{1}$$

for $(t_1, t_2, \ldots, t_{n-1}, x) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_{n-1} \times \mathbb{R}$.

Theorem

Suppose an equation (1) *satisfies the following conditions.*
i The function f is defined in a neighbourhood U of the point $(t_1^0, t_2^0, \ldots, t_{n-1}^0, x) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa \times \cdots \times \mathbb{T}_{n-1}^k \times \mathbb{R}$ and is continuous in U together with its partial derivatives $f_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{n-1}, x)$, $i \in \{1, \ldots, n-1\}$, and $\frac{\partial f}{\partial x}(t_1, \ldots, t_{n-1}, x)$.

Theorem

ii $f(t_1^0, t_2^0, \ldots, t_{n-1}^0, x^0) = 0.$

iii $\frac{\partial f}{\partial x}(t_1^0, t_2^0, \ldots, t_{n-1}^0, x^0) \neq 0.$ *Then the following statements are true.*

a There is a "rectangle"

$$\mathcal{N} = \Big\{ (t_1^0, t_2^0, \ldots, t_{n-1}^0, x) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa \times \cdots \times \mathbb{T}_{n-1}^k \times \mathbb{R} : \\ |t_i - t_i^0| < \delta_i, \quad i = 1, 2, \ldots, n-1, \quad |x - x^0| < \delta' \Big\} \tag{2}$$

Theorem

belonging to U such that $\mathcal{M} \cap \mathcal{N}$ is described by a uniquely determined single-valued function

$$x = \psi(t_1, t_2, \ldots, t_{n-1}) \quad \textit{for} \quad (t_1^0, t_2^0, \ldots, t_{n-1}^0, x) \in \mathcal{N}^0,$$

where

Theorem

$$\begin{aligned}\mathcal{M} &= \Big\{(t_1, t_2, \ldots, t_{n-1}, x) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_{n-1} \times \mathbb{R} : \\ & f(t_1, t_2, \ldots, t_{n-1}, x) = 0\Big\}, \\ \mathcal{N}^0 &= \Big\{(t_1, t_2, \ldots, t_{n-1}) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_{n-1} : \\ & |t_i - t_i^0| < \delta_i, \quad i = 1, 2, \ldots, n-1\Big\}.\end{aligned}$$

Theorem

b $x^0 = \psi(t_1^0, t_2^0, \ldots, t_{n-1}^0)$.

c The function $\psi(t_1, t_2, \ldots, t_{n-1})$ *is continuous in* $\mathcal{N}^0$.

d The function $\psi(t_1, t_2, \ldots, t_{n-1})$ *has partial delta derivatives*

$$\psi_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{n-1}), \quad i = 1, 2, \ldots, n-1 \quad on \quad \mathcal{N}^0.$$

Proof.

a Without loss of generality, we can suppose that U is an open "rectangle" of the form

$$U = \Big\{(t_1, t_2, \ldots, t_{n-1}, x) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_{n-1} \times \mathbb{R} :$$

$$|t_i - t_i^0| < \tilde{a}_i, \quad i = 1, 2, \ldots, n-1, \quad |x - x^0| < b\Big\},$$

and we can assume that

$$\frac{\partial}{\partial x} f(t_1^0, t_2^0, \ldots, t_{n-1}^0, x) > 0.$$

Because $\frac{\partial}{\partial x} f(t_1, t_2, \ldots, t_{n-1}, x)$ is continuous on U, we also have □

Proof.

$$\frac{\partial}{\partial x} f(t_1, t_2, \ldots, t_{n-1}, x) > 0 \tag{3}$$

in a small neighbourhood $U_1 \subset U$ of the point $(t_1^0, t_2^0, \ldots, t_{n-1}^0, x^0)$ and

$$\begin{aligned} U_1 &= \Big\{(t_1, t_2, \ldots, t_{n-1}, x) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_{n-1} \times \mathbb{R} : \\ &\quad |t_i - t_i^0| < a_i, \quad i = 1, 2, \ldots, n-1, \quad |x - x^0| < b_1 \Big\}. \end{aligned}$$

$\square$

Proof.

