Time Scales Analysis
Lecture 28
Multiple Delta Integrals

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Definition

Suppose

$$P = P_1 \times P_2 \times \ldots \times P_n \quad \text{and} \quad Q = Q_1 \times Q_2 \times \ldots \times Q_n,$$

where $P_i, Q_i \in \mathcal{P}([a_i, b_i))$, $i \in \{1, 2, \ldots, n\}$, are two Δ-partitions of

$$R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n).$$

If P_i is generated by a set

Definition

$$\{t_i^0, t_i^1, \ldots, t_i^{k_i}\}, \quad \text{where} \quad a_i = t_i^0 < t_i^1 < \ldots < t_i^{k_i} = b_i,$$

and Q_i is generated by a set

$$\{\tau_i^0, \tau_i^2, \ldots, \tau_i^{p_i}\}, \quad \text{where} \quad a_i = \tau_i^0 < \tau_i^1 < \ldots < \tau_i^{p_i} = b_i,$$

then, by

Definition

$$P + Q = (P_1 + Q_1) \times (P_2 + Q_2) \times \ldots \times (P_n + Q_n),$$

we denote the Δ-partition of R generated by

$$P_i + Q_i = \{t_i^0, t_i^1, \ldots, t_i^{k_i}\} \cup \{\tau_i^0, \tau_i^1, \ldots, \tau_i^{p_i}\}, \quad i = 1, 2, \ldots, n.$$

Remark

Obviously, $P + Q$ is a refinement of both P and Q.

Theorem

If f is a bounded function on R and if P and Q are any two Δ-partitions of R, then

$$L(f, P) \leq U(f, Q),$$

i.e., every lower sum is less than or equal to every upper sum.

Proof.

Since $P+Q$ is a Δ-partition of R, which is a refinement of both P and Q, applying Theorem **??**, we get

$$L(f,P) \leq L(f,P+Q) \leq U(f,P+Q) \leq U(f,Q),$$

i.e., $L(f,P) \leq U(f,Q)$. □

Theorem

If f is a bounded function on R, then $L(f) \leq U(f)$.

Proof.

Let $P \in \mathcal{P}(R)$. Then

$$L(f, P) \leq U(f, Q) \quad \text{for all} \quad Q \in \mathcal{P}(R).$$

Hence,

$$L(f, P) \leq \inf_{Q \in \mathcal{P}(R)} U(f, Q) = U(f).$$

Because $P \in \mathcal{P}(R)$ was arbitrarily chosen, we conclude that

$$\sup_{P \in \mathcal{P}(R)} L(f, P) \leq U(f),$$

i.e.,

$$L(f) \leq U(f),$$

completing the proof. □

Theorem

If $L(f, P) = U(f, P)$ for some $P \in \mathcal{P}(R)$, then the function f is Δ-integrable over R and

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = L(f, P) = U(f, P).$$

Proof.

The result follows from the inequality

$$L(f, P) \leq L(f) \leq U(f) \leq U(f, P).$$

□

Theorem

A bounded function f on R is Δ-integrable if and only if for each $\varepsilon > 0$, there exists $P \in \mathcal{P}(R)$ such that

$$U(f, P) - L(f, P) < \varepsilon. \tag{1}$$

Proof.

Let f be Δ-integrable on R. Then

$$L(f) = U(f).$$

Using the definitions of $L(f)$ and $U(f)$, it follows that there exist $P, Q \in \mathcal{P}(R)$ such that

$$L(f, P) > L(f) - \frac{\varepsilon}{2} \quad \text{and} \quad U(f, P) < U(f) + \frac{\varepsilon}{2}.$$

Let $S = P + Q$, which is a refinement of both P and Q. Thus, employing Theorem **??**, we find

$$U(f, S) \leq U(f, Q) \quad \text{and} \quad L(f, S) \geq L(f, P)$$

and □

Proof.

$$\begin{aligned} U(f,S) - L(f,S) &\leq U(f,Q) - L(f,P) \\ &< -L(f) + \frac{\varepsilon}{2} + U(f) + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Conversely, suppose that for every $\varepsilon > 0$, the inequality (1) holds for some $P \in \mathcal{P}(R)$. Therefore, □

Proof.

$$
\begin{aligned}
U(f) &\leq U(f,P) \\
&= U(f,P) - L(f,P) + L(f,P) \\
&< \varepsilon + L(f,P) \\
&\leq \varepsilon + L(f).
\end{aligned}
$$

□

Proof.

Since $\varepsilon > 0$ was arbitrarily chosen, we get

$$U(f) \leq L(f).$$

From the last inequality and from Theorem 5, we conclude that $U(f) = L(f)$, i.e., f is Δ-integrable on R. The proof is complete. □

Remark

Let $\mathbb{T}$ *be a time scale with forward jump operator* σ*. We note that for every* $\delta > 0$*, there exists at least one partition* $P_1 \in \mathcal{P}([a, b])$ *generated by a set*

$$\{t_0, t_1, t_2, \ldots, t_n\} \subset [a, b], \quad \textit{where} \quad a = t_0 < t_1 < \ldots < t_n = b,$$

such that for each $i \in \{1, 2, \ldots, n\}$ *either*

$$t_i - t_{i-1} < \delta$$

or

$$t_i - t_{i-1} > \delta \quad \textit{and} \quad \sigma(t_{i-1}) = t_i.$$

Definition

We denote by $P_\delta([a,b))$ the set of all $P_1 \in \mathcal{P}([a,b))$ that possess the property indicated in Remark 0.2. Further, by $\mathcal{P}_\delta(R)$, we denote the set of all $P \in \mathcal{P}(R)$ such that

$$P = P_1 \times P_2 \times \ldots \times P_n, \quad \text{where} \quad P_i \in \mathcal{P}_\delta([a_i, b_i)), \quad i = 1, 2, \ldots, n.$$

Theorem

Let $P^0 \in \mathcal{P}(R)$ *be given by*

$$P^0 = P_1^0 \times P_2^0 \times \ldots \times P_n^0$$

in which $P_i^0 \in \mathcal{P}([a_i, b_i))$, $i \in \{1, 2, \ldots, n\}$, *is generated by a set*

$$A_i^0 = \{t_{i_0}^0, t_{i_1}^0, \ldots, t_{i_{n_i}}^0\} \subset [a_i, b_i], \quad \text{where} \quad a_i = t_{i_0}^0 < t_{i_1}^0 < \ldots < t_{i_{n_i}}^0 = b_i.$$

Then, for each $P \in \mathcal{P}_\delta(R)$, *we have*

$$L(f, P^0 + P) - L(f, P) \leq (M - m)D^{n-1}(n_1 + n_2 + \ldots + n_n - n)\delta$$

and

$$U(f, P) - U(f, P + P^0) \leq (M - m)D^{n-1}(n_1 + n_2 + \ldots + n_n - n)\delta,$$

where $D = \max_{i \in \{1,2,\ldots,n\}}\{b_i - a_i\}$, *and* M *and* m *are defined as above.*

Proof.

Suppose the partition P is given by

$$P = P_1 \times P_2 \times \ldots \times P_n$$

in which $P_i \in \mathcal{P}_\delta([a_i, b_i))$ is generated by a set

$$A_i = \{t_0^i, t_1^i, \ldots, t_{p_i}^i\} \subset [a_i, b_i],$$

where

$$a_i = t_0^i < t_1^i < \ldots < t_{p_i}^i = b_i, \quad i = 1, 2, \ldots, n.$$

Let $Q = P^0 + P = Q_1 \times Q_2 \times \ldots \times Q_n$, where $Q_i \in \mathcal{P}([a_i, b_i))$, $i = 1, 2, \ldots, n$, are generated by the sets □

Proof.

$$B_i = A_i^0 + A_i.$$

We suppose that there exists $i \in \{1, 2, \dots, n\}$ such that B_i has one more point, say t', than A_i and $B_l = A_l$, $l \neq i$, $l \in \{1, 2, \dots, n\}$. Then $t' \in (t^i_{k_i-1}, t^i_{k_i})$ for some $k_i \in \{1, 2, \dots, p_i\}$, where $t^i_{k_i} - t^i_{k_i-1} \leq \delta$. If $t^i_{k_i} - t^i_{k_i-1} \geq \delta$, then, using

$$P_i \in \mathcal{P}_\delta([a_i, b_i)),$$

we have $\sigma(t^i_{k_i-1}) = t^i_{k_i}$ and $(t^i_{k_i-1}, t^i_{k_i}) = \emptyset$. □

Proof.

