Long-time dynamics of water-wave models

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Math Seminar

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Outline

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Introduction

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A wave on a bounded domain

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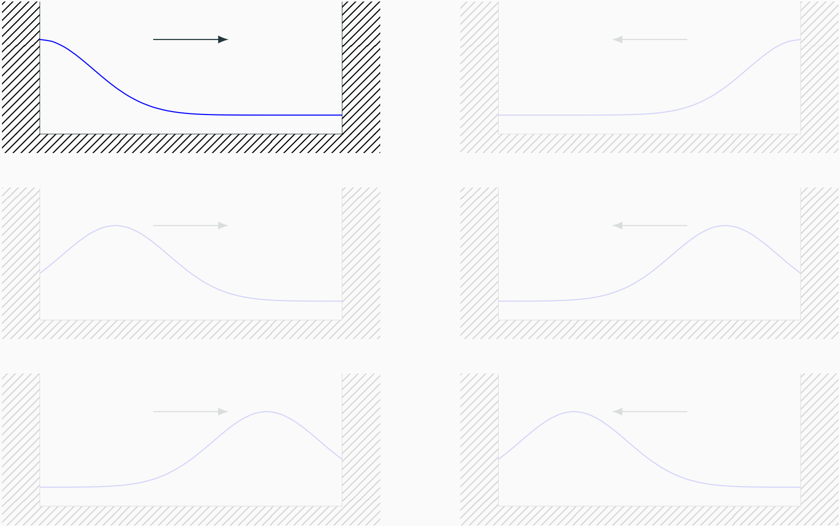

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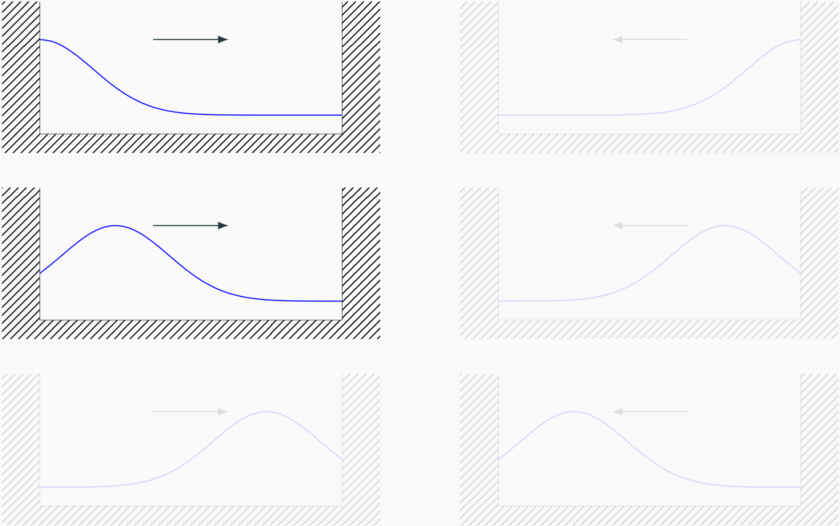

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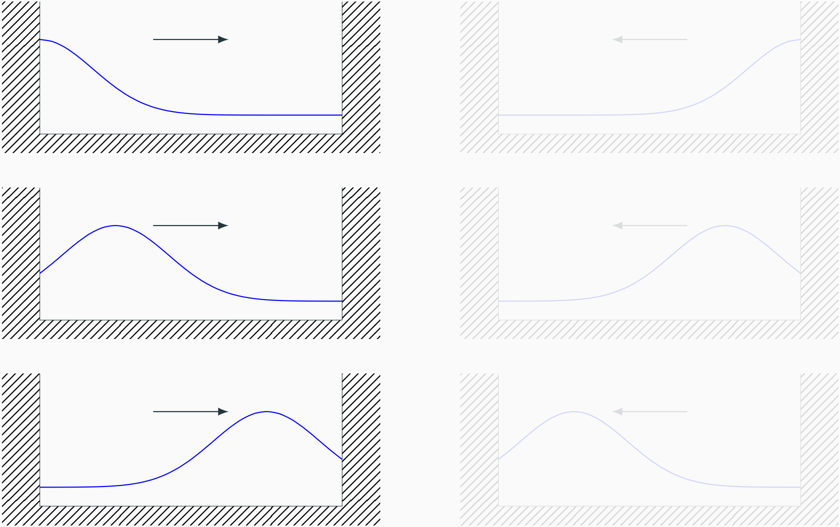

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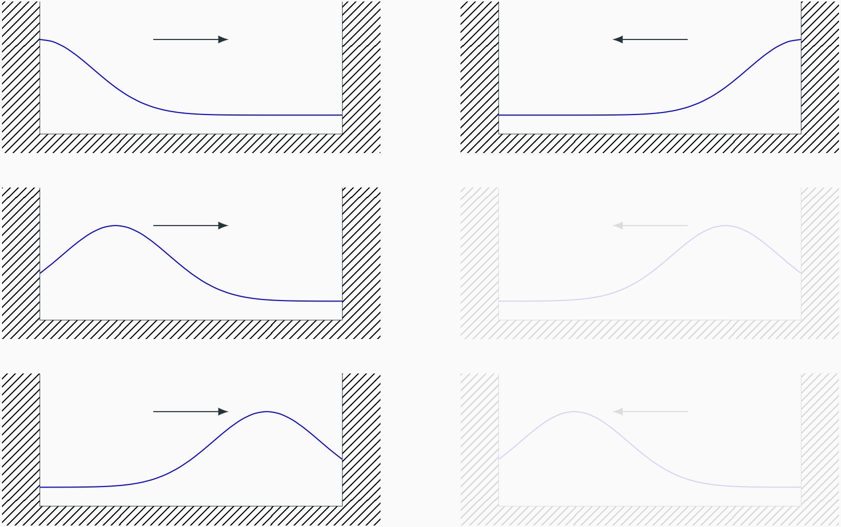

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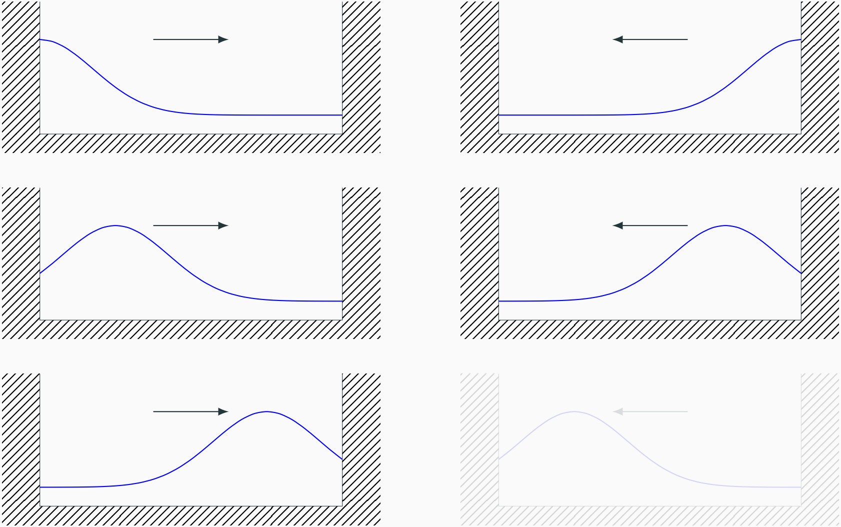

