Enumeration problem for Pell-Abel equations

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History and background

N.H.Abel, Sur l'intégration la formule différentielle $\frac{\rho dx}{\sqrt{R}}$, R et ρ étant des fonctions entières, J.Reine u. Angewand. Math., 1, pp. 105–144., 1826.

The reincarnation of diophantine Pell equation for polynomials

$$P^2(x) - D(x)Q^2(x) = 1 \tag{PA}$$

with the given $D(x) := \prod_{e \in \mathsf{E}} (x - e)$ monic complex polynomial of degree $\deg D = |\mathsf{E}| := 2g + 2$ without multiple roots.

Neighbours and Friends

Approximation theory (Least deviation problems)

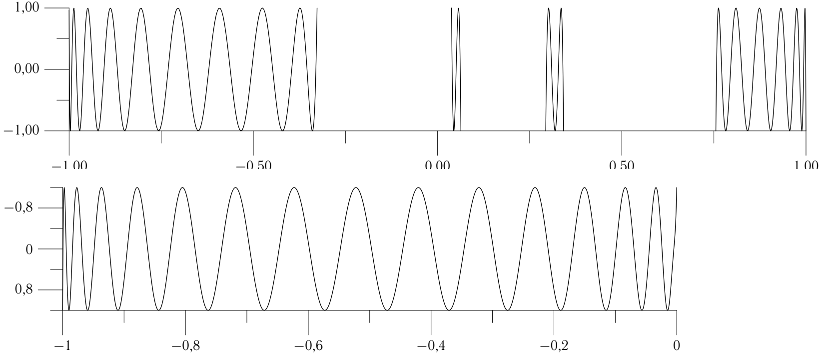


Figure: Four-band degree 29 Chebyshev polynomial (top).
Rescaled optimal stability polynomial of degree 31 and touching number 3 at zero (bottom)

Neighbours and Friends

Electrical engineering (multiband filters): optimization of gain-frequency characteristic for multiband electrical filters of various implementation

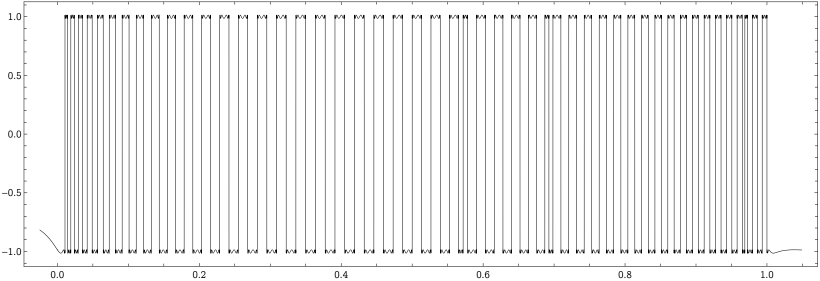


Figure: Optimal magnitude response of a filter with 99 bands

Other Neighbours and Friends

Reduction of Abelian functions (N.H.Abel, V.Enolskii, E.Belokolos, D.Masser, U.Zannier)
Algebraic geometry:
Spaces of differentials stratified by multiplicities of zeros (M.Kontsevich, A.Zorich, Q.Gendron..),
Betti maps (U.Zannier, F.Barroero, L.Capuano,..),
Frobenuis homomorphisms (J.-P.Serre)
Dynamical systems: Elliptic billiards, Poncelet porism (V.Dragovic, M.Radnovic, ..)
Spectral theory for periodic Jacobi matrices (N.I.Achiezer, P.Yuditskii, M.Sodin, A.Aptekarev, A. Poltoratski, C. Remling, ..)
Dessin d'Enfants and Belyi maps, Shabat polynomials (G.Shabat, E.Kreines, N.Adrianov, ..)

Enumeration problem statement

Trival solution $(P, Q) = (\pm 1, 0)$.
Nontrivial (P, Q) with minimal $\deg P =: n$ is called **primitive**.
It generates all the other solutions via the group structure:

$$(P, Q) * (P', Q') = (PP' + DQQ', PQ' + QP')$$

($=$ special case of Brahmagupta's identity, VI century AD)

Problem: Find the number of connected components in the space of PA equations with given degree coefficient $\deg D(x) =: 2g + 2$ admitting primitive solution of fixed degree $n := \min \deg P$.

Trivial cases:
$n < g + 1$, no nontrivial solutions
$g = 0$, e.g. $D(x) = x^2 - 1$, then $(P, Q) = (x, 1)$,
primitive solution with degree $n = 1$.