Now, denoting by $m_{k_1 k_2 \ldots k_n}$, $m^{(1)}_{k_1 k_2 \ldots k_n}$, and $m^{(2)}_{k_1 k_2 \ldots k_n}$ the infimum of f on

$$R_{k_1 k_2 \ldots k_n} = [t_{k_1 - 1}, t_{k_1}) \times [t_{k_2 - 1}, t_{k_2}) \times \ldots \times [t_{k_n - 1}, t_{k_n}),$$

$$R^{(1)}_{k_1 k_2 \ldots k_n} = [t_{k_1 - 1}, t_{k_1}) \times \ldots \times [t_{k_{i-1} - 1}, t_{k_{i-1}}) \times [t_{k_i - 1}, t') \times [t_{k_{i+1} - 1}, t_{k_{i+1}})$$

□

Proof.

$$\times \ldots \times [t_{k_n-1}, t_{k_n}),$$

$$R^{(2)}_{k_1 k_2 \ldots k_n} = [t_{k_1-1}, t_{k_1}) \times \ldots \times [t_{k_{i-1}-1}, t_{k_{i-1}}) \times [t', t_{k_i}) \times [t_{k_{i+1}-1}, t_{k_{i+1}})$$

$$\times \ldots \times [t_{k_n-1}, t_{k_n}),$$

respectively, we have □

Proof.

$$m^{(1)}_{k_1 k_2 \ldots k_n} \geq m_{k_1 k_2 \ldots k_n},$$

$$m^{(2)}_{k_1 k_2 \ldots k_n} \geq m_{k_1 k_2 \ldots k_n},$$

$$m^{(1)}_{k_1 k_2 \ldots k_n} - m_{k_1 k_2 \ldots k_n} \leq M - m,$$

$$m^{(2)}_{k_1 k_2 \ldots k_n} - m_{k_1 k_2 \ldots k_n} \leq M - m,$$

and

$$m(R_{k_1 k_2 \ldots k_n}) = m(R^{(1)}_{k_1 k_2 \ldots k_n}) + m(R^{(2)}_{k_1 k_2 \ldots k_n}),$$

Proof.

so that

$$L(f,Q) - L(f,P)$$

$$= \sum_{j_{k_1}=1}^{p_1} \sum_{j_{k_2}=1}^{p_2} \cdots \sum_{j_{k_{i-1}}=1}^{p_{i-1}} \sum_{j_{k_{i+1}}=1}^{p_{i+1}} \cdots \sum_{j_{k_n}=1}^{p_n} \Big(m^{(1)}_{j_{k_1} j_{k_2} \ldots j_{k_n}} m(R^{(1)}_{j_{k_1} j_{k_2} \ldots j_{k_n}})$$

$$+ m^{(2)}_{j_{k_1} j_{k_2} \ldots j_{k_n}} m(R^{(2)}_{j_{k_1} j_{k_2} \ldots j_{k_n}}) - m_{j_{k_1} j_{k_2} \ldots j_{k_n}} m(R_{j_{k_1} j_{k_2} \ldots j_{k_n}}) \Big)$$

□

Proof.

$$= \sum_{j_{k_1}=1}^{p_1} \sum_{j_{k_2}=1}^{p_2} \cdots \sum_{j_{k_{i-1}}=1}^{p_{i-1}} \sum_{j_{k_{i+1}}=1}^{p_{i+1}} \cdots \sum_{j_{k_n}=1}^{p_n} \Big((m^{(1)}_{j_{k_1} j_{k_2} \dots j_{k_n}} - m_{j_{k_1} j_{k_2} \dots j_{k_n}})$$

$$\times m(R^{(1)}_{j_{k_1} j_{k_2} \dots j_{k_n}}) + (m^{(2)}_{j_{k_1} j_{k_2} \dots j_{k_n}} - m_{j_{k_1} j_{k_2} \dots j_{k_n}}) m(R^{(2)}_{j_{k_1} j_{k_2} \dots j_{k_n}}) \Big)$$

□

Proof.

$$
\begin{aligned}
&\leq (M-m)\sum_{j_{k_1}=1}^{p_1}\sum_{j_{k_2}=1}^{p_2}\cdots\sum_{j_{k_{i-1}}=1}^{p_{i-1}}\sum_{j_{k_{i+1}}=1}^{p_{i+1}}\cdots\sum_{j_{k_n}=1}^{p_n}\Big(m(R^{(1)}_{j_{k_1}j_{k_2}\dots j_{k_n}}) \\
&\quad +m(R^{(2)}_{j_{k_1}j_{k_2}\dots j_{k_n}})\Big) \\
&= (M-m)(t_{k_i}-t_{k_i-1})
\end{aligned}
$$

□

Proof.

$$\times \sum_{j_{k_1}=1}^{p_1} \sum_{j_{k_2}=1}^{p_2} \cdots \sum_{j_{k_{i-1}}=1}^{p_{i-1}} \sum_{j_{k_{i+1}}}^{p_{i+1}} \cdots \sum_{j_{k_n}=1}^{p_n} (t_{j_{k_1}} - t_{j_{k_1-1}})(t_{j_{k_2}} - t_{j_{k_2-1}}) \ldots (t_{j_{k_n}} - t_{j}$$

$$\leq \quad (M - m) D^{n-1} \delta.$$

Since B_i has at most $n_i - 1$ points that are not in A_i, an induction argument shows that

$$L(f, Q) - L(f, P) \leq (M - m)(n_1 + n_2 + \cdots + n_n - n) D^{n-1} \delta.$$

The proof of the other inequality is similar. □

Theorem

A bounded function f on R is Δ-integrable if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P \in \mathcal{P}_\delta(R) \quad \textit{implies} \quad U(f, P) - L(f, P) < \varepsilon. \tag{2}$$

Proof.

Suppose that for each $\varepsilon > 0$, there exists $\delta > 0$ such that (2) holds. Because $P \in \mathcal{P}(R)$, we have that (1) holds. Hence, using Theorem 7, we conclude that f is Δ-integrable on R.

Suppose that f is Δ-integrable over R. Let $\varepsilon > 0$ be arbitrarily chosen. Hence, by Theorem 7, it follows that there exists $P^0 \in \mathcal{P}(R)$ such that

$$U(f, P^0) - L(f, P^0) < \frac{\varepsilon}{2}.$$

□

Proof.

Let D be as in Theorem 9 and P^0 be determined as in Theorem 9. We choose

$$\delta = \frac{\varepsilon}{4(M-m)D^{n-1}(n_1+n_2+\ldots+n_n-n)}.$$

Then, using Theorem 9, for each $P \in \mathcal{P}_\delta(R)$, we have

$$\begin{aligned} L(f, P^0+P) - L(f,P) &\leq (M-m)D^{n-1}(n_1+n_2+\cdots+n_n-n)\delta \\ &= \frac{\varepsilon}{4}, \\ U(f,P) - U(f, P^0+P) &\leq (M-m)D^{n-1}(n_1+n_2+\cdots+n_n-n)\delta \\ &= \frac{\varepsilon}{4}. \end{aligned}$$

□

Proof.

Using this and

$$L(f,P^0) \leq L(f,P^0+P) \quad \text{and} \quad U(f,P^0+P) \leq U(f,P^0),$$

we obtain

$$L(f,P^0) - L(f,P) \leq \frac{\varepsilon}{4} \quad \text{and} \quad U(f,P) - U(f,P^0) \leq \frac{\varepsilon}{4},$$

i.e.,

$$-L(f,P) \leq \frac{\varepsilon}{4} - L(f,P^0) \quad \text{and} \quad U(f,P) \leq \frac{\varepsilon}{4} + U(f,P^0).$$

Therefore, □

Proof.

$$
\begin{aligned}
U(f,P) - L(f,P) &\leq \frac{\varepsilon}{4} - L(f,P^0) + \frac{\varepsilon}{4} + U(f,P^0) \\
&= \frac{\varepsilon}{2} + U(f,P^0) - L(f,P^0) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Thus, we have verified (2). □

Theorem

For every bounded function f on R, the Darboux Δ-sums $L(f,P)$ and $U(f,P)$ evaluated for $P \in \mathcal{P}_\delta(R)$ have limits as $\delta \to 0$, uniformly with respect to P, and

$$\lim_{\delta\to 0} L(f,P) = L(f) \quad \textit{and} \quad \lim_{\delta\to 0} U(f,P) = U(f).$$

Proof.