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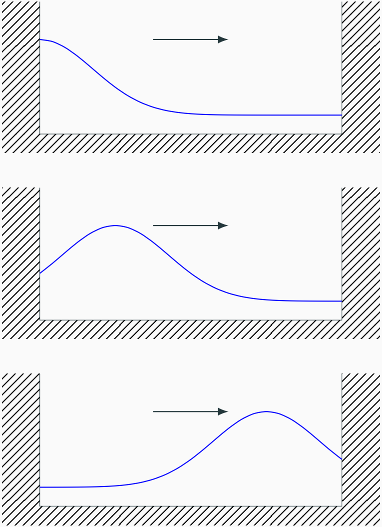
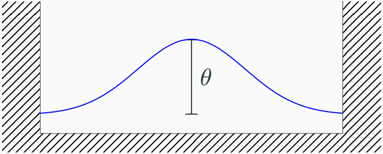


A wave on a bounded domain

A wave's potential energy is related to its position or configuration within a medium. It is often associated with particle displacement or wave amplitude for mechanical waves, such as ocean or sound waves.



The connection between potential energy and the $L^2(\Omega)$ norm arises from the fact that the potential energy of a wave system is often proportional to the square of the amplitude or displacement.

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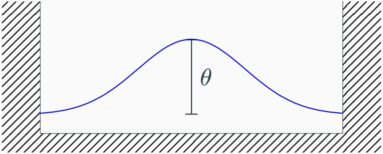

The connection between potential energy and the $L^2(\Omega)$ norm arises from the fact that the potential energy of a wave system is often proportional to the square of the amplitude or displacement.

A wave on a bounded domain

In a perfect world, free from losses, friction, or dissipation, energy would remain conserved as the wave evolves—remaining constant over time.

However, the real world is far from ideal. Mechanisms, both internal and external, influence the wave's behavior and, consequently, its energy. Understanding these factors is essential for addressing real-world applications. This brings us to the question:

How can we analyze the behavior of such systems?

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Introduction

Control and Stabilization: A general context

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Let $T > 0$, H and V be Hilbert spaces. Given $v \in L^2(0, T; V)$ (control) and $u_0 \in H$, we consider the solution $u\colon [0, T] \to H$ (state) of

$$\text{Control System} \quad \begin{cases} u_t = Au + Bv, \\ u(0) = u_0. \end{cases}$$

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The solution of the Control System is understood in the mild sense, that is,

$$u(t) = S(t)u_0 + \int_0^T S(t-s)Bu(s)\,ds, \quad 0 \le t \le T.$$

- $S(t)$: semigroup generated by A.
- $A\colon D(A) \to H$: linear operator.
- $B \in \mathcal{L}(V, D(A^*)')$: control operator.

Control System $\begin{cases} u_t = Au + Bv, \\ u(0) = u_0. \end{cases}$

Controllability

The possibility to find a time $T > 0$ and a control v allowing to bring the state u from a given initial state u_0 to a final state u_T at time T.

Control and Stabilization: A general context

By taking $v(t) = K(u(t))$, our control system turns into the Feedback system

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Stabilization (by feedback)

Now, our objective is to bring the state u close to the state u_T, that is,

$$\|u(t) - u_T\| \to 0 \quad \text{when} \quad t \to +\infty.$$

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Let $\bar{u}$ be an equilibrium solution for the uncontrolled case ($A\bar{u} \equiv 0$ and $K \equiv 0$), we said that $\bar{u}$ is

Exponentially stable

If there exists $\lambda > 0$ such that $\forall u_0 \in H$, the solution u satisfies

$$\|u(t) - \bar{u}\| \leq e^{-\lambda t}\|u_0 - \bar{u}\|.$$

Feedback damping input

Observe that a control input as the feedback allows a well-behaved solution and consequently, gives an exponential decay.

However, if we look at some scenarios in the real world, there exist inputs that may affect the behavior of the solutions.

Here, we focus on the **delay input**. Essentially, the meaning is related to postponing, hindering, or causing the system to act more slowly than normal.

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How does the delay work?

Systems of differential equations that are dependent on previous states are called delay differential equations (DDEs) or also referred to as retarded functional differential equations (RFDEs).

The simplest DDE

$$\begin{cases} u'(t) = -u(t-\tau), & t \geq 0, \\ u(t) = 1, & t \in [-\tau, 0]. \end{cases}$$

The Method of Steps looks for the solution $u(t)$ in each interval $[(n-1)\tau, n\tau]$ and allows to obtain the unique solution given by

$$u(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t-(k-1)\tau]^k}{k!}, \quad t \in [(n-1)\tau, n\tau]$$

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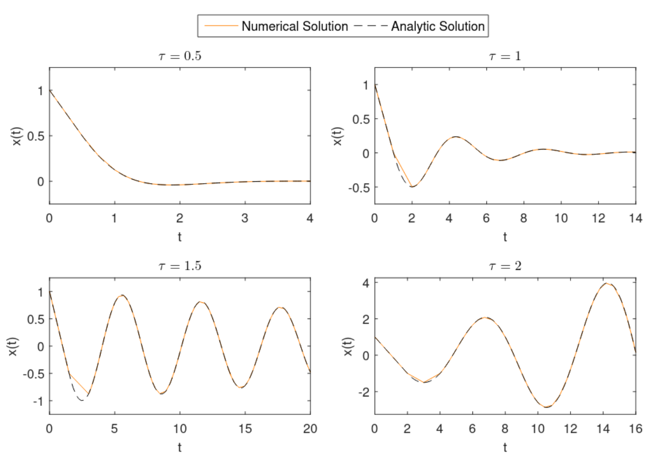

-Behavior of the system under different delays-

Introduction

Some comments on the used frameworks

Lyapunov's approach

- Define an appropriate Lyapunov Functional, namely V. This functional should ideally be positive-definite and designed as an equivalent energy perturbation.
- Ensure that there exists a constant $\lambda > 0$ such that

$$V'(t) \leq -\lambda V(t).$$

- The aforementioned inequality indicates that $V(t)$ decays over time, ideally leading to exponential stability.
- It is challenging to find suitable Lyapunov functionals. However, allows dealing directly with the nonlinear problem, gives an explicit decay rate, and obtains optimality results.
- Restrictions, such as the size of the interval lengths and small enough initial data, appear.