Since f is continuous in U, we have that f is continuous in U_1. Also, the function $f(t_1^0, t_2^0, \ldots, t_{n-1}^0, x)$ of the single variable x is continuous on the closed interval $[x^0 - b_1, x^0 + b_1]$. Hence, using (3), we have that $f(t_1^0, t_2^0, \ldots, t_{n-1}^0, x)$ is strongly increasing in $[x^0 - b_1, x^0 + b_1]$ and

$$f(t_1^0, t_2^0, \ldots, t_{n-1}^0, x^0) = 0.$$

Therefore,

$$f(t_1^0, t_2^0, \ldots, t_{n-1}^0, x_0 - b_1) < 0 \quad \text{and} \quad f(t_1^0, t_2^0, \ldots, t_{n-1}^0, x^0 + b_1) > 0. \quad (4)$$

□

Proof.

By the continuity of f, there is a sufficiently small number $\delta > 0$ with $\delta < \min\{a_1, a_2, \ldots, a_{n-1}\}$ such that (4) holds for all

$$\begin{aligned}(t_1, t_2, \ldots, t_{n-1}) \in \mathcal{N}^0 \quad &= \quad \Big\{(t_1, t_2, \ldots, t_{n-1}) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_{n-1} : \\ &\qquad |t_i - t_i^0| < \delta, \quad i = 1, 2, \ldots, n-1\Big\}.\end{aligned}$$

Now, we choose an arbitrary $(t_1, t_2, \ldots, t_{n-1}) \in \mathcal{N}^0$, fix it temporarily, and consider the function $f(t_1, t_2, \ldots, t_{n-1}, x)$ on the real variable x in the interval $[x^0 - b_1, x^0 + b_1] \subset \mathbb{R}$. $\square$

Proof.

This function is continuous, strictly increasing, and assumes values of the opposite signs at the end points of this interval. Therefore, there is a single value $x \in (x^0 - b_1, x^0 + b_1)$, denoted by

$$x = \psi(t_1, t_2, \ldots, t_{n-1}),$$

for which

$$f(t_1, t_2, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{n-1})) = 0.$$

Thus, letting $\delta' = b_1$, $\delta_i = \delta$, we see that in the neighbourhood $\mathcal{N}$ of the point

$$(t_1^0, t_2^0, \ldots, t_{n-1}^0, x^0),$$

defined by (2), (1) determines x as a unique function of $t_1, t_2, \ldots, t_{n-1}$: $\square$

Proof.

$$x = \psi(t_1, t_2, \ldots, t_{n-1}).$$

b From the condition ii, it follows that

$$x^0 = \psi(t_1^0, t_2^0, \ldots, t_{n-1}^0).$$

c We will show that the function ψ is a continuous function in $\mathcal{N}^0$. To do so, it is enough to prove that it is continuous at the point $(t_1^0, t_2^0, \ldots, t_{n-1}^0)$. Let $\varepsilon' \in (0, \delta')$ be arbitrarily chosen. There exists $\varepsilon \in (0, \delta)$ such that for the rectangle

Proof.

$$\mathcal{N}_* = \Big\{(t_1, t_2, \ldots, t_{n-1}, x) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_{n-1} \times \mathbb{R} :$$

$$|t_i - t_i^0| < \varepsilon, \quad i = 1, 2, \ldots, n-1, \quad |x - x^0| < \varepsilon' \Big\},$$

there exists a function $x = \psi_*(t_1, t_2, \ldots, t_{n-1})$ for

$$(t_1, t_2, \ldots, t_{n-1}) \in \mathcal{N}_*^* = \Big\{(t_1, t_2, \ldots, t_{n-1}) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_{n-1} :$$

$$|t_i - t_i^0| < \varepsilon, \quad i = 1, 2, \ldots, n-1 \Big\},$$

□

Proof.