We fix $\varepsilon > 0$ and choose a partition $P^0 \in \mathcal{P}(R)$ so that

$$L(f) - L(f, P^0) < \varepsilon \quad \text{and} \quad U(f, P^0) - U(f) < \varepsilon.$$

Let P^0 be described as in Theorem 9. Then, for any $P \in \mathcal{P}_\delta(R)$, using Theorem 9, we have

$$L(f, P^0 + P) - L(f, P) \leq (M - m)D^{n-1}(n_1 + n_2 + \cdots + n_n - n)\delta$$

and

$$U(f, P) - U(f, P^0 + P) \leq (M - m)D^{n-1}(n_1 + n_2 + \cdots + n_n - n)\delta.$$

We take □

Proof.

$$\delta = \frac{\varepsilon}{(M-m)D^{n-1}(n_1+n_2+\cdots+n_n-n)}.$$

Because $P^0 + P$ is a refinement of P^0, we have

$$L(f,P^0) \le L(f,P^0+P) \quad \text{and} \quad U(f,P^0+P) \le U(f,P^0).$$

Thus,

$$\begin{aligned} L(f) - \varepsilon < L(f,P^0) &\le L(f,P^0+P) \\ &\le L(f), \\ U(f) &\le U(f,P^0+P) \end{aligned}$$

□

Proof.

$$\leq \quad U(f, P^0)$$

$$< \quad \varepsilon + U(f).$$

Hence,

$$L(f, P^0 + P) - L(f, P^0) < \varepsilon \quad \text{and} \quad U(f, P^0) - U(f, P^0 + P) < \varepsilon.$$

Therefore,

$$\begin{aligned} |L(f) - L(f, P)| &= |L(f) - L(f, P^0) + L(f, P^0) - L(f, P^0 + P) \\ &\quad + L(f, P^0 + P) - L(f, P)| \end{aligned}$$

□

Proof.

$$
\begin{aligned}
&\leq |L(f) - L(f,P^0)| + |L(f,P^0) - L(f,P^0+P)| \\
&\quad + |L(f,P+P^0) - L(f,P)| \\
&< \varepsilon + \varepsilon + (M-m)D^{n-1}(n_1 + n_2 + \cdots + n_n - n)\delta \\
&\leq \varepsilon + \varepsilon + \varepsilon \\
&= 3\varepsilon
\end{aligned}
$$

and

□

Proof.

$$
\begin{aligned}
|U(f,P) - U(f)| &= |U(f,P) - U(f,P+P^0) + U(f,P+P^0) - U(f,P^0) \\
&\quad + U(f,P^0) - U(f)| \\
&\leq |U(f,P) - U(f,P+P^0)| + |U(f,P+P^0) - U(f,P^0)| \\
&\quad + |U(f,P^0) - U(f)| \\
&< (M-m)D^{n-1}(n_1 + n_2 + \cdots + n_n - n)\delta + \varepsilon + \varepsilon \\
&\leq \varepsilon + \varepsilon + \varepsilon \\
&= 3\varepsilon,
\end{aligned}
$$

Definition

Let f be a bounded function on R and $P \in \mathcal{P}(R)$. In each "rectangle" $R_{j_1 j_2 \dots j_n}$, $1 \le j_i \le k_i$, $i = 1, 2, \dots, n$, choose a point $\xi_{j_1 j_2 \dots j_n}$ and form the sum

$$S = \sum_{i=1}^{n} \sum_{j_i=1}^{k_i} f(\xi_{j_1 j_2 \dots j_n})(t_1^{j_i} - t_1^{j_i-1}) \dots (t_n^{j_n} - t_n^{j_n-1}). \tag{3}$$

We call S a *Riemann Δ-sum* of f corresponding to $P \in \mathcal{P}(R)$.

Definition

We say that f is *Riemann Δ-integrable* over R if there exists a number I such that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S - I| < \varepsilon$$

for every Riemann Δ-sum S of f corresponding to any $P \in \mathcal{P}_\delta(R)$, independent of the choice of the point $\xi_{j_1 j_2 \dots j_n} \in R_{j_1 j_2 \dots j_n}$ for $1 \leq j_i \leq k_i$, $i = 1, 2, \dots, n$.

Definition

The number I is called the *Riemann Δ-integral* of f over R.
We write

$$I = \lim_{\delta \to 0} S.$$

Theorem

The Riemann Δ-integral is well defined.

Proof.

Suppose that f is Riemann Δ-integrable over R and there are two numbers I_1 and I_2 such that for every $\varepsilon > 0$, there exists $\delta > 0$ so that

$$|S - I_1| < \frac{\varepsilon}{2} \quad \text{and} \quad |S - I_2| < \frac{\varepsilon}{2}$$

for every Riemann Δ-sum S of f corresponding to any $P \in \mathcal{P}_\delta(R)$, independent of the way in which $\xi_{j_1 j_2 \ldots j_n} \in R_{j_1 j_2 \ldots j_n}$ for $1 \le j_i \le k_i$, $i = 1, 2, \ldots, n$, is chosen. Therefore, □

Proof.

$$\begin{aligned} |I_1 - I_2| &= |I_1 - S + S - I_2| \\ &\leq |S - I_1| + |S - I_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Consequently, $I_1 = I_2$. □

Remark

Note that in the Riemann definition of the integral, we need not assume the boundedness of f in advance. However, it follows that the Riemann integrability of a function f over R implies its boundedness on R.

Theorem

A bounded function on R is Riemann Δ-integrable if and only if it is Darboux Δ-integrable, in which case the value of the integrals are equal.

Proof.

Suppose that f is Darboux Δ-integrable over R in the sense of Definition **??**. Let $\varepsilon > 0$ and $\delta > 0$ be chosen so that (1) of Theorem 7 holds. Using the definition of S, we have

$$L(f, P) \le S \le U(f, P).$$

Also,

$$\begin{aligned} U(f, P) &< L(f, P) + \varepsilon \\ &\le L(f) + \varepsilon \\ &= \int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n + \varepsilon, \end{aligned}$$

□

Proof.

$$
\begin{aligned}
L(f,P) &> U(f,P) - \varepsilon \\
&\geq U(f) - \varepsilon \\
&= \int_R f(t_1, t_2, \dots, t_n)\Delta_1 t_1 \Delta_2 t_2 \dots \Delta_n t_n - \varepsilon.
\end{aligned}
$$

Hence,

$$
S - \int_R f(t_1, t_2, \dots, t_n)\Delta_1 t_1 \Delta_2 t_2 \dots \Delta_n t_n \leq U(f,P)
$$

□

Proof.

$$
\begin{aligned}
& -\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \\
< & \int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n + \varepsilon \\
& -\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \\
= & \ \varepsilon
\end{aligned}
$$

and

$$
S - \int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \geq L(f, P)
$$

□

Proof.

$$
\begin{aligned}
& - \int_R f(t_1, t_2, \ldots, t_n) \Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \\
> \ & \int_R f(t_1, t_2, \ldots, t_n) \Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n - \varepsilon \\
& - \int_R f(t_1, t_2, \ldots, t_n) \Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \\
= \ & -\varepsilon.
\end{aligned}
$$

Consequently,

$$
\left| S - \int_R f(t_1, t_2, \ldots, t_n) \Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \right| < \varepsilon.
$$

□

Proof.

