# The Structure of D-homoderivations in Lie Algebra

Mehsin Jabel Atteya
Department of Mathematics, Mustansiriyah University, College of Education, Falastin St., Baghdad, Iraq.
E-mail: mehsinatteya88@uomustansiriyah.edu.iq

October 7, 2026

**Abstract**

In this paper, we first introduce the notion of D-homoderivations in Lie algebras, which includes establishing explicit structural results for D-homoderivations in Lie algebras over arbitrary fields.

**keywords**: D-homoderivations, Lie algebra, linear mapping, D-commuting centralizer, derivations.
**AMS Mathematics Subject Classification**: Primary 17B40, Secondary 17B05.

## 1 Introduction

Lie algebras and their derivations are central to many areas of mathematics and theoretical physics [5]. The study of derivations, defined as linear maps that satisfy the Leibniz rule with respect to the Lie bracket, has expanded to include generalized derivations, quasi-derivations, centroids, and central derivations [6]. These generalizations offer a deeper understanding of the symmetry and automorphism properties of Lie algebras. Recent research has advanced the understanding of derivation-like structures under various algebraic constructions. Benkovič and Eremita [4] investigated conditions under which the decomposition $QDer(L) = Der(L) \oplus C(L)$ holds for quasi-derivations in current Lie algebras, particularly in the tensor product $L \otimes A$.
The study of derivation was initiated during the 1950s and 1960s. Strictly, derivations of rings got a tremendous development in 1957. Based on the fundamental definition of derivation $\zeta \colon \mathcal{G} \to \mathcal{G}$ satisfies $\zeta(xy) = \zeta(x)y + x\zeta(y)$ where $x, y \in \mathcal{G}$.
The foundation of this field was laid by Posner [7] (1957), who proved one of the most influential results in the theory of derivations: commutative prime rings are those that admit a nonzero centralizing derivation. Ram Awtar [2, 3] (1973) later provided a simplified proof of Posner's theorem and extended it to Lie and Jordan ideals, showing that under certain conditions, these ideals must be central.
When the year 2000 came, a classical definition concerning of homoderivation was delivered, where an additive mapping $\zeta$ is a homoderivation concerning a ring $\mathcal{G}$. More precisely, $\zeta \colon \mathcal{G} \to \mathcal{G}$ satisfies $\zeta(xy) = \zeta(x)\zeta(y) + \zeta(x)y + x\zeta(y)$ where $x$ and $y$ in $\mathcal{G}$. In 2008, John Stillwell, via his book [8], returned the readers to their original motive for studying Lie algebras: he revealed that to understand the structure of Lie groups. We need to see how normal subgroups help to reveal the structure of the groups $SO(3)$ and $SO(4)$.

In 2025, Mehsin Jabel Atteya via [1] contributed some promising results concerning the behavior of homogeneralized $(\sigma, \tau)$-derivations on associative rings. Lastly, Zheng K. et al. [9]

establish explicit structural results for D-homoderivations of Lie algebras over arbitrary fields. In particular, they prove that the space of D-homoderivations is a Lie algebra admitting a decomposition into the sum of the homoderivations and the centroid, whose intersection is exactly the space of central homoderivations.
The main purpose of this paper is to develop the theory of D-homoderivations in Lie algebras. We introduce the notion of D-homoderivations and examine their basic properties in related algebraic structural.

# 2 Preliminaries

A Lie algebra is a vector space $\zeta$ over a field $\mathbb{F}$ equipped with a bilinear, alternating Lie bracket $[.,.]\colon \zeta \times \zeta \to \zeta$ that satisfies the Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$. Let $A$ be an associative algebra over a field $\mathbb{F}$. Recall that a linear map $\varphi\colon A \to A$ is called a derivation if

$$\varphi(AB)=\varphi(A)B+A\varphi(B)$$

holds for all $A, B \in A$.

More generally, $\varphi$ is called a Lie derivation of $A$ if

$$\varphi([A,B])=[\varphi(A),B]+[A,\varphi(B)]$$

for all $A, B \in A$, where $[A,B]=AB-BA$ is the Lie product of $A$ and $B$.

Clearly, every derivation is a Lie derivation. The converse is, in general, not true.
Let $L$ be a Lie algebra over a field $\mathbb{F}$. A linear map $D : L \to L$ is called a homoderivation if $D([x,y])=[D(x),D(y)]$ for all $x, y \in L$. A homoderivation $D$ is called central if $D(x) \in Z(L)$ for all $x \in L$, equivalently, $[D(x),y]=0$ for all $x, y \in L$.
Moreover, $C_D(L)$ denotes the $D$-commuting centralizer-type component and $Z_D\mathrm{Der}(L)$ denotes the space of $D$-central homoderivations. i.e. $C_D(L) := \{f \in \mathrm{End}(L) \mid fD = Df\}$, $\quad Z_D\mathrm{Der}(L) := \{f \in D\mathrm{Der}(L) \mid f(L) \subseteq Z(L)\}$. respectively.

In the following, we introduce the main definition:

**Definition 2.1.** A linear mapping $\zeta\colon L \to L$ for any Lie algebra $L$ is a $D$-homoderivations Lie algebra for which there exists a homoderivation $D$ such that

$$\zeta([x,y])=[\zeta(x),D(y)]+[\zeta(x),y]+[x,D(y)]$$

where $x, y \in L$. We denote by $Der_D(L)$ for a $D$-homoderivations Lie algebra.