which describes the set $\mathcal{M} \cap \mathcal{N}_*$. Since $\mathcal{N}_* \subset \mathcal{N}$, we have that

$$\psi(t_1, t_2, \ldots, t_{n-1}) = \psi_*(t_1, t_2, \ldots, t_{n-1})$$

for $(t_1, t_2, \ldots, t_{n-1}) \in \mathcal{N}_*^*$. Hence, for any sufficiently small $\varepsilon' > 0$, there exists $\varepsilon > 0$ such that

$$|\psi(t_1, t_2, \ldots, t_{n-1}) - \psi(t_1^0, t_2^0, \ldots, t_{n-1}^0)| < \varepsilon'$$

provided that $|t_i - t_i^0| < \varepsilon$, $i \in \{1, 2, \ldots, n-1\}$, i.e., the function ψ is continuous at the point $(t_1^0, t_2^0, \ldots, t_{n-1}^0)$. $\square$

Proof.

d We take $(t_1, t_2, \ldots, t_{n-1}) \in \mathcal{N}^0$. *Let* $i \in \{1, 2, \ldots, n-1\}$ *be arbitrarily chosen.*

First case. $\sigma_i(t_i) > t_i$. *Since the function* ψ *is continuous at* $(t_1, t_2, \ldots, t_{n-1})$, *it has a partial delta derivative* $\psi_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{n-1})$ *with*

$$\psi_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{n-1}) = \frac{\psi_i^{\sigma_i}(t) - \psi(t)}{\sigma_i(t_i) - t_i}.$$

$\square$

Proof.

Second case. $t_i = \sigma_i(t_i)$. Let

$$(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_n) \in \mathcal{N}^0, \quad t_i' \neq t_i.$$

We have that

$$\begin{aligned} f(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)) &= 0, \\ f(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_n)) &= 0. \end{aligned}$$

$\square$

Proof.

Thus,

$$f(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}))$$

$$- f(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}))$$

$$= f(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1})) \tag{5}$$

□

Proof.

$$- f(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}))$$

$$= \frac{\partial f}{\partial x}(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \theta)$$

$$\times \left(\psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}) - \psi(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-}\right.$$

where θ is a real number between $\square$

Proof.

$$\psi(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}) \quad \text{and} \quad \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}$$

Also, by the mean value theorem for delta derivatives, we have

$$f_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{i-1}, \xi', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}))$$

$$\times(t_i' - t_i)$$

$$\leq \quad f(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}))$$

$\square$

Proof.

$$-f(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}$$

$$\leq \quad f_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{i-1}, \xi'', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-}$$

$$\times(t_i' - t_i).$$

Hence, using (5), we get

$$f_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{i-1}, \xi', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots$$

$$\times(t_i' - t_i)$$

$\square$

Proof.

$$\leq \frac{\partial}{\partial x} f(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \theta)$$

$$\times \left(\psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}) - \psi(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots,\right.$$

$$\leq f_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{i-1}, \xi'', t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}$$

$$\times (t_i' - t_i).$$

□

Proof.

Dividing the last inequality by

$$\frac{\partial}{\partial x} f(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}, \theta)(t_i - t_i')$$

and using (3) and the continuity of $f_{t_i}^{\Delta_i}$ and $\frac{\partial f}{\partial x}$, we see that

$$\psi_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{n-1})$$

$$= \lim_{t_i' \to t_i} \frac{\psi(t_1, t_2, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_{n-1}) - \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots}{t_i' - t_i}$$

$\square$

Proof.

$$= \quad -\frac{f_{t_i}^{\Delta_i}(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-}}{\frac{\partial f}{\partial x}(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-1}, \psi(t_1, t_2, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_{n-}}$$

The proof is complete. □

Let $\mathbb{T}_i$, $i \in \{1, 2, \ldots, n\}$, be time scales. For $i \in \{1, 2, \ldots, n\}$, let σ_i, ρ_i, and Δ_i denote the forward jump operator, the backward jump operator, and the delta differentiation, respectively, on $\mathbb{T}_i$. Suppose $a_i < b_i$ are points in $\mathbb{T}_i$ and $[a_i, b_i)$ is the half-closed bounded interval in $\mathbb{T}_i$, $i \in \{1, \ldots, n\}$. Let us introduce a "rectangle" in $\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_n$ by

$$
\begin{aligned}
R &= [a_1, b_1) \times [a_2, b_2) \times \ldots [a_n, b_n) \\
&= \{(t_1, t_2, \ldots, t_n) : \; t_i \in [a_i, b_i), \; i = 1, 2, \ldots, n\}.
\end{aligned}
$$