Now, we suppose that f is Riemann Δ-integrable in the sense of Definition 12. Select $P \in \mathcal{P}_\delta(R)$ of the type (**??**) and (**??**). For each $i \in \{1, 2, \ldots, n\}$ and $1 \leq j_i \leq k_i$, we choose $\xi_{j_1 j_2 \ldots j_n} \in R_{j_1 j_2 \ldots j_n}$ so that

$$M_{j_1 j_2 \ldots j_n} - \varepsilon < f(\xi_{j_1 j_2 \ldots j_n}) < m_{j_1 j_2 \ldots j_n} + \varepsilon.$$

The Riemann Δ-sum S for this choice of the points $\xi_{j_1 j_2 \ldots j_n}$ satisfies □

Proof.

$$U(f,P) - \varepsilon \prod_{i=1}^{n}(b_i - a_i) < S < L(f,P) + \varepsilon \prod_{i=1}^{n}(b_i - a_i)$$

as well as

$$-\varepsilon < S - I < \varepsilon.$$

Thus,

$$\begin{aligned} L(f) &\geq L(f,P) \\ &> S - \varepsilon \prod_{i=1}^{n}(b_i - a_i) \\ &> I - \varepsilon - \varepsilon \prod_{i=1}^{n}(b_i - a_i) \end{aligned}$$

and

Proof.

$$\begin{aligned} U(f) &\leq U(f,P) \\ &< S + \varepsilon \prod_{i=1}^{n}(b_i - a_i) \\ &< I + \varepsilon + \varepsilon \prod_{i=1}^{n}(b_i - a_i). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$L(f) \geq I \quad \text{and} \quad U(f) \leq I,$$

i.e.,

$$I \leq L(f) \leq U(f) \leq I.$$

This completes the proof. □

Remark

In the definition of

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n$$

with $R = [a_1, b_1) \times \ldots \times [a_n, b_n)$*, we assumed that* $a_i < b_i$*,* $i \in \{1, \ldots, n\}$*. We extend the definition to the case* $a_i = b_i$ *for some* $i \in \{1, 2, \ldots, n\}$ *by setting*

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = 0 \tag{4}$$

if $a_i = b_i$ *for some* $i \in \{1, \ldots, n\}$*.*

Theorem

Let $a = (a_1, \ldots, a_n) \in \Lambda^n$ *and* $b = (b_1, \ldots, b_n) \in \Lambda^n$ *with* $a_i \le b_i$ *for all* $i \in \{1, \ldots, n\}$. *Every constant function*

$$f(t_1, t_2, \ldots, t_n) = A \quad \text{for} \quad (t_1, t_2, \ldots, t_n) \in R = [a_1, b_1) \times \ldots \times [a_n, b_n)$$

is Δ*-integrable over* R *and*

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = A \prod_{i=1}^{n} (b_i - a_i). \tag{5}$$

Proof.

We assume that $a_i < b_i$ for all $i \in \{1, \dots, n\}$. Consider a partition P of R of the type (**??**) and (**??**). Since

$$M_{j_1 j_2 \dots j_n} = m_{j_1 j_2 \dots j_n} = A \quad \text{for all} \quad 1 \le j_i \le k_i, \quad i \in \{1, \dots, n\},$$

we have that

$$U(f, P) = L(f, P) = A \prod_{i=1}^{n} (b_i - a_i).$$

Hence, using Theorem 7, it follows that f is Δ-integrable and (5) holds. If $a_i = b_i$ for some $i \in \{1, \dots, n\}$, then (5) follows by (4). Note that every Riemann Δ-sum of f associated with P is also equal to $A \prod_{i=1}^{n} (b_i - a_i)$. □

Theorem

Let $t^0 = (t_1^0, \ldots, t_n^0) \in \Lambda^n$. *Every function* $f : \Lambda^n \to \mathbb{R}$ *is* Δ*-integrable over*

$$R = R(t^0) = [t_1^0, \sigma_1(t_1^0)) \times \ldots \times [t_n^0, \sigma_n(t_n^0)),$$

and

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = \prod_{i=1}^{n} \mu_i(t_i^0) f(t_1^0, \ldots, t_n^0). \qquad (6)$$

Proof.

If $\mu_i(t_i^0) = 0$ for some $i \in \{1, \dots, n\}$, then (6) is obvious as both sides of (6) are equal to zero in this case. If $\mu_i(t_i^0) > 0$ for all $i \in \{1, \dots, n\}$, then a single partition P of $R(t^0)$ is

$$[t_1^0, \sigma_1(t_1^0)) \times \dots \times [t_n^0, \sigma_n(t_n^0)) = \{(t_1^0, \dots, t_n^0)\}.$$

Consequently, we have

$$U(f, P) = L(f, P) = \prod_{i=1}^{n} \mu_i(t_i^0) f(t_1^0, \dots, t_n^0).$$

Therefore, Theorem 7 shows that f is Δ-integrable over $R(t^0)$ and (6) holds. Note that the Riemann Δ-sum associated with the above partition is also equal to the right-hand side of (6). □

Theorem

Let $a = (a_1, a_2, \ldots, a_n) \in \Lambda^n$ *and* $b = (b_1, b_2, \ldots, b_n) \in \Lambda^n$ *with* $a_i \leq b_i$ *for all* $i \in \{1, 2, \ldots, n\}$. *If* $\mathbb{T}_i = \mathbb{R}$ *for every* $i \in \{1, 2, \ldots, n\}$, *then every bounded function* f *on* $R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n)$ *is* Δ*-integrable if and only if* f *is Riemann integrable on* R *in the classical sense, and in this case*

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = \int_R f(t_1, t_2, \ldots, t_n)\mathrm{d}t_1\mathrm{d}t_2 \ldots \mathrm{d}t_n,$$

where the integral on the right-hand side is the ordinary Riemann integral.

Proof.

Clearly, Definition **??** and Definition 12 of the Δ-integral coincide in the case $\mathbb{T}_i = \mathbb{R}$, $i \in \{1, 2, \ldots, n\}$, with the usual Darboux and Riemann definitions of the integral, respectively. Note that the classical definitions of Darboux's and Riemann's integral do not depend on whether the rectangles of the partition are taken closed, half-closed, or open. Moreover, if $\mathbb{T}_i = \mathbb{R}$, $i \in \{1, 2, \ldots, n\}$, then $\mathcal{P}_\delta(\mathbb{R})$ consists of all partitions of R with norm (mesh) less than or equal to $\delta\sqrt{n}$. □

Theorem

Let $a = (a_1, a_2, \ldots, a_n) \in \Lambda^n$ *and* $b = (b_1, b_2, \ldots, b_n) \in \Lambda^n$ *with* $a_i \leq b_i$ *for all* $i \in \{1, 2, \ldots, n\}$. *If* $\mathbb{T}_i = \mathbb{Z}$ *for all* $i \in \{1, 2, \ldots, n\}$, *then every function defined on* $R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n)$ *is* Δ*-integrable over* R, *and*

Theorem

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n$$

$$= \begin{cases} 0 & \textit{if } a_i = b_i \textit{ for some } i \in \{1, \\ \sum_{r_1=a_1}^{b_1-1} \sum_{r_2=a_2}^{b_2-1} \ldots \sum_{r_n=a_n}^{b_n-1} f(r_1, r_2, \ldots, r_n) & \textit{otherwise.} \end{cases}$$

Proof.

Let $b_i = a_i + p_i$, $p_i \in \mathbb{N}$, $i \in \{1, 2, \ldots, n\}$. Consider the partition P^* of R given by (**??**) and (**??**) with $k_i = p_i$, $i \in \{1, 2, \ldots, n\}$, and

$$t_i^0 = a_i, \quad t_i^1 = a_i + 1, \quad \ldots, \quad t_i^{k_i} = a_i + p_i.$$

Thus, $R_{j_1 j_2 \ldots j_n}$ contains the single point $(t_1^{j_1 - 1}, t_2^{j_2 - 1}, \ldots, t_n^{j_n - 1})$. Therefore,

$$U(f, P^*) = L(f, P^*) = \sum_{r_1 = a_1}^{b_1 - 1} \sum_{r_2 = a_2}^{b_2 - 1} \cdots \sum_{r_n = a_n}^{b_n - 1} f(r_1, r_2, \ldots, r_n).$$

Hence, Theorem 7 shows that f is Δ-integrable over R and (7) holds for $a_i < b_i$, $i \in \{1, 2, \ldots, n\}$. If $a_i = b_i$ for some $i \in \{1, 2, \ldots, n\}$, then the relation (4) shows the validity of (7). □

Example

Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$. We consider

$$I = \int_0^4 \int_1^8 t_2(2t_1 + 1)\Delta_1 t_1 \Delta_2 t_2.$$

Here,

$$f(t_1, t_2) = t_2(2t_1 + 1), \quad (t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2,$$

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = t_2 + 1, \quad t_2 \in \mathbb{T}_2.$$

We note that

Example

$$
\begin{aligned}
(t_2 t_1^2)_{t_1}^{\Delta_1} &= t_2 (t_1^2)_{t_1}^{\Delta_1} \\
&= t_2(\sigma_1(t_1) + t_1) \\
&= t_2(t_1 + 1 + t_1) \\
&= t_2(2t_1 + 1).
\end{aligned}
$$