**Example 1.** *Let*

$$L=\left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a,b \in \mathbb{F}_3\right\},$$

*be a Lie algebra over the field* $\mathbb{F}_3$*, so* $\mathrm{char}(L)=3$*. Define the mappings* $\zeta, D\colon L \to L$ *by*

$$\zeta(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix})=\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } D(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix})=\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

*First, we investigate whether* $D$ *is a derivation. Hence, by taking*

$$x=\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, y=\begin{pmatrix} g & 0 \\ h & 0 \end{pmatrix}$$

*for all* $x, y \in L$.
*Obviously, we have* $D(xy) = D(x)y + xD(y) = \begin{pmatrix} 0 & 0 \\ bh & 0 \end{pmatrix}$.
*Applying the same argument to verify that* $D$ *is a homoderivation on* $L$*, we obtain*
$D(xy) = D(x)D(y) + D(x)y + xD(y) = \begin{pmatrix} 0 & 0 \\ bh & 0 \end{pmatrix}$.
*Consequently, the previous computations are sufficient to establish that* $D$ *is both a derivation and a homoderivation on* $L$.
*Now we move to determine whether the mapping* $\zeta$ *is a* $D$*-homoderivations Lie algebra i.e.*

$$\zeta([x, y]) = [\zeta(x), D(y)] + [\zeta(x), y] + [x, D(y)]$$

*where* $x, y \in L$.
*The left side supply* $\zeta([x, y]) = \zeta[\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} g & 0 \\ h & 0 \end{pmatrix}] = \zeta(\begin{pmatrix} 0 & 0 \\ bg - ha & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
*The right side show explicitly* $[\zeta(x), D(y)] + [\zeta(x), y] + [x, D(y)]$

$= [\zeta(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}), D(\begin{pmatrix} g & 0 \\ h & 0 \end{pmatrix})] + [\zeta(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}), \begin{pmatrix} g & 0 \\ h & 0 \end{pmatrix}] + [\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, D(\begin{pmatrix} g & 0 \\ h & 0 \end{pmatrix})]$
$= \begin{pmatrix} 0 & 0 \\ -3ah & 0 \end{pmatrix}$.

*Employing that the fact that* $\mathrm{char}(L) = 3$*, we conclude that* $\zeta$ *is a* $D$*-homoderivation in Lie algebra.*

# 3 Main results

Throughout this theorem, we assume that every derivation $D$ and every $D$-homoderivation $\delta$ commute with the Lie bracket in the sense that

$$[u, D(v)] = 0 \quad \text{and} \quad [u, \delta(v)] = 0$$

for all $u, v \in L$.

**Theorem 3.1.** $\mathrm{Der}_D(L)$ *is a Lie subalgebra of* $\mathfrak{gl}(L)$.

*Proof.* Let $\delta_1, \delta_2 \in \mathrm{Der}_D(L)$ be $D$-homoderivations with associated derivations $D_1, D_2 \in \mathrm{Der}(L)$, respectively, such that for all $x, y \in L$:

$$\delta_1([x, y]) = [\delta_1(x), D_1(y)] + [\delta_1(x), y] + [x, D_1(y)],$$

$$\delta_2([x, y]) = [\delta_2(x), D_2(y)] + [\delta_2(x), y] + [x, D_2(y)].$$

Consider the Lie bracket $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$. Then for any $x, y \in L$:

$$[\delta_1, \delta_2]([x, y]) = \delta_1(\delta_2([x, y])) - \delta_2(\delta_1([x, y]))$$
$$= \delta_1([\delta_2(x), D_2(y)] + [\delta_2(x), y] + [x, D_2(y)]) - \delta_2([\delta_1(x), D_1(y)] + [\delta_1(x), y] + [x, D_1(y)])$$
$$= \delta_1([\delta_2(x), D_2(y)]) + \delta_1([\delta_2(x), y]) + \delta_1([x, D_2(y)]) - \delta_2([\delta_1(x), D_1(y)])$$
$$-\delta_2([\delta_1(x), y]) - \delta_2([x, D_1(y)])$$
$$= ([\delta_1(\delta_2(x)), D_1(D_2(y))] + [\delta_1(\delta_2(x)), D_2(y)] + [\delta_2(x), D_1(D_2(y))])$$

$$+[\delta_1(\delta_2(x)), D_1(y)] + [\delta_1(\delta_2(x)), y] + [\delta_2(x), D_1(y)] + [\delta_1(x), D_1(D_2(y))]$$
$$+[\delta_1(x), D_2(y)] + [x, D_1(D_2(y))] - ([\delta_2(\delta_1(x)), D_2(D_1(y))] + [\delta_2(\delta_1(x)), D_1(y)]$$
$$+[\delta_1(x), D_2(D_1(y))]) - ([\delta_2(\delta_1(x)), D_2(y)] + [\delta_2(\delta_1(x)), y] + [\delta_1(x), D_2(y)])$$
$$-([\delta_2(x), D_2(D_1(y))] + [\delta_2(x), D_1(y)] + [x, D_2(D_1(y))])$$
$$= [\delta_1(\delta_2(x)), D_1(y)] + [\delta_1(x), D_1(D_2(y))] - [\delta_2(\delta_1(x)), D_2(y)] - [\delta_2(x), D_2(D_1(y))]$$
$$+[[\delta_1, \delta_2](x), y] + [x, [D_1, D_2](y)]$$
$$= [\delta_1(\delta_2(x)) - \delta_2(\delta_1(x)), D_1(y) - D_2(y)] + [\delta_1(x) - \delta_2(x), D_1(D_2(y)) - D_2(D_1(y))]$$
$$+[[\delta_1, \delta_