Let

$$
a_i = t_i^0 < t_i^1 < \ldots < t_i^{k_i} = b_i.
$$

Definition

We call the collection of intervals

$$P_i = \left\{[t_i^{j_i-1}, t_i^{j_i}) : \; j_i = 1, \ldots, k_i\right\}, \quad i = 1, 2, \ldots, n,$$

a Δ_i-partition of $[a_i, b_i)$ and denote the set of all Δ_i-partitions of $[a_i, b_i)$ by $P_i([a_i, b_i))$.

Definition

Let

$$R_{j_1 j_2 \ldots j_n} = [t_1^{j_1 - 1}, t_1^{j_1}) \times [t_2^{j_2 - 1}, t_2^{j_2}) \times \ldots \times [t_n^{j_n - 1}, t_n^{j_n}) \qquad (6)$$

$$1 \le j_i \le k_i, \quad i = 1, 2, \ldots, n.$$

We call the collection

$$P = \{R_{j_1 j_2 \ldots j_n} : \ 1 \le j_i \le k_i, \ i = 1, 2, \ldots, n\} \qquad (7)$$

a Δ-*partition* of R,

Definition

generated by the Δ_i-partitions P_i of $[a_i, b_i)$, and we write

$$P = P_1 \times P_2 \times \ldots \times P_n.$$

The set of all Δ-partitions of R is denoted by $\mathcal{P}(R)$. Moreover, for a bounded function $f : R \to \mathbb{R}$, we set

$$\begin{aligned} M &= \sup\{f(t_1, t_2, \ldots, t_n) : (t_1, t_2, \ldots, t_n) \in R\}, \\ m &= \inf\{f(t_1, t_2, \ldots, t_n) : (t_1, t_2, \ldots, t_n) \in R\}, \end{aligned}$$

Definition

$$
\begin{aligned}
M_{j_1 j_2 \ldots j_n} &= \sup\{f(t_1, t_2, \ldots, t_n) : (t_1, t_2, \ldots, t_n) \in R_{j_1 j_2 \ldots j_n}\}, \\
m_{j_1 j_2 \ldots j_n} &= \inf\{f(t_1, t_2, \ldots, t_n) : (t_1, t_2, \ldots, t_n) \in R_{j_1 j_2 \ldots j_n}\}.
\end{aligned}
$$

Definition

The *upper Darboux Δ-sum* $U(f,P)$ and the *lower Darboux Δ-sum* $L(f,P)$ with respect to P are defined by

$$U(f,P) = \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \dots \sum_{j_n=1}^{k_n} M_{j_1 j_2 \dots j_n} (t_1^{j_1} - t_1^{j_1-1})(t_2^{j_2} - t_2^{j_2-1}) \dots (t_n^{j_n} - t_n^{j_n-1})$$

and

$$L(f,P) = \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \dots \sum_{j_n=1}^{k_n} m_{j_1 j_2 \dots j_n} (t_1^{j_1} - t_1^{j_1-1})(t_2^{j_2} - t_2^{j_2-1}) \dots (t_n^{j_n} - t_n^{j_n-1}).$$

Remark

We note that

$$\begin{aligned} U(f,P) &\leq M\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\ldots(t_n^{j_n}-t_n^{j_n-1}) \\ &\leq M(b_1-a_1)(b_2-a_2)\ldots(b_n-a_n) \end{aligned}$$

and

Remark

$$
\begin{aligned}
L(f,P) &\geq m\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\ldots(t_n^{j_n}-t_n^{j_n-1})\\
&\geq M(b_1-a_1)(b_2-a_2)\ldots(b_n-a_n),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
m(b_1-a_1)(b_2-a_2)\ldots(b_n-a_n) &\leq L(f,P)\\
&\leq U(f,P)\\
&\leq M(b_1-a_1)(b_2-a_2)\ldots(b_n-a_n). \qquad (8)
\end{aligned}
$$

Definition

The *upper Darboux Δ-integral* $U(f)$ of f over R and the *lower Darboux Δ-integral* $L(f)$ of f over R are defined by

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}(R)\} \quad \text{and} \quad L(f) = \sup\{L(f, P) : P \in \mathcal{P}(R)\}.$$

From (8), it follows that $U(f)$ and $L(f)$ are finite real numbers.