Therefore,

Example

$$\int_1^8 t_2(2t_1+1)\Delta_1 t_1 = t_2 t_1^2\Big|_{t_1=1}^{t_1=8} = 64t_2 - t_2 = 63t_2.$$

Hence,

$$I = \int_0^4 63t_2\Delta_2 t_2 = 63\int_0^4 t_2\Delta_2 t_2.$$

Since

Example

$$
\begin{aligned}
\frac{1}{2}\left((t_2^2)^{\Delta_2}_{t_2} - 1\right) &= \frac{1}{2}(\sigma_2(t_2) + t_2 - 1) \\
&= \frac{1}{2}(t_2 + 1 + t_2 - 1) \\
&= t_2,
\end{aligned}
$$

we get

Example

$$
\begin{aligned}
I &= 63\int_0^4 \frac{1}{2}\left((t_2^2)^{\Delta_2}_{t_2} - 1\right)\Delta_2 t_2 \\
&= \frac{63}{2}\int_0^4 (t_2^2)^{\Delta_2}_{t_2}\Delta_2 t_2 - \frac{63}{2}\int_0^4 \Delta_2 t_2 \\
&= \frac{63}{2} t_2^2\Big|_{t_2=0}^{t_2=4} - 126 \\
&= 504 - 126 \\
&= 378.
\end{aligned}
$$

Example

Let $\mathbb{T}_1 = \mathbb{Z}$ and $\mathbb{T}_2 = 2^{\mathbb{N}}$. We consider

$$I = \frac{1}{2}\sin\frac{1}{2}\int_0^3\int_2^8 t_2\cos\left(t_1 + \frac{1}{2}\right)\Delta_1 t_1 \Delta_2 t_2.$$

Here,

$$f(t_1, t_2) = \frac{1}{2}t_2\cos\left(t_1 + \frac{1}{2}\right)\sin\frac{1}{2}, \quad (t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2,$$

$$\sigma_1(t_1) = t_1 + 1, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = 2t_2, \quad t_2 \in \mathbb{T}_2.$$

Since

Example

$$
\begin{aligned}
(\sin t_1)^{\Delta_1}_{t_1} &= \frac{\sin \sigma_1(t_1) - \sin t_1}{\sigma_1(t_1) - t_1} \\
&= \frac{\sin(t_1 + 1) - \sin t_1}{t_1 + 1 - t_1} \\
&= \frac{1}{2} \sin \frac{1}{2} \cos\left(t_1 + \frac{1}{2}\right),
\end{aligned}
$$

we get

Example

$$\begin{aligned}
\frac{1}{2}\sin\frac{1}{2}\int_2^8 t_2\cos\left(t_1+\frac{1}{2}\right)\Delta_1 t_1 &= t_2\int_2^8 (\sin t_1)_{t_1}^{\Delta_1}\Delta_1 t_1 \\
&= t_2 \sin t_1\Big|_{t_1=2}^{t_1=8} \\
&= t_2(\sin 8-\sin 2) \\
&= 2t_2\sin 3\cos 5.
\end{aligned}$$

Therefore,

Example

$$I = 2\sin 3\cos 5\int_0^3 t_2\Delta_2 t_2.$$

Because

$$(t_2^2)_{t_2}^{\Delta_2} = \sigma_2(t_2) + t_2 = 2t_2 + t_2 = 3t_2,$$

we get

$$t_2 = \frac{1}{3}(t_2^2)_{t_2}^{\Delta_2}.$$

Consequently,

Example

$$
\begin{aligned}
I &= 2\sin 3\cos 5\int_0^3 \frac{1}{3}(t_2^2)_{t_2}^{\Delta_2}\Delta_2 t_2 \\
&= \frac{2}{3}\sin 3\cos 5\int_0^3 (t_2^2)_{t_2}^{\Delta_2}\Delta_2 t_2 \\
&= \frac{2}{3}\sin 3\cos 5\, t_2^2\Big|_{t_2=0}^{t_2=3} \\
&= 6\sin 3\cos 5.
\end{aligned}
$$

Example

Let $\mathbb{T}_1 = 3\mathbb{Z}$ and $\mathbb{T}_2 = 3^{\mathbb{N}}$. We consider

$$I = \int_3^9 \int_{-3}^{12} (t_1 t_2 + 2t_1 + t_2 + 3)\Delta_1 t_1 \Delta_2 t_2.$$

Here,

$$f(t_1, t_2) = t_1 t_2 + 2t_1 + t_2 + 3, \quad (t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2,$$

$$\sigma_1(t_1) = t_1 + 3, \quad t_1 \in \mathbb{T}_1, \quad \sigma_2(t_2) = 3t_2, \quad t_2 \in \mathbb{T}_2.$$

We note that

Example

$$(t_1^2)_{t_1}^{\Delta_1} = \sigma_1(t_1) + t_1 = 2t_1 + 3,$$

whereupon

$$t_1 = \frac{(t_1^2)_{t_1}^{\Delta_1} - 3}{2},$$

$$t_1 t_2 + 2t_1 + t_2 + 3 = \frac{(t_1^2)_{t_1}^{\Delta_1} - 3}{2}(t_2 + 2) + t_2 + 3,$$

and

$$\int_{-3}^{12} f(t_1, t_2)\Delta_1 t_1 = \int_{-3}^{12} \left(\frac{(t_1^2)_{t_1}^{\Delta_1} - 3}{2}(t_2 + 2) + (t_2 + 3) \right) \Delta_1 t_1$$

Example

$$
\begin{aligned}
&= \frac{t_2+2}{2}\int_{-3}^{12}(t_1^2)_{t_1}^{\Delta_1}\Delta_1 t_1 + \int_{-3}^{12}\left(-\frac{3}{2}(t_2+2)+t_2+3\right)\Delta_1 t_1 \\
&= \frac{t_2+2}{2}t_1^2\Big|_{t_1=-3}^{t_1=12} - \frac{t_2}{2}\int_{-3}^{12}\Delta_1 t_1 \\
&= \frac{t_2+2}{2}(144-9) - \frac{15}{2}t_2 \\
&= \frac{135}{2}(t_2+2) - \frac{15}{2}t_2 \\
&= 60t_2 + 135.
\end{aligned}
$$

Example

Therefore,

$$I = \int_3^9 (60t_2 + 135)\Delta_2 t_2.$$

Since

$$(t_2^2)_{t_2}^{\Delta_2} = \sigma_2(t_2) + t_2 = 3t_2 + t_2 = 4t_2,$$

we get

$$t_2 = \frac{1}{4}(t_2^2)_{t_2}^{\Delta_2}.$$

Consequently,

Example

$$
\begin{aligned}
I &= \int_3^9 \left(60 \cdot \frac{1}{4}(t_2^2)_{t_2}^{\Delta_2} + 135\right) \Delta_2 t_2 \\
&= 15 \int_3^9 (t_2^2)_{t_2}^{\Delta_2} \Delta_2 t_2 + 135 \int_3^9 \Delta_2 t_2 \\
&= 15 t_2^2 \Big|_{t_2=3}^{t_2=9} + 135 \cdot 6 \\
&= 15 \cdot (81 - 9) + 810 \\
&= 1890.
\end{aligned}
$$

Note that Λ^n is a complete metric space with the metric D defined by

$$d(t,s) = \sqrt{\sum_{i=1}^{n}(t_i - s_i)^2} \quad \text{for} \quad t = (t_1, t_2, \dots, t_n), \quad s = (s_1, s_2, \dots, s_n) \in \Lambda^n$$

and also with the equivalent metric

$$d(t,s) = \max_{i \in \{1,2,\dots,n\}} \{|t_i - s_i|\}.$$

Definition

A function $f : \Lambda^n \to \mathbb{R}$ is said to be continuous at $t \in \Lambda^n$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \varepsilon$$

for all points $s \in \Lambda^n$ satisfying $d(t,s) < \delta$.

Remark

If t is an isolated point of Λ^n, then every function $f : \Lambda^n \to \mathbb{R}$ is continuous at t. In particular, if $\mathbb{T}_i = \mathbb{Z}$ for all $i \in \{1, 2, \ldots, n\}$, then every function $f : \Lambda^n \to \mathbb{R}$ is continuous at each point of Λ^n.

Theorem

Every continuous function on $K = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ is Δ-integrable over $R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n)$.