Main theorem

AB + QG :

Theorem

PA equation has no primitive solutions of degree $n < g + 1$ *or* $n > 1$ *when* $g = 0$. *Otherwise, the sought for number of components is equal to* $[m/2] + 1$ *if* $n + g$ *is odd and* $[(m+1)/2]$ *if* $n + g$ *is even. Here* $m = min(g, n - g - 1)$ *and* $[\cdot]$ *denotes the integer part.*

Applications:

1) Given degree g-extremal polynomials (U.Zannier: Almost Belyi map), number of components.

2) Topological Hurwitz numbers for special branching passport

3) Space of genus g hyperelliptic curves with a pair of points in involution mapped to n-torsion points of Jacobian.

4) Moduli space of primitive k-differentials with a unique zero of order $2k$ on curves of genus 2.

Solvability criterion

Consider the affine hyperelliptic curve

$$M := \{(x,w) \in \mathbb{C}^2 : \qquad w^2 = D(x) = \prod_{e\in E}(x-e)\} \tag{1}$$

and its two-point compactification $M_c := M \cup \{\infty_\pm\}$,
$w/x^{g+1}(\infty_\pm) = \pm 1$.
M_c bears the unique **3rd kind abelian differential**

$$d\eta_M = (x^g + \dots)w^{-1}dx \tag{2}$$

with purely imaginary periods and $Res\ d\eta_M|_{\infty_\pm} = \mp 1$.

Theorem

PA equation admits a nontrivial solution with deg $P = n$ iff **all the periods** *of $d\eta_M$ lie in the same* **lattice** $2\pi i\mathbb{Z}/n$*.*
If the latter holds the solution has the representation:

$$\begin{matrix} P(x) = & \pm\cos \\ Q(x) = & \pm i w^{-1}\sin \end{matrix} \left(in \int_{(e,0)}^{(x,w)} d\eta_M \right)$$

Criterion proof sketch

1) Once PA holds, $(P(x) + wQ(x)) = n(\infty_- - \infty_+)$ and $d\eta = \frac{1}{n} d \log(P + wQ)$
2) Conversely, if the lattice condition holds, $P(x)$ and $Q(x)$ are polynomials and Pythagoras Theorem $sin^2 z + cos^2 z = 1$ for complex z reads as PA equation.

Now we need a tool for the effective control of the periods map $\int_C d\eta_M$, $C \in H_1(M, \mathbb{Z})$, during the deformation of the curve M. Graph calculus can be used in a number of problems. In our case it gives the upper bound for the number of components.

Global width function

One immediately checks that the normalization conditions of $d\eta_M$ imply that the *width function*

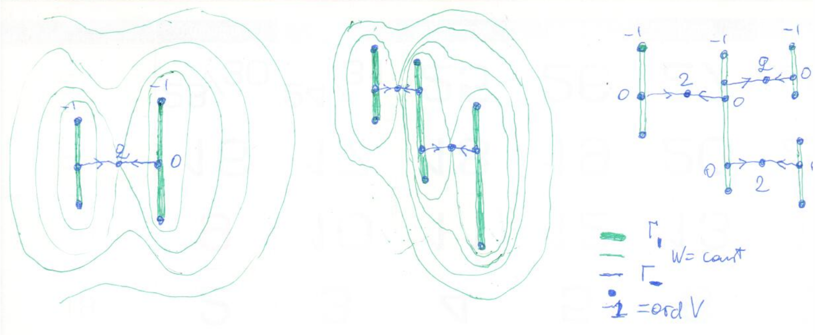
$$W(x) := |Re \int_{(e,0)}^{(x,w)} d\eta_M|, \qquad x \in \mathbb{C}, \tag{3}$$

obeys the following easily checked properties:

- W is single valued on the plane,
- W is harmonic outside its zero set $\Gamma_{\text{I}} := \{x \in \mathbb{C} : W(x) = 0\}$,
- W has a logarithmic pole at infinity,
- W vanishes at each branch point $e \in \mathrm{E}$

Graph calculus

To each hyperelliptic curve M we associate a weighted planar graph with edges of two types: 'horizontal' and 'vertical'. Typical graphs for $g = 1, 2, 3$.



Combinatorial formula for order of $(d\eta)^2$ at vertex V:

$$\mathrm{ord}(V) := d_{|}(V) + 2d_{in}(V) - 2,$$

Formal construction of a Graph

- Vertical edges are arcs of the zero set of $W(x)$; they are segments of the vertical foliation $d\eta_M^2 < 0$ and are not oriented.
- Horizontal edges are all segments of the horizontal foliation $(d\eta_M)^2 > 0$ (or steepest descent lines for $W(x)$) connecting saddle points of function W to the zero level set of W; oriented by the growth of $W(x)$.
- Each edge is equipped with its length in the metric $|d\eta_M|$ of quadratic differential.
- Vertexes of the graph Γ: all finite points of the divisor of $(d\eta_M)^2$ considered on the plane as well as points in $\Gamma_{|} \cap \Gamma_{\_}$ – projections of the saddle points of W to its zero set along the horizontal leaves.