Definition

We say that f is Δ-*integrable* over R provided $L(f) = U(f)$. In this case, we write

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n$$

for this common value. We call this integral the *Darboux* Δ-*integral*.

Remark

For a given rectangle

$$V = [c_1, d_1) \times [c_2, d_2) \times \ldots \times [c_n, d_n) \subset \Lambda^n,$$

the "area" of V*, i.e.,* $(d_1 - c_1)(d_2 - c_2)\ldots(d_n - c_n)$*, is denoted by* $m(V)$.

Definition

Let $P, Q \in \mathcal{P}(R)$ and

$$P = P_1 \times P_2 \times \ldots \times P_n, \quad Q = Q_1 \times Q_2 \times \ldots \times Q_n,$$

where $P_i, Q_i \in \mathcal{P}([a_i, b_i))$. We say that Q is a *refinement* of P provided Q_i is a refinement of P_i for all $i \in \{1, 2, \ldots, n\}$.

Theorem

Let f be a bounded function on R. If P and Q are Δ-partitions of R and Q is a refinement of P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P),$$

i.e., refining of a partition increases the lower sum and decreases the upper sum.

Proof.

For every Δ-partition Q of R, we have

$$L(f, Q) \leq U(f, Q).$$

Now, we prove

$$L(f, P) \leq L(f, Q) \quad \text{and} \quad U(f, Q) \leq U(f, P).$$

To this end, let

$$P = \{R_1, R_2, \ldots, R_N\}.$$

Because Q is a refinement of P, there exists $k \in \{1, 2, \ldots, N\}$ such that

Proof.

$$Q = \{R_1, R_2, \ldots, R_{k-1}, R_k', R_k'', R_{k+1}, \ldots, R_N\},$$

where

$$R_k = R_k' \cup R_k''.$$

Define

$$\begin{aligned} m_k &= \inf_{(t_1,t_2,\ldots,t_n)\in R_k} f(t_1, t_2, \ldots, t_n), \\ m_k^{(1)} &= \inf_{(t_1,t_2,\ldots,t_n)\in R_k'} f(t_1, t_2, \ldots, t_n), \\ m_k^{(2)} &= \inf_{(t_1,t_2,\ldots,t_n)\in R_k''} f(t_1, t_2, \ldots, t_n), \end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
M_k &= \sup_{(t_1,t_2,\ldots,t_n)\in R_k} f(t_1,t_2,\ldots,t_n),\\
M_k^{(1)} &= \sup_{(t_1,t_2,\ldots,t_n)\in R_k'} f(t_1,t_2,\ldots,t_n),\\
M_k^{(2)} &= \sup_{(t_1,t_2,\ldots,t_n)\in R_k''} f(t_1,t_2,\ldots,t_n).
\end{aligned}
$$

We note that

$$m_k \le m_k^{(1)}, \quad m_k \le m_k^{(2)}$$

and $\square$

Proof.

$$M_k \geq M_k^{(1)}, \quad M_k \geq M_k^{(2)}.$$

Thus,

$$\begin{aligned} L(f,Q) - L(f,P) &= m_k^{(1)} m(R_k') + m_k^{(2)} m(R_k'') - m_k m(R_k) \\ &\geq m_k m(R_k') + m_k m(R_k'') - m_k m(R_k) \\ &= 0 \end{aligned}$$

and □

Proof.

$$
\begin{aligned}
U(f,P) - U(f,Q) &= M_k m(R_k) - M_k^{(1)} m(R_k') - M_k^{(2)} m(R_k'') \\
&= M_k m(R_k') + M_k m(R_k'') - M_k^{(1)} m(R_k') - M_k^{(2)} m(R_k'') \\
&\geq M_k^{(1)} m(R_k') + M_k^{(2)} m(R_k'') - M_k^{(1)} m(R_k') - M_k^{(2)} m(R \\
&= 0,
\end{aligned}
$$

which completes the proof. □