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is continuous, it is uniformly continuous on the compact subset K of Λ^n. Therefore, there exists $\delta > 0$ such that

$$\begin{cases} t = (t_1, t_2, \ldots, t_n), \quad t' = (t'_1, t'_2, \ldots, t'_n) \in R \quad \text{and} \quad \max_{i \in \{1,2,\ldots,n\}}\{|t_i - \\ \\ \quad \text{implies} \quad |f(t) - f(t')| < \frac{\varepsilon}{(2^n-1)\prod_{i=1}^{n}(b_i - a_i + 1)}. \end{cases} \tag{8}$$

□

Proof.

Consider $P \in \mathcal{P}(R)$ given by (**??**) and (**??**). Let

$$\begin{aligned}
\tilde{R}_{j_1 j_2 \ldots j_n} &= [t_1^{j_1-1}, \sigma_1(t_1^{j_1-1})] \times [t_2^{j_2-1}, \sigma_2(t_2^{j_2-1})] \times \ldots \times [t_n^{j_n-1}, \sigma_n(t_n^{j_n-1})], \\
\tilde{M}_{j_1 j_2 \ldots j_n} &= \sup\{f(t_1, t_2, \ldots, t_n) : (t_1, t_2, \ldots, t_n) \in \tilde{R}_{j_1 j_2 \ldots j_n}\}, \\
\tilde{m}_{j_1 j_2 \ldots j_n} &= \inf\{f(t_1, t_2, \ldots, t_n) : (t_1, t_2, \ldots, t_n) \in \tilde{R}_{j_1 j_2 \ldots j_n}\}.
\end{aligned}$$

$\square$

Proof.

Then, since $R_{j_1 j_2 \dots j_n} \subset \tilde{R}_{j_1 j_2 \dots j_n}$, we have

$$\tilde{m}_{j_1 j_2 \dots j_n} \leq m_{j_1 j_2 \dots j_n} \leq M_{j_1 j_2 \dots j_n} \leq \tilde{M}_{j_1 j_2 \dots j_n}$$

for $1 \leq j_i \leq k_i$, $i = 1, 2, \dots, n$. Therefore, taking into account that f assumes its maximum and minimum on each compact rectangle $\tilde{R}_{j_1 j_2 \dots j_n}$, (8) shows □

Proof.

$$U(f,P)-L(f,P)=\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}(M_{j_1j_2\ldots j_n}-m_{j_1j_2\ldots j_n})$$

$$\times(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\ldots(t_n^{j_n}-t_n^{j_n-1})$$

$$\leq\ \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}(\tilde{M}_{j_1j_2\ldots j_n}-\tilde{m}_{j_1j_2\ldots j_n})$$

$$\times(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\ldots(t_n^{j_n}-t_n^{j_n-1})$$

□

Proof.

$$= \sum_{t_1^{j_1} - t_1^{j_1-1} \le \delta} \sum_{t_2^{j_2} - t_2^{j_2-1} \le \delta} \cdots \sum_{t_n^{j_n} - t_n^{j_n-1} \le \delta} (\tilde{M}_{j_1 j_2 \ldots j_n} - \tilde{m}_{j_1 j_2 \ldots j_n})$$

$$\times (t_1^{j_1} - t_1^{j_1-1})(t_2^{j_2} - t_2^{j_2-1}) \ldots (t_n^{j_n} - t_n^{j_n-1})$$

$$+ \sum_{t_1^{j_1} - t_1^{j_1-1} \le \delta} \sum_{t_2^{j_2} - t_2^{j_2-1} \le \delta} \cdots \sum_{t_n^{j_n} - t_n^{j_n-1} > \delta} (\tilde{M}_{j_1 j_2 \ldots j_n} - \tilde{m}_{j_1 j_2 \ldots j_n})$$

$$\times (t_1^{j_1} - t_1^{j_1-1})(t_2^{j_2} - t_2^{j_2-1}) \ldots (t_n^{j_n} - t_n^{j_n-1})$$

$\square$

Proof.

$$+\cdots$$

$$+\sum_{t_1^{j_1}-t_1^{j_1-1}>\delta}\ \sum_{t_2^{j_2}-t_2^{j_2-1}>\delta}\cdots\sum_{t_n^{j_n}-t_n^{j_n-1}>\delta}(\tilde{M}_{j_1j_2\ldots j_n}-\tilde{m}_{j_1j_2\ldots j_n})$$

$$\times(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\ldots(t_n^{j_n}-t_n^{j_n-1})$$

$$\leq \frac{\varepsilon(2^n-1)}{(2^n-1)\prod_{i=1}^{n}(b_i-a_i+1)}$$

□

Proof.

$$\times \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} (t_1^{j_1} - t_1^{j_1-1})(t_2^{j_2} - t_2^{j_2-1}) \ldots (t_n^{j_n} - t_n^{j_n-1})$$

$$= \frac{\varepsilon \prod_{i=1}^{n}(b_i - a_i)}{\prod_{i=1}^{n}(b_i - a_i + 1)}$$

$$< \varepsilon.$$

Thus, $U(f, P) - L(f, P) < \varepsilon$. Hence, Theorem 7 yields that f is Δ-integrable. □

Definition

We say that a function $\phi : [\alpha, \beta] \to \mathbb{R}$ satisfies the *Lipschitz condition* if there exists a constant $B > 0$, a so-called *Lipschitz constant*, such that

$$|\phi(u) - \phi(v)| \leq B|u - v| \quad \text{for all} \quad u, v \in [\alpha, \beta].$$

Example

Let $\phi : [0,1] \to \mathbb{R}$ be defined by $\phi(x) = x^2 + 1$, $x \in [0,1]$. Then, for $x, y \in [0,1]$, we have

$$
\begin{aligned}
|\phi(x) - \phi(y)| &= |x^2 + 1 - y^2 - 1| \\
&= |x^2 - y^2| \\
&= |x - y||x + y| \\
&\leq |x - y|(|x| + |y|) \\
&\leq 2|x - y|,
\end{aligned}
$$

i.e., ϕ satisfies the Lipschitz condition with Lipschitz constant $B = 2$.

Example

Let $\phi : [0, \pi] \to \mathbb{R}$ be defined by $\phi(x) = \sin x$, $x \in [0, \pi]$. Then, for $x, y \in [0, \pi]$, we have

$$\begin{aligned}
|\phi(x) - \phi(y)| &= |\sin x - \sin y| \\
&= 2\left|\sin\frac{x-y}{2}\cos\frac{x+y}{2}\right| \\
&= 2\left|\sin\frac{x-y}{2}\right|\left|\cos\frac{x+y}{2}\right| \\
&\leq 2\frac{|x-y|}{2} \\
&= |x-y|,
\end{aligned}$$

i.e., ϕ satisfies the Lipschitz condition with constant $B = 1$.

Example

Let $\phi : [0,3] \to \mathbb{R}$ be defined by $\phi(x) = \frac{1}{x+4}$, $x \in [0,3]$. Then, for $x, y \in [0,3]$, we have

$$\begin{aligned} |\phi(x) - \phi(y)| &= \left| \frac{1}{x+4} - \frac{1}{y+4} \right| \\ &= \left| \frac{y+4-x-4}{(x+4)(y+4)} \right| \\ &= \frac{|x-y|}{(x+4)(y+4)} \\ &\leq \frac{1}{16}|x-y|, \end{aligned}$$

i.e., ϕ satisfies the Lipschitz condition with Lipschitz constant $L = \frac{1}{16}$.

Theorem

Let $\phi : [\alpha, \beta] \to \mathbb{R}$ *be differentiable. Then* ϕ *satisfies the Lipschitz condition with Lipschitz constant* B *if and only if*

$$|\phi'(x)| \leq B \quad \textit{for all} \quad x \in [\alpha, \beta].$$

Proof.

1. Suppose ϕ satisfies the Lipschitz condition with Lipschitz constant B. Then, for every $x, y \in [\alpha, \beta]$, we have

$$|\phi(x) - \phi(y)| \leq B|x - y|,$$

whereupon □

Proof.

$$|\phi'(x)| \leq B \quad \text{for all} \quad x \in [\alpha, \beta].$$

2. Suppose $|\phi'(x)| \leq B$ for all $x \in [\alpha, \beta]$. Then, for $x, y \in [\alpha, \beta]$, using the mean value theorem, we have that there exists $\xi \in [\alpha, \beta]$ so that

$$|\phi(x) - \phi(y)| = |\phi'(\xi)||x - y| \leq B|x - y|,$$

i.e., ϕ satisfies the Lipschitz condition with Lipschitz constant B. □

Example

Let $\phi : [0,2] \to \mathbb{R}$ be defined by $\phi(x) = \arctan x$, $x \in [0,2]$. We note that ϕ is continuously differentiable on $[0,2]$ and

$$\phi'(x) = \frac{1}{1+x^2}, \quad |\phi'(x)| \leq 1 \quad \text{for all} \quad x \in [0,2].$$

Consequently, ϕ satisfies the Lipschitz condition with Lipschitz constant $B = 1$.