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Duality approach (Observability)

- The observability inequality provides a quantitative measure of how well the state of a system can be inferred over a finite time from certain inputs.
- If a system is observable, then it can often be controlled by manipulating inputs that influence the observed state.

$$\text{Stabilization} \equiv \text{Observability} \equiv \text{Controllability}$$

- A contradiction argument is used frequently, leading to some unique continuation problems that will require the use of sophisticated arguments.
- This approach gives a generic (but global) decay rate where the restriction on the size of the interval will not appear.
- Usually, linear and nonlinear problems should be treated separately.

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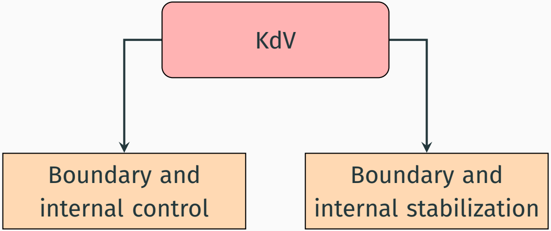
Introduction

Water waves models involved and outstanding results

Korteweg and de-Vries (1895)

$$u_t + 6uu_x + u_{xxx} = 0.$$



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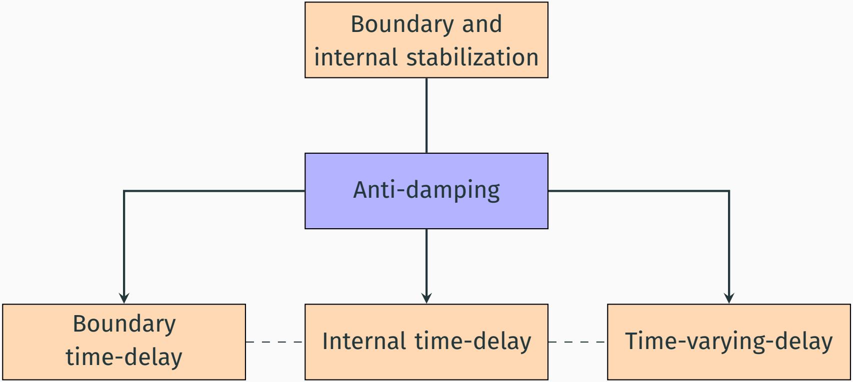
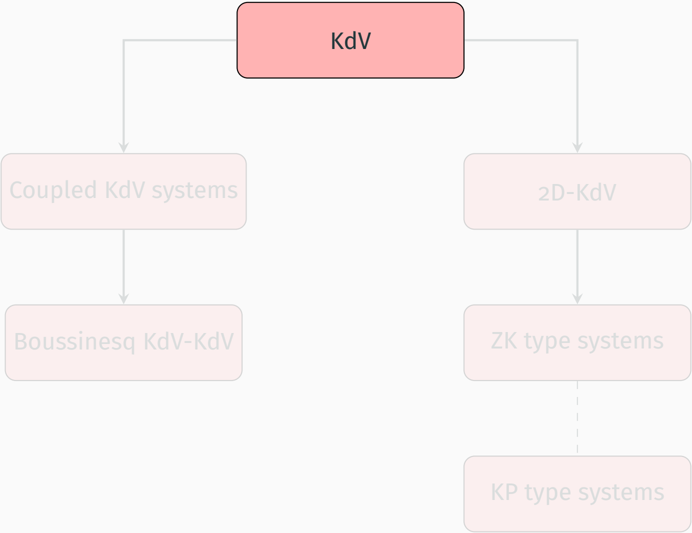


KdV-type equations



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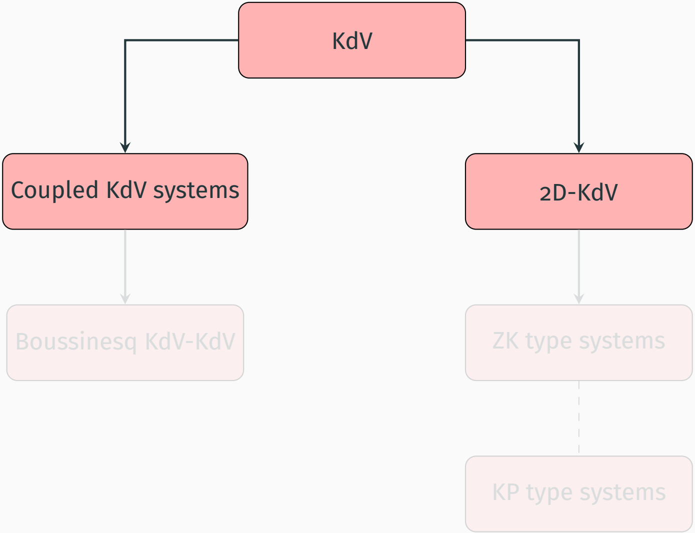

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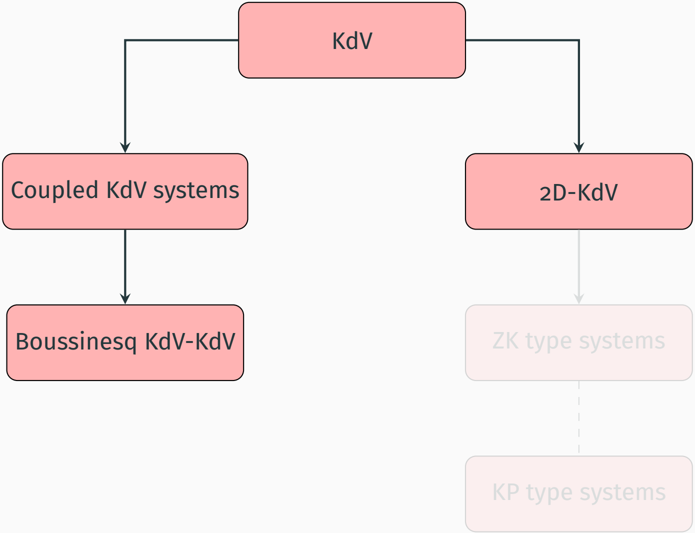