Axiomatic description of Graphs

- (T1) The graph Γ is a tree.
- (T2) Horizontal edges leaving the same vertex are separated by a vertical or an incoming edge.
- (T3) If $\mathrm{ord}(V) = 0$ then $V \in \Gamma_{-} \cap \Gamma_{|}$.
- (W1) Width function increases along oriented edges and $W(V) = 0$ if V lies on the vertical part of the graph.
- (W2) The weights of vertical edges are positive and their total sum is π.

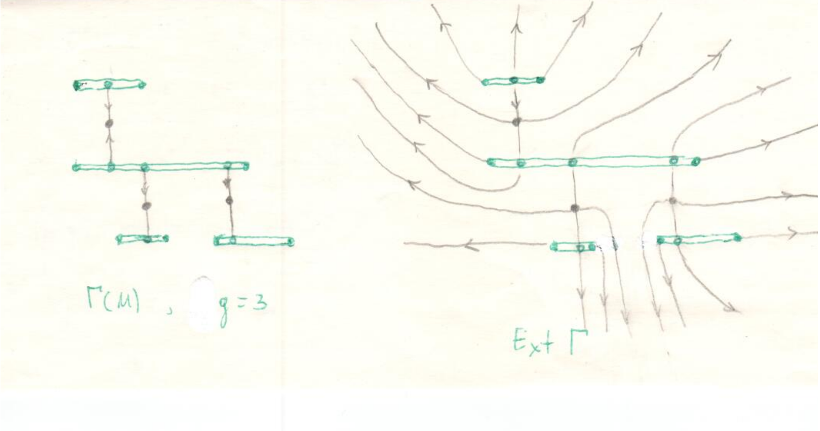
Theorem

Each planar graph satisfying 5 axioms stems from a unique element $M(E) \in \mathcal{H}$.

The branching divisor E *real analytically depends on positive weights of the graph*

NB: Number of graphs of fixed genus = Good enumeration problem!

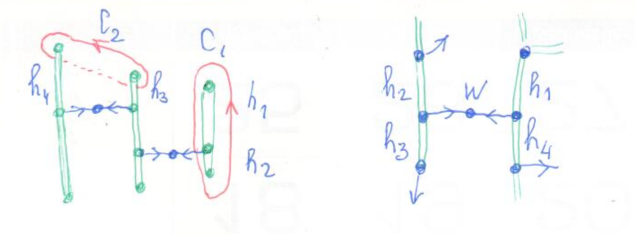
Reconstruction of a curve from its graph: sketch



Draw all horizontal leaves through vertexes of the graph. Abelian integral $\eta(x)$ maps each component of the complement to the extended graph to a horizontal half-stripe of known height. To reconstruct the curve, we glue half-stripes in a way dictated by combinatorics and weights of the graph $Ext\Gamma$.

Graphs, periods map and isoperiodic deformation

Periods of $d\eta_M$ are integer combinations of weights of vertical edges:



$$\int_{C_1} d\eta_M = 2i(h_1 + h_2), \qquad \int_{C_2} d\eta_M = 2i(h_3 + h_4).$$

Local **isoperiodic deformation** of a typical graph:

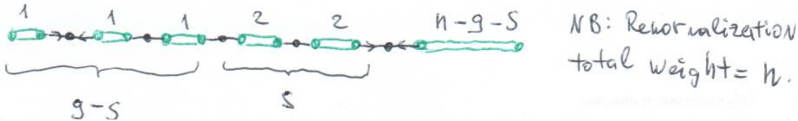
$$W \to W + \delta W; \qquad h_s \to h_s + (-1)^s \delta h, \quad s = 1, \ldots, 4.$$

Isoperiodic deformation to standard forms

Several standard forms of graphs may be used. E.g. one can take the graphs embedded to a line. They correspond to curves associated to multiband Chebyshev polynomials.