Example

Let $\phi : [0,1] \to \mathbb{R}$ be defined by $\phi(x) = \log(1+x^2)$, $x \in [0,1]$. We note that ϕ is continuously differentiable on $[0,1]$ and

$$\phi'(x)| = \left|\frac{2x}{1+x^2}\right| \leq 2 \quad \text{for all} \quad x \in [0,1].$$

Therefore, ϕ satisfies the Lipschitz condition with Lipschitz constant $B = 2$.

Example

Let

$$\phi(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \in (0,1], \\ \\ 0 & \text{if} \quad x = 0. \end{cases}$$

We assume that the function ϕ satisfies the Lipschitz condition with Lipschitz constant B. Then for all $x \in (0,1]$ and $y = 0$, we have

$$\frac{1}{x} \leq B,$$

which is a contradiction.

Theorem

Let f be bounded and Δ-integrable over

$$R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n)$$

and let M and m be its supremum and infimum over R, respectively. If $\phi : [m, M] \to \mathbb{R}$ is a function satisfying the Lipschitz condition, then the composite function $h = \phi \circ f$ is Δ-integrable over R.

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is Δ-integrable over R, there exists $P \in \mathcal{P}(R)$ given by (**??**) and (**??**) such that □

Proof.

$$U(f,P) - L(f,P) < \frac{\varepsilon}{B},$$

where B is a Lipschitz constant for ϕ. Let $M_{j_1 j_2 \dots j_n}$ and $m_{j_1 j_2 \dots j_n}$ be the supremum and infimum of f on $R_{j_1 j_2 \dots j_n}$, respectively, and let $M^*_{j_1 j_2 \dots j_n}$ and $m^*_{j_1 j_2 \dots j_n}$ be the corresponding numbers for h. Then, for every

$$(t_1^{j_1}, t_2^{j_2}, \dots, t_n^{j_n}), \quad (t_1'^{j_1}, t_2'^{j_2}, \dots, t_n'^{j_n}) \in R_{j_1 j_2 \dots j_n},$$

we have □

Proof.

$$
\begin{aligned}
& h(t_1^{j_1}, t_2^{j_2}, \ldots, t_n^{j_n}) - h(t_1'^{j_1}, t_2'^{j_2}, \ldots, t_n'^{j_n}) \\
\leq \;& |h(t_1^{j_1}, t_2^{j_2}, \ldots, t_n^{j_n}) - h(t_1'^{j_1}, t_2'^{j_2}, \ldots, t_n'^{j_n})| \\
= \;& |\phi(h(t_1^{j_1}, t_2^{j_2}, \ldots, t_n^{j_n})) - \phi(h(t_1'^{j_1}, t_2'^{j_2}, \ldots, t_n'^{j_n}))| \\
\leq \;& B|f(t_1^{j_1}, t_2^{j_2}, \ldots, t_n^{j_n}) - f(t_1'^{j_1}, t_2'^{j_2}, \ldots, t_n'^{j_n})| \\
\leq \;& B(M_{j_1 j_2 \ldots j_n} - m_{j_1 j_2 \ldots j_n}).
\end{aligned}
$$

□

Proof.

Hence,

$$M^*_{j_1j_2\ldots j_n} - m^*_{j_1j_2\ldots j_n} \le B(M_{j_1j_2\ldots j_n} - m_{j_1j_2\ldots j_n})$$

because there exist two sequences

$$(t^{j_1}_{1p}, t^{j_2}_{2p}, \ldots, t^{j_n}_{np}), (t'^{j_1}_{1p}, t'^{j_2}_{2p}, \ldots, t'^{j_n}_{np}) \in R_{j_1j_2\ldots j_n}$$

such that

$$h(t^{j_1}_{1p}, t^{j_2}_{2p}, \ldots, t^{j_n}_{np}) \to M^*_{j_1j_2\ldots j_n}, \quad h(t'^{j_1}_{1p}, t'^{j_2}_{2p}, \ldots, t'^{j_n}_{np}) \to m^*_{j_1j_2\ldots j_n}$$

as $p \to \infty$. □

Proof.

Consequently,

$$U(h,P)-L(h,P)=\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\dots\sum_{j_n=1}^{k_n}(M^*_{j_1j_2\dots j_n}-m^*_{j_1j_2\dots j_n})$$

$$\times(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\dots(t_n^{j_n}-t_n^{j_n-1})$$

$$\leq\quad B\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\dots\sum_{j_n=1}^{k_n}(M_{j_1j_2\dots j_n}-m_{j_1j_2\dots j_n})$$

□

Proof.

$$
\begin{aligned}
& \times(t_1^{j_1} - t_1^{j_1-1})(t_2^{j_2} - t_2^{j_2-1})\dots(t_n^{j_n} - t_n^{j_n-1}) \\
= \;& B(U(f,P) - L(f,P)) \\
< \;& \varepsilon.
\end{aligned}
$$

By Theorem 7, h is Δ-integrable. □

Theorem

Let f be a bounded function that is Δ-integrable over

$$R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n).$$

If $a_i', b_i' \in [a_i, b_i]$ with $a_i' < b_i'$ for all $i \in \{1, 2, \ldots, n\}$, then f is Δ-integrable over $R' = [a_1', b_1') \times [a_2', b_2') \times \ldots \times [a_n', b_n')$.

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is Δ-integrable over R, there exists $P \in \mathcal{P}(R)$ given by (**??**) and (**??**) so that □

Proof.

$$U(f,P) - L(f,P) < \varepsilon.$$

Let $P' \in \mathcal{P}(R)$ be such that

$$P' = P \cup \{\{a'_1, b'_1\} \times \{a'_2, b'_2\} \times \ldots \times \{a'_n, b'_n\}\}.$$

Then P' is a refinement of P. Therefore,

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P).$$

Hence,

$$U(f,P') - L(f,P') \le U(f,P) - L(f,P) < \varepsilon.$$

□

Proof.

Now, consider $P'' \in \mathcal{P}(R)$ consisting of all subrectangles of P' belonging to R'. If $\tilde{U}$ and $\tilde{L}$ are the upper and lower Δ-sums of f on R' associated with the partition P'', then

$$\tilde{U} - \tilde{L} \leq U(f, P') - L(f, P') < \varepsilon.$$

Hence, by Theorem 7, f is Δ-integrable over R'. □

Theorem

Let f be a bounded function that is Δ-integrable on

$$R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n).$$

If $\alpha \in \mathbb{R}$, then αf is Δ-integrable on R and

$$\int_R \alpha f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = \alpha \int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta \tag{9}$$

Proof.

If $\alpha = 0$, then (9) is obvious as both sides of (9) are equal to zero in this case. □

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. We assume $\alpha \neq 0$.
1. Let $\alpha > 0$. Since f is Δ-integrable over R, there exists $P \in \mathcal{P}(R)$ given by (**??**) and (**??**) so that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{\alpha}.$$

Thus,

$$U(\alpha f, P) - L(\alpha f, P) = \alpha U(f, P) - \alpha L(f, P) < \varepsilon.$$

Hence, by Theorem 7, αf is Δ-integrable over R. Also, we have

$$\alpha L(f, P) = L(\alpha f, P) \leq U(\alpha f, P) = \alpha U(f, P),$$

whereupon □

Proof.

$$\alpha L(f) = L(\alpha f) \leq U(\alpha f) = \alpha U(f).$$

From here, using that $L(f) = U(f)$, we conclude (9).
2. Let $\alpha < 0$. Since f is Δ-integrable over R, there exists $P \in \mathcal{P}(R)$ such that

$$\frac{\varepsilon}{\alpha} < U(f, P) - L(f, P) < -\frac{\varepsilon}{\alpha}.$$

Thus,

$$U(\alpha f, P) - L(\alpha f, P) \leq -\alpha(U(f, P) - L(f, P)) < \varepsilon,$$

and hence, by Theorem 7, αf is Δ-integrable over R. As in the previous case, we get (9). The proof is complete. □

Theorem

If f and g are bounded functions that are Δ-integrable over

$$R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n),$$

then $f + g$ is Δ-integrable over R and

$$\int_R (f+g)(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n$$

$$= \int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n + \int_R g(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n$$

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f and g are Δ-integrable over R, there exists $P \in \mathcal{P}(R)$ given by (**??**) and (**??**) such that

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g,P) - L(g,P) < \frac{\varepsilon}{2}.$$

Because

$$U(f+g,P) \leq U(f,P) + U(g,P) \quad \text{and} \quad L(f+g,P) \geq L(f,P) + L(g,P), \tag{11}$$

we find □

Proof.

$$\begin{aligned} U(f+g,P) - L(f+g,P) &\leq U(f,P) + U(g,P) - L(f,P) - L(g,P) \\ &= U(f,P) - L(f,P) + U(g,P) - L(g,P) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, by Theorem 7, it follows that $f + g$ is Δ-integrable over R. □

Proof.