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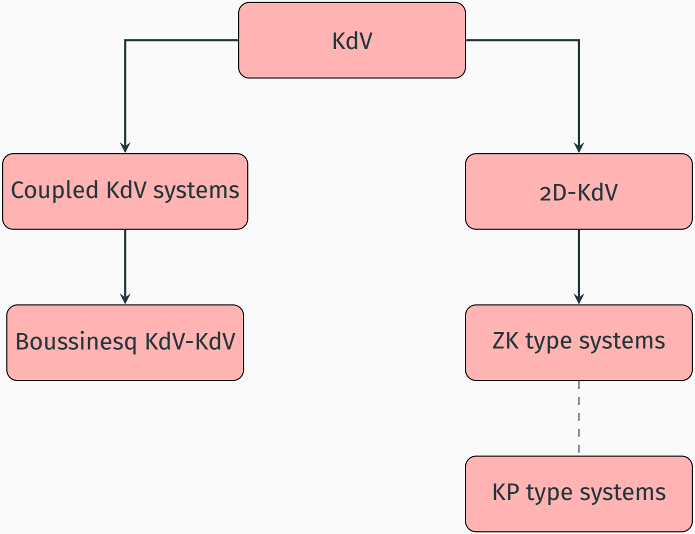


Boundary stabilization for coupled KdV-KdV type systems with time-delay

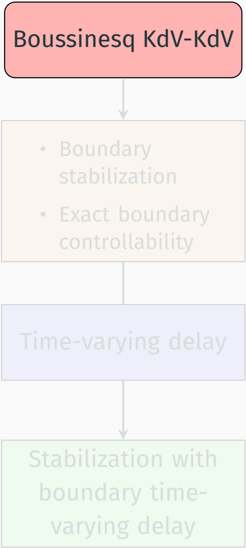
To understand the interaction between the KdV waves that arises

Boussinesq KdV-KdV

$$\begin{cases} \eta_t + \omega_x + \omega_{xxx} = -(\eta\omega)_x, \\ \omega_t + \eta_x + \eta_{xxx} = -\omega\omega_x. \end{cases}$$

Both systems are nonlinear dispersive PDEs that describe bidirectional wave propagation.

Coupled KdV systems



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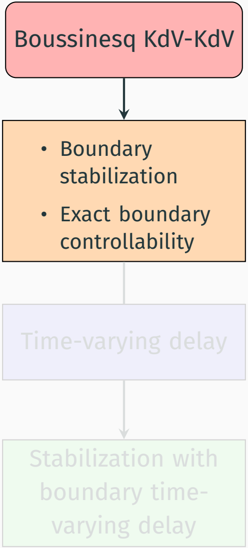

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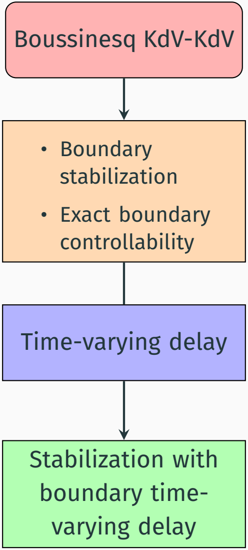

Boundary stabilization for coupled KdV-KdV type systems with time-delay

On the boundary stabilization of the KdV-KdV system with time-dependent delay

Stabilization of the KdV-KdV system

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On the boundary stabilization of the KdV–KdV system with time-dependent delay

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A R T I C L E I N F O

Keywords:
KdV–KdV equation
Stabilization
Decay rate
Lyapunov approach

A B S T R A C T

The boundary stabilization problem of the Boussinesq KdV–KdV type system is investigated in this paper. An appropriate boundary feedback law consisting of a linear combination of a damping mechanism and a delay term is designed. Then, considering time-varying delay feedback together with a smallness restriction on the length of the spatial domain and the initial data, we show that the problem under consideration is well-posed. The proof combines Kato's approach and the fixed-point argument. Last but not least, we prove that the energy of the linearized KdV–KdV system decays exponentially by employing the Lyapunov method.

Let us consider the Boussinesq system of the KdV-KdV type

$$\begin{cases} \eta_t + \omega_x + \omega_{xxx} = -(\eta\omega)_x, \\ \omega_t + \eta_x + \eta_{xxx} = -\omega\omega_x, \end{cases}$$

with the boundary conditions

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0, & t > 0, \\ \omega(t,0) = \omega(t,L) = \omega_x(t,L) = 0, & t > 0. \end{cases}$$

Stabilization of the KdV-KdV system

- The *Kato smoothing effect* does not hold for this boundary conditions.
- We do not have any control over the energy,

$$E_0(t) = \frac{1}{2}\int_0^L (\eta^2 + \omega^2)dx$$

in the sense that the time derivative

$$E_0'(t) = -\int_0^L (\eta\omega)_x \eta \, dx$$

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Some questions arises

- **Question A:** Is there a suitable set of boundary conditions so that the Kato smoothing effect can be revealed?
- **Question B:** Is there a feedback control law strong enough to allow describing a well behavior of the solutions in the presence of a time-dependent delay?
- **Question C:** If the answer to these previous questions is yes, does $E_0(t) \to 0$ as $t \to \infty$? If this is the case, can we give an explicit decay rate?

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Stabilization of the KdV-KdV system

To answer these questions, we design a boundary feedback law

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = \omega(t,0) = \omega(t,L) = 0, & t > 0 \\ \omega_x(t,L) = -\alpha\eta_x(t,L) + \beta\eta_x(t-\tau(t),L), & t > 0, \end{cases}$$

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where $\tau(t)$ is the time-varying delay, while α and β are feedback gains.

i. The feedback mechanism will guarantee the Kato smoothing effect, which is paramount to proving the well-posedness of the system.

ii. The time-varying delay feedback, together with the damping mechanism, gives the stabilization of the Boussinesq KdV-KdV system with an explicit decay rate.

Assumption 1

We assume that there exists M and $d < 1$ positive constants such that

$$\begin{cases} 0 < \tau(0) \leq \tau(t) \leq M, \quad \dot{\tau}(t) \leq d < 1, \quad t \geq 0, \\ \tau \in W^{2,\infty}([0,T]), \quad T > 0. \end{cases}$$

Furthermore, the feedback gains are related to

$$\alpha > \frac{|\beta|}{2}\left(\frac{2-d}{1-d}\right).$$

Under the Assumption 1, the Energy

$$E(t) = \frac{1}{2}\int_0^L \eta^2 + \omega^2\, dx + \frac{|\beta|}{2}\tau(t)\int_0^1 \eta_x^2(t-\tau(t)\rho, L)\, d\rho.$$

for the linearized system, it decays exponentially

Theorem 1

Let $0 < L < \sqrt{3}\pi$. Then, for two constants μ_1 and μ_2 small enough, there exist

$$0 < r < \frac{\mu_1(3\pi^2 - L^2)}{2\pi^2 L^{\frac{1}{2}}(2L+1)}, \qquad \kappa \leq 1 + \max\{2\mu_1 L, \mu_2\},$$

and

$$\lambda \leq \min\left\{ \frac{\mu_1\left(3\pi^2 - L^2\right) - 2r\pi^2 L^{\frac{1}{2}}(2L+1)}{L^2(1+2\mu_1 L)}, \frac{\mu_2(1-d)}{M(1+\mu_2)} \right\}$$

such that the the total energy $E(t)$ decays exponetially, provided that the initial data of the solution (η, ω) obeys $\|(\eta_0, \omega_0; z_0)\|_H \leq r$.

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and

$$\lambda \leq \min\left\{ \frac{\mu_1\left(3\pi^2 - L^2\right) - 2r\pi^2 L^{\frac{1}{2}}(2L+1)}{L^2(1+2\mu_1 L)}, \frac{\mu_2(1-d)}{M(1+\mu_2)} \right\}$$

such that the the total energy $E(t)$ decays exponetially, provided that the initial data of the solution (η, ω) obeys $\|(\eta_0, \omega_0; z_0)\|_H \leq r$.

Well-posedness

First, we look for the Cauchy problem for linear evolution equations

$$\frac{d}{dt}U(t) = A(t)U(t), \quad U(0) = U_0, \quad t > 0,$$

where $A(t)\colon D(A(t)) \subset H \to H$ is densely defined. If $D(A(t))$ is independent of time t, i.e., $D(A(t)) = D(A(0))$, for $t > 0$.

Theorem 2 (Kato, Linear evolution equations of hyperbolic type, 1970)

Assume that:

1. $\mathcal{Z} = D(A(0))$ *is a dense subset of* H *and* $D(A(t)) = D(A(0))$*, for all* $t > 0$.
2. $A(t)$ *generates a strongly continuous semigroup on* H*. Moreover, the family* $\{A(t) \colon t \in [0, T]\}$ *is stable with stability constants* $C,\ m$ *independent of* t.
3. $\partial_t A(t)$ *belongs to* $L^\infty_*([0, T], B(\mathcal{Z}, H))$*, the space of equivalent classes of essentially bounded, strongly measure functions from* $[0, T]$ *into the set* $B(\mathcal{Z}, H)$ *of bounded operators from* $\mathcal{Z}$ *into* H.

Then, Cauchy problem has a unique solution $U \in C([0, T], \mathcal{Z}) \cap C^1([0, T], H)$ *for any initial data in* $\mathcal{Z}$.

Well-posedness

Let us define $z(t,\rho) = \eta_x(t-\tau(t)\rho, L)$, such that

$$\begin{cases} \tau(t)z_t(t,\rho) + (1-\dot{\tau}(t)\rho)z_\rho(t,\rho) = 0, & t>0, \rho\in(0,1), \\ z(t,0) = \eta_x(t,L),\ z(0,\rho) = z_0(-\tau(0)\rho) & t>0,\ \rho\in(0,1). \end{cases} \tag{1}$$

Consider $H = L^2(0,L)\times L^2(0,L)\times L^2(0,1)$ equipped with the inner product

$$\langle (\eta,\omega,z), (\tilde{\eta},\tilde{\omega},\tilde{z})\rangle_t = \langle (\eta,\omega), (\tilde{\eta},\tilde{\omega})\rangle_{X_0} + |\beta|\tau(t)\langle z,\tilde{z}\rangle_{L^2(0,1)}. \tag{2}$$

Let $U = (\eta,\omega;z)^T$ and $A(t)\colon D(A(t))\subset H\to H$ given by

$$A(t)(\eta,\omega,z) := \left(-\omega_x-\omega_{xxx}, -\eta_x-\eta_{xxx}, \frac{\dot{\tau}(t)\rho-1}{\tau(t)}z_\rho\right) \tag{3}$$

with a domain defined by

$$D(A(t)) = \left\{ \begin{matrix} (\eta,\omega)\in\left[H^3(0,L)\cap H_0^1(0,L)\right]^2 \\ z\in H^1(0,1) \end{matrix} \middle| \begin{matrix} \eta_x(0)=0, z(0)=\eta_x(L), \\ \omega_x(L) = -\alpha\eta_x(L)+\beta z(1). \end{matrix} \right\}, \tag{4}$$

Let us define $z(t,\rho) = \eta_x(t - \tau(t)\rho, L)$, such that

$$\begin{cases} \tau(t) z_t(t,\rho) + (1 - \dot{\tau}(t)\rho) z_\rho(t,\rho) = 0, & t > 0, \rho \in (0,1), \\ z(t,0) = \eta_x(t,L),\ z(0,\rho) = z_0(-\tau(0)\rho) & t > 0,\ \rho \in (0,1). \end{cases} \tag{1}$$

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Well-posedness

- By definition $D(A(t)) = D(A(0))$, for all $t > 0$.
- Integrating by parts,

$$\langle A(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq \frac{1}{2}\begin{pmatrix}\eta_x(L)\\ z(1)\end{pmatrix}^T \Phi_{\alpha,\beta}\begin{pmatrix}\eta_x(L)\\ z(1)\end{pmatrix}$$

where

$$\kappa(t) = \frac{(\dot{\tau}(t)^2+1)^{\frac{1}{2}}}{2\tau(t)} \quad \text{and} \quad \Phi_{\alpha,\beta} = \begin{pmatrix}-2\alpha+|\beta| & \beta\\ \beta & |\beta|(d-1)\end{pmatrix}.$$

- $\Phi_{\alpha,\beta}$ negative definite matrix implies

$$\langle A(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq 0.$$

Thereby, $\tilde{A}(t) = A(t) - \kappa(t)I$ is dissipative.

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Claim

- $A(t)$ is maximal, or equivalently, for some $\lambda > 0$ we have that $\lambda I - A(t)$ is surjective.
- $\tilde{A}$ and $\tilde{A}^*$ are dissipative.

This follows using the idea of Capistrano-Filho, Pazoto, and Rosier, (ESAIM:COCV) 2019, which ensures the well-posedness by using the Lumer-Phillips Theorem.

Proposition (Kato Smoothing Effect)

Let Assumption 1 be satisfied. Then, the map

$$(\eta_0, \omega_0; z_0) \in H \mapsto (\eta, \omega; z) \in \mathcal{B} \times C(0, T; L^2(0, 1))$$

is well-defined, continuous, and fulfills

$$\|(\eta, \omega)\|_{X_0}^2 + |\beta| \|z\|_{L^2(0,1)}^2 \leq \|(\eta_0, \omega_0)\|_{X_0}^2 + |\beta| \|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2,$$

Furthermore, for every $(\eta_0, \omega_0, z_0) \in H$, we have that

$$\|\eta_x(\cdot, L)\|_{L^2(0,T)}^2 + \|z(\cdot, 1)\|_{L^2(0,T)}^2 \leq \|(\eta_0, \omega_0)\|_{X_0}^2 + \|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2.$$

Proposition (Kato Smoothing Effect)

Moreover, the Kato smoothing effect is verified

$$\int_0^T \int_0^L \eta_x^2 + \omega_x^2 \, dx \, dt \leq C(L, T, \alpha) \left(\|(\eta_0, \omega_0)\|_{X_0}^2 + \|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2 \right).$$

Finally, for the initial data, we have the following estimates

$$\begin{aligned} \|(\eta_0, \omega_0)\|_{X_0}^2 \leq & \frac{1}{T} \|(\eta, \omega)\|_{L^2(0,T;X_0)}^2 \\ & + (2\alpha + |\beta|) \|\eta_x(\cdot, L)\|_{L^2(0,T)}^2 + |\beta| \|z(\cdot, 1)\|_{L^2(0,1)}^2 \end{aligned}$$

and

$$\|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2 \leq C_1(d, M) \left(\|z(T, \cdot)\|_{L^2(0,1)} + \|z(\cdot, 1)\|_{L^2(0,T)}^2 \right).$$

Regularity estimates

The proof relies in:

- The suitable choice of the feedback gains that allows obtaining the symmetric negative definite matrix $\Phi_{\alpha,\beta}$.
- The application of a symmetric Morawetz multipliers. $\square$

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- The suitable choice of the feedback gains that allows obtaining the symmetric negative definite matrix $\Phi_{\alpha,\beta}$.
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- We can ensure the existence of solutions to the KdV–KdV system with source terms in $L^1(0, T; X_0)$.
- The solutions of the system are only local.
- Due to a lack of *a priori* L^2-estimate, the problem of the global existence of solutions is difficult to address in the energy space for the nonlinear system with a delay term.

Exponential decay for the solutions throughout Lyapunov's approach.

Let us consider $V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t)$ where $\mu_1, \mu_2 \in \mathbb{R}^+$ will be chosen later and

$$\begin{cases} V_1(t) = \dfrac{1}{2}\displaystyle\int_0^L x\eta(t,x)\omega(t,x)\,dx, \\ V_2(t) = \dfrac{|\beta|}{2}\tau(t)\displaystyle\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)\,d\rho. \end{cases}$$

Some calculations give

$$\begin{aligned} V'(t) + \lambda V(t) &= E'(t) + \mu_1 V_1'(t) + \mu_2 V_2'(t) + \lambda E(t) + \lambda\mu_1 V_1(t) + \lambda\mu_2 V_2(t) \\ &= M + S_1 + S_2. \end{aligned}$$

Linear stabilization result

Here,

$$M = \frac{1}{2}\left\langle \Psi_{\mu_1,\mu_2}(\eta_x(t,L),\eta_x(t-\tau(t),L)),(\eta_x(t,L),\eta_x(t-\tau(t),L))\right\rangle,$$

with

$$\Psi_{\mu_1,\mu_2} = \Phi_{\alpha,\beta} + L\mu_1\begin{pmatrix}\alpha^2+1 & -\alpha\beta\\ -\alpha\beta & \beta^2\end{pmatrix} + |\beta|\mu_2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

and

$$\begin{aligned}S_1 =& \frac{\mu_1}{2}\int_0^L\left(\omega^2+\eta^2\right)dx - \frac{3\mu_1}{2}\int_0^L\left(\omega_x^2+\eta_x^2\right)dx + \frac{\lambda}{2}\int_0^L\left(\eta^2+\omega^2\right)dx\\ &+\mu_1\lambda\int_0^L x\eta\omega dx,\\ S_2 =& -\mu_2\frac{|\beta|}{2}\int_0^1(1-\dot{\tau}(t)\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho + \frac{\lambda|\beta|}{2}\tau(t)\int_0^1\eta_x^2(t-\tau(t)\rho,L)d\rho\\ &+\frac{\mu_2|\beta|\lambda}{2}\tau(t)\int_0^1(1-\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho,\end{aligned}$$

Linear stabilization result

- As $\Phi_{\alpha,\beta}$ is negative definite, the continuity of the trace and determinant allows us to obtain that Ψ_{μ_1,μ_2} is negative definite and then $M \leq 0$.
- Simples calculations allow to choose explicitly the values of μ_1 and μ_2 such that Ψ_{μ_1,μ_2} is negative definite.
- $S_1 < 0$ and $S_2 < 0$ whenever

$$\lambda \leq \min\left\{ \frac{\mu_1(3\pi^2 - L^2)}{L^2(1+\mu_1)}, \frac{\mu_2(1-d)}{M(1+\mu_2)} \right\}.$$

Some remarks:

- Choosing $\mu_1 \in \left[0, \frac{(2\alpha-|\beta|)(1-d)-|\beta|}{L(1-d)(1+\alpha^2)}\right)$, we obtain the largest possible value for λ.
- Contrary to some relevant results, we do not have critical lengths as long as $L \in (0, \sqrt{3}\pi)$, and the **decay rate is obtained explicitly**.
- Taking more regular initial data to obtain the global solution of the nonlinear system can not be applied due to the time-dependent delay.

Stabilization for KP-type systems

Two dimensional KdV

It is natural to try to extend our understanding from one to two dimensions; in this sense, the introduction of a term that captures wave interaction in a wider plane appears

$$u_t + uu_x + u_{xxx} + \partial_x^{\pm 1} u_{yy} = 0,$$

KdV with a non-local term![1]

[1] where $\partial_x^{\pm 1}$ is the inverse spatial derivative with respect to x, meaning it integrates the function in the x direction

The KP equation

$$u_t + uu_x + u_{xxx} \pm \partial_x^{-1} u_{yy} = 0$$

where the negative term will refer to it as KP-I and the positive term will refer to it as KP-II and distinguish between the focusing and defocusing cases, respectively.

In the sequel, we consider the KP-II within a rectangular domain $\Omega := (0, L) \times (0, L)$, $L > 0$

$$u_t + u_x + u_{xxx} + \partial_x^{-1}(u_{yy}) = 0, \quad (x, y) \in \Omega,\ t \in (0, T). \tag{5}$$

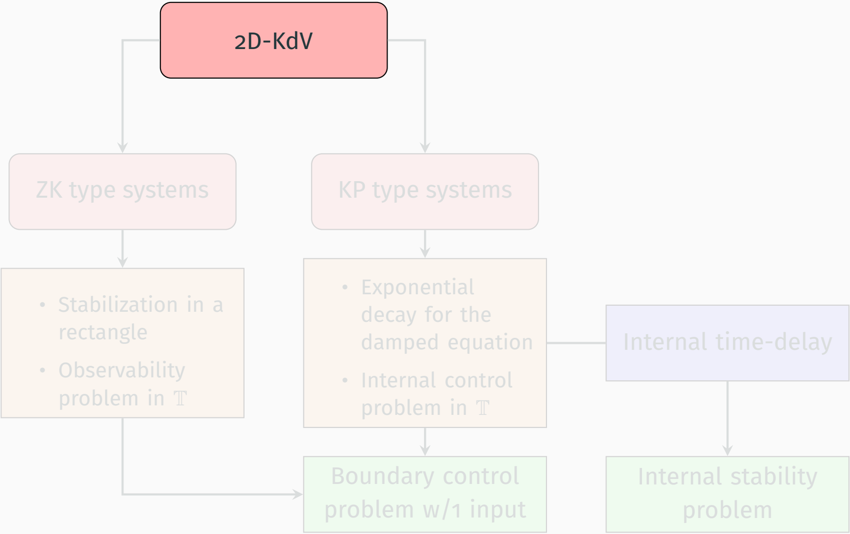
the operator ∂_x^{-1} is defined by $\partial_x^{-1} u(x, y, t) = -\int_x^L u(s, y, t)\, ds$, or equivalently,

$$\partial_x^{-1}\varphi(x, y, t) = \psi(x, y, t) \quad \text{s.t.} \quad \psi(L, y, t) = 0 \quad \text{and} \quad \psi_x(x, y, t) = \varphi(x, y, t).$$

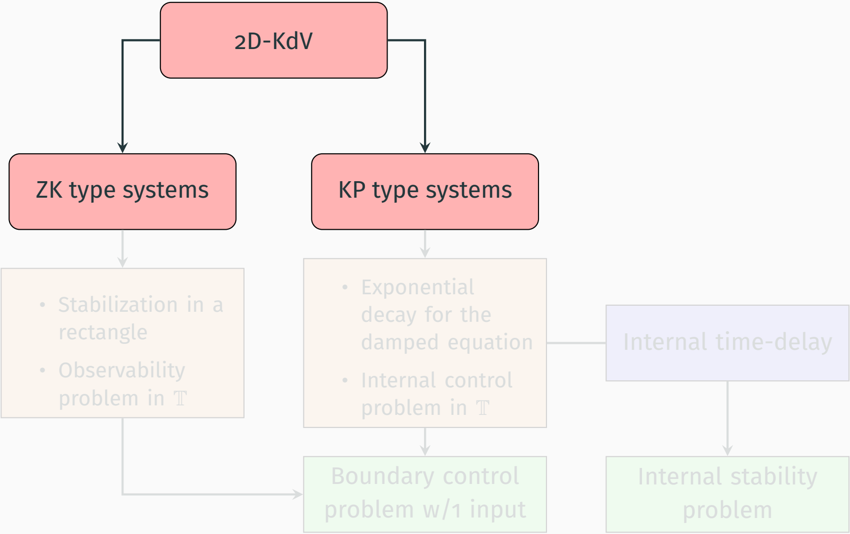
Two dimensional KdV

-Transversal waves at Île de Ré, France -

Two dimensional KdV



Two dimensional KdV



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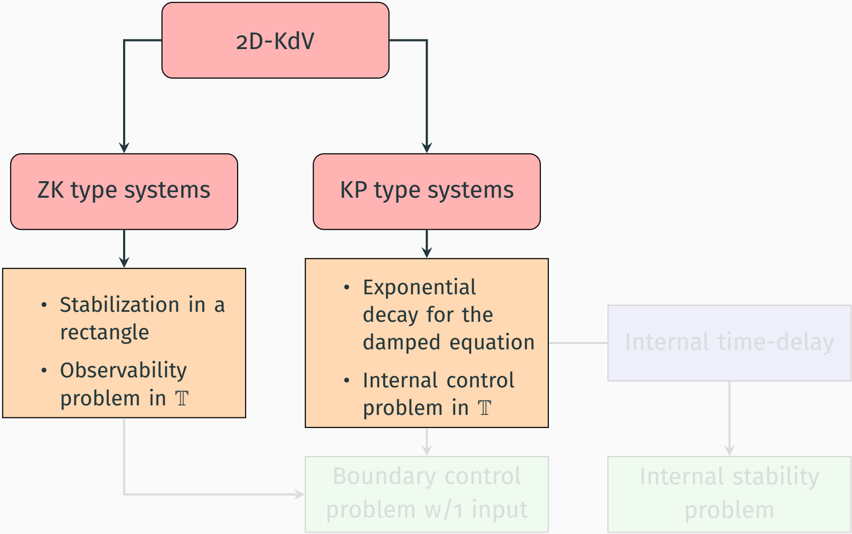

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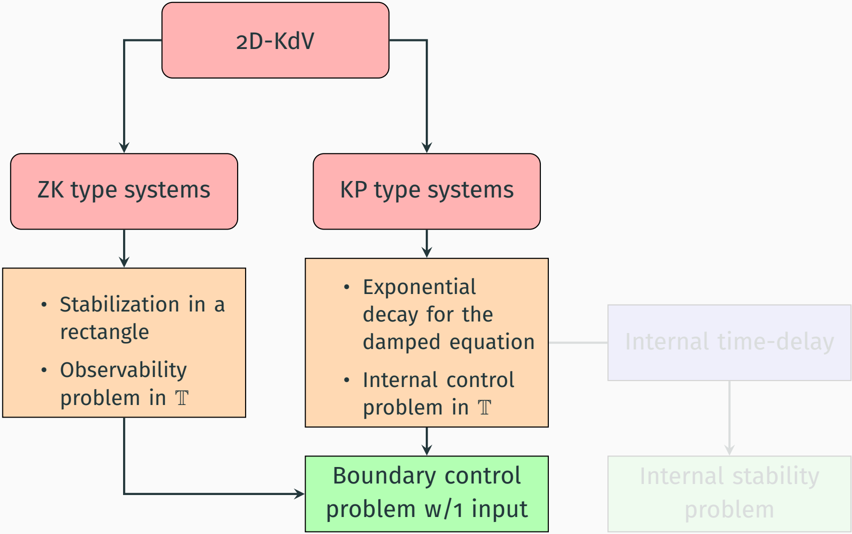

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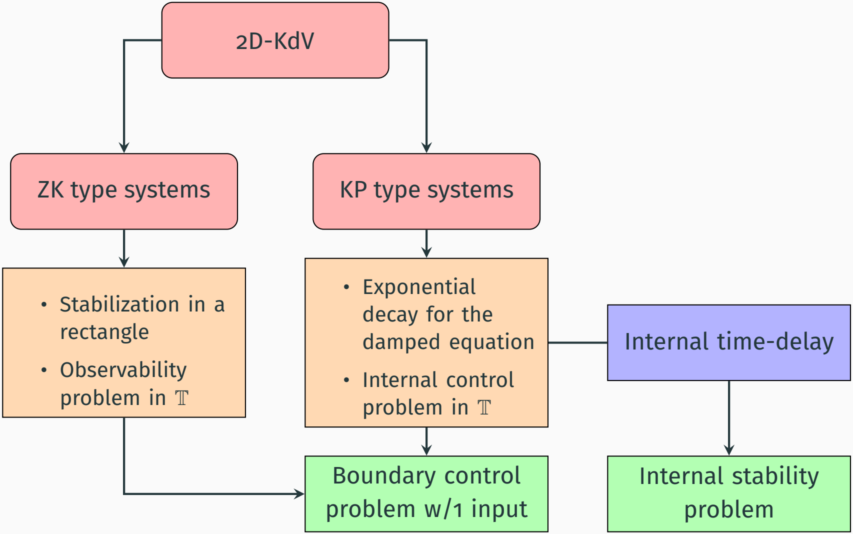

Stabilization for KP-type systems

Stabilization of the Kawahara–Kadomtsev–Petviashvili equation with time-delayed feedback

Exponential stabilization for the K-KP equation

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Stabilization of the Kawahara–Kadomtsev–Petviashvili equation with time-delayed feedback

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Results of stabilization for the higher order of the Kadomtsev-Petviashvili equation are presented in this manuscript. Precisely, we prove with two different approaches that under the presence of a damping mechanism and an internal delay term (anti-damping) the solutions of the Kawahara–Kadomtsev–Petviashvili equation are locally and globally exponentially stable. The main novelty of this work is that we present the optimal constant, as well as the minimal time, that ensures that the energy associated with this system goes to zero exponentially.

Now, we deal with the stabilization problem for the Higher-order KP-II with localized damping and delay terms posed on a bounded domain $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$:

$$\partial_t u + \alpha \partial_x^3 u + \beta \partial_x^5 u + \gamma \partial_x^{-1} \partial_y^2 u + a(x,y)u + b(x,y)u(x,y,t-h) = -\frac{1}{2}\partial_x(u^2) \quad (6)$$

and the equilibrium problem

$$\partial_t u + \alpha \partial_x^3 u + \beta \partial_x^5 u + \gamma \partial_x^{-1} \partial_y^2 u + a(x,y)\left(\mu_1 u + \mu_2 u(x,y,t-h)\right) = -\frac{1}{2}\partial_x(u^2) \quad (7)$$

where $\mu_1 > \mu_2$ are positive real numbers, $h > 0$ is the time delay, and $a(x,y), b(x,y) \in L^\infty(\Omega)$ non-negative real functions such that

$$a(x,y) \geq a_0 > 0 \quad \text{a.e. in } \omega \subset \Omega.$$

Both systems are equipped with homogeneous boundary conditions and initial data $u_0(x,y), z_0(x,y,t)$.

In this context, the energy associated with the solutions of the K-KP-II system

$$\begin{aligned} E_u(t) =& \frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy \\ &+\frac{h}{2}\int_0^L\int_0^L\int_0^1 b(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy. \end{aligned} \tag{6}$$

and

$$E_u(t) = \frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy + \frac{\xi}{2}\int_0^L\int_0^L\int_0^1 a(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy \tag{7}$$

is a non-increasing function where $h\mu_2 < \xi < h(2\mu_1 - \mu_2)$. Then, we focus on answering the next question:

Does $E_u(t) \to 0$ as $t \to \infty$? If this is the case, can we give the decay rate?

Theorem 3 (Optimal local stabilization)

Let $L > 0$, $\xi > 1$, $0 < \mu < 1$ and T_0 given by

$$T_0 = \frac{1}{2\theta} \ln\left(\frac{2\xi\kappa}{\mu}\right) + 1, \tag{8}$$

with $\theta = \frac{3\alpha\eta}{(1+2\eta L)L^2}$, $\kappa = 1 + \max\left\{2\eta L, \frac{\sigma}{\xi}\right\}$ and $\eta \in \left(0, \frac{\xi - 1}{2L(1+2\xi)}\right)$ satisfying

$$\frac{2\alpha\eta}{(2+2\eta L)L^2} = \frac{\sigma}{2h(\xi+\sigma)}$$

where $\sigma = \xi - 1 - 2L\eta(1+2\xi)$. Let $T_{\min} > 0$ given by

$$T_{\min} := -\frac{1}{\nu} \ln\left(\frac{\mu}{2}\right) + \left(\frac{2\|b\|_\infty}{\nu} + 1\right) T_0, \text{ with } \nu = \frac{1}{T_0} \ln\left(\frac{1}{(\mu+\varepsilon)}\right).$$

Then, there exists $\delta > 0$, $r > 0$, $C > 0$ and γ, depending on $T_{\min}, \xi, L, h$, such that if $\|b\|_\infty \leq \delta$, then for every $(u_0, z_0) \in \mathcal{H} = L^2(\Omega) \times L^2(\Omega \times (0,1))$ satisfying $\|(u_0, z_0)\|_{\mathcal{H}} \leq r$, the energy satisfies $E_u(t) \leq Ce^{-\gamma t} E_u(0)$, for all $t > T_{\min}$.

Some comments about the future

Coupled KdV-KdV systems

- How about the nonlinear problem for the Boussinesq KdV-KdV? Due to the lack of L^2-estimates, some hidden regularity properties will be required. An approach using the W_{bdr} operator will work.
- Extension to higher-order Boussinesq systems? Bautista, Capistrano-Filho, Chentouf and Sierra Fonseca (Asymptotic Analysis, 2026).
- As boundary damping is essential in the Boussinesq KdV-KdV, the interaction between internal and boundary feedback mechanisms will be analyzed. S. Majumdar, H. Parada for the saturated Kawahara equation.

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- Controllability for the Higher-order KP equation and characterization of their critical lengths. (In progress)
- Is it possible to design some feedback mechanism aimed at avoiding the critical length phenomena? F. A. Gallego and J. R. Munoz, Boundary Exponential Stabilization for the Linear KP-II equation without Critical Size Restrictions. (Submitted - arXiv:2511.17830 [math.AP])
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- Another type of stabilization is either bounded or unbounded domains.

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THANK YOU VERY MUCH!!!