Lemma

1. *Any graph corresponding to PA equation admitting degree n primitive solution, may be transformed by isoperiodical transformations to a linear graph $\Gamma(g, n, s)$ with $0 \leq s \leq min(g-1, n-g-1)$.*
2. *Linear graphs (see Fig. below) $\Gamma(g, n, s)$ and $\Gamma(g, n, s-1)$ are in the same component if $n-g-s$ is odd.*
3. *Their total number is exactly as in the Main theorem.*



The theorem gives the upper bound for the sought number of connected components in the space of PA's

Lower bound: invariants of isoperiodic deformation

Solution $P(x)$ at a zero of the polynomial D may be either $+1$ or -1. Hence, the set E of zeros of $D(x)$ splits into two subsets $\mathsf{E}^{\pm}$. The degree partition invariant of solvable PA equation is the unordered pair $(|\mathsf{E}^{-}|, |\mathsf{E}^{+}|)$ computed for the primitive solution $\pm P(x)$. This invariant may be extracted from the graph Γ of solvable equation and takes different values at all nonequivalent standard graphs.

Lemma

The degree partition invariant $(|\mathsf{E}^{-}|, |\mathsf{E}^{+}|)$ of (PA) a degree n primitive solution satisfies: $n \geq |\mathsf{E}^{\pm}| > 0$, $|\mathsf{E}^{\pm}|$ and n have the same parity.

The number of admissible degree partitions is equal to the number of components in the main theorem. This invariant may be also expressed as the number of orbits of reduced mod 2 Burau representation of braids on $2g+1$ strands acting on binary arrays of length $2g+2$.

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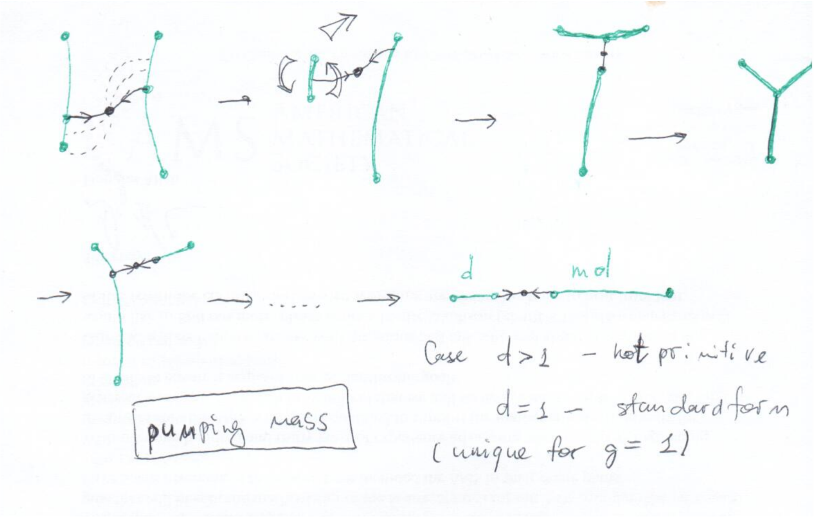
Thanks for the patience!

Bonus track: for the most stubborn

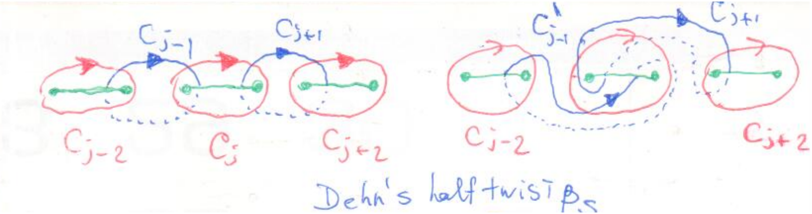


Bonus track 2: Reduced Burau representation

Let τ be an isoperiodic path connecting surfaces M_1 and M_2, $C \in H_1(M_1, \mathbb{Z})$ then we have the equality of the periods

$$\int_C d\eta_1 = \int_{C\cdot\tau} d\eta_2$$

where $d\eta_j$ is the differential associated to the curve M_j, $j = 1, 2$, and $C \cdot \tau$ is the cycle C transported from M_1 by GM connection.



Loops in the moduli space are braids on $2g + 2$ strands and they act on the homologies of curves by Burau representation

$$(C_1, \dots, C_{s-1}, C_s, C_{s+1}, \dots)\cdot\beta_s := (C_1, \dots, C_{s-1}+C_s, C_s, C_{s+1}-C_s, \dots),$$

Bonus track 2: Reduced Burau representation

Lemma

If two linear graphs Γ are connected by an isoperiodic transformation, then two binary strings

$$\{b_s := \frac{n}{2\pi i}\int_{C_s} d\eta \quad mod\ 2\}_{s=0}^{2g}$$

live in the same orbit of Burau representation reduced modulo 2.

Straight calculation shows that all above graphs live in different orbits of braid group action, hence lower bound = upper bound. Main theorem holds.