From (11), we get

$$U(f+g) \leq U(f) + U(g) \quad \text{and} \quad L(f+g) \geq L(f) + L(g),$$

whereupon

$$U(f) + U(g) = L(f) + L(g) = U(f+g) = L(f) + L(g),$$

i.e., (10) holds. □

Corollary

If f and g be bounded Δ-integrable over

$$R = [a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n)$$

and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is Δ-integrable over R and

$$\int_R (\alpha f + \beta g)(t_1, t_2, \ldots, t_n) \Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n$$

$$= \alpha \int_R f(t_1, t_2, \ldots, t_n) \Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n + \beta \int_R g(t_1, t_2, \ldots, t_n) \Delta_1 t_1 \Delta_2 t_2 \ldots$$

Proof.

Since f and g are Δ-integrable over R, by Theorem 49, we get that αf and βg are Δ-integrable over R and

$$\int_R \alpha f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = \alpha \int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots$$

$$\int_R \beta g(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n = \beta \int_R g(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots$$

From here and from Theorem 50, □

Proof.

we find that $\alpha f+\beta g$ is Δ-integrable over R and

$$\int_R (\alpha f+\beta g)(t_1,t_2,\dots,t_n)\Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n$$

$$= \int_R \alpha f(t_1,t_2,\dots,t_n)\Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n+\int_R \beta g(t_1,t_2,\dots,t_n)\Delta_1\Delta_2 t_2 .$$

$$= \alpha\int_R f(t_1,t_2,\dots,t_n)\Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n+\beta\int_R g(t_1,t_2,\dots,t_n)\Delta_1 t_1\Delta_2 t$$

completing the proof. □

Theorem

If f and g are bounded functions that are Δ-integrable over R with

$$f(t_1, t_2, \ldots, t_n) \leq g(t_1, t_2, \ldots, t_n) \quad \textit{for all} \quad (t_1, t_2, \ldots, t_n) \in R,$$

then

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \leq \int_R g(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n$$

Proof.

By Corollary 51, we have that $g - f$ is Δ-integrable over R. □

Proof.

Since $g - f$ is nonnegative on R, we have

$$L(g - f, P) \geq 0 \quad \text{for all} \quad P \in \mathcal{P}(R).$$

Hence, by Corollary 51, we get

$$\begin{aligned} 0 &\leq L(g - f, P) \\ &\leq \int_R (g(t_1, t_2, \ldots, t_n) - f(t_1, t_2, \ldots, t_n))\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \\ &= \int_R g(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n - \int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 . \end{aligned}$$

which completes the proof. □

Theorem

If f is a bounded function that is Δ-integrable over R, then so is $|f|$, and

$$\left|\int_R f(t_1, t_2, \dots, t_n)\Delta_1 t_1 \Delta_2 t_2 \dots \Delta_n t_n\right| \leq \int_R |f(t_1, t_2, \dots, t_n)|\Delta_1 t_1 \Delta_2 t_2 \dots \Delta$$

Proof.

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is Δ-integrable on R, there exists $P \in \mathcal{P}(R)$ given by (**??**) and (**??**) so that

$$U(f, P) - L(f, P) < \varepsilon.$$

□

Proof.

Let

$$\overline{M}_{j_1 j_2 \dots j_n} = \sup\{|f(t_1, t_2, \dots, t_n)| : (t_1, t_2, \dots, t_n) \in R_{j_1 j_2 \dots j_n}\},$$

$$\overline{m}_{j_1 j_2 \dots j_n} = \inf\{|f(t_1, t_2, \dots, t_n)| : (t_1, t_2, \dots, t_n) \in R_{j_1 j_2 \dots j_n}\}.$$

Thus,

$$\overline{M}_{j_1 j_2 \dots j_n} - \overline{m}_{j_1 j_2 \dots j_n}$$

$$= \sup\{|f(t_1, t_2, \dots, t_n)| - |f(t_1', t_2', \dots, t_n')| : (t_1, t_2, \dots, t_n), (t_1', t_2', \dots, t_n'$$

□

Proof.

$$\leq \sup\{|f(t_1, t_2, \ldots, t_n) - f(t_1', t_2', \ldots, t_n')| : (t_1, t_2, \ldots, t_n), (t_1', t_2', \ldots, t_n')$$

$$= M_{j_1 j_2 \ldots j_n} - m_{j_1 j_2 \ldots j_n}.$$

Therefore,

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \varepsilon.$$

Hence, using Theorem 7, we get that $|f|$ is Δ-integrable over R. Since □

Proof.

$$-f(t_1, t_2, \ldots, t_n) \leq |f(t_1, t_2, \ldots, t_n)| \quad \text{and} \quad f(t_1, t_2, \ldots, t_n) \leq |f(t_1, t_2, \ldots,$$

for all $(t_1, t_2, \ldots, t_n) \in R$, using Theorem 52, we get

$$-\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \leq \int_R |f(t_1, t_2, \ldots, t_n)|\Delta_1 t_1 \Delta_2 t_2 \ldots$$

and

$$\int_R f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n \leq \int_R |f(t_1, t_2, \ldots, t_n)|\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n$$

which completes the proof. □

Theorem

If f is a bounded function that is Δ-integrable over R, then so is f^2.

Proof.

Since $f^2 = |f||f|$, without loss of generality, we assume that f is nonnegative over R. Let

$$M_f = \sup\{f(t_1, t_2, \dots, t_n) : \ (t_1, t_2, \dots, t_n) \in R\}.$$

Let $\varepsilon > 0$ be arbitrarily chosen. Since f is Δ-integrable over R, there exists $P \in \mathcal{P}(R)$ given by (**??**) and (**??**) so that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{2M_f + 1}.$$

Hence,

$$U(f^2, P) - L(f^2, P)$$

$$= \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \dots \sum_{j_n=1}^{k_n} M^2_{j_1 j_2 \dots j_n} (t_1^{j_1} - t_1^{j_1 - 1})(t_2^{j_2} - t_2^{j_2 - 1}) \dots (t_n^{j_n} - t_n^{j_n - 1})$$

Proof.

$$-\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\dots\sum_{j_n=1}^{k_n} m_{j_1j_2\dots j_n}^2(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\dots(t_n^{j_n}-t_n^{j_n-1})$$

$$= \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\dots\sum_{j_n=1}^{k_n}(M_{j_1j_2\dots j_n}+m_{j_1j_2\dots j_n})(M_{j_1j_2\dots j_n}-m_{j_1j_2\dots j_n})$$

$$\times(t_1^{j_1}-t_1^{j_1-1})(t_2^{j_2}-t_2^{j_2-1})\dots(t_n^{j_n}-t_n^{j_n-1})$$

$$\leq 2M_f\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\dots\sum_{j_n=1}^{k_n}(M_{j_1j_2\dots j_n}-m_{j_1j_2\dots j_n})$$

□

Proof.

$$\times(t_1^{j_1} - t_1^{j_1-1})(t_2^{j_2} - t_2^{j_2-1})\dots(t_n^{j_n} - t_n^{j_n-1})$$

$$= \quad 2M_f(U(f,P) - L(f,P))$$

$$< \quad 2M_f\frac{\varepsilon}{2M_f+1}$$

$$< \quad \varepsilon,$$

which completes the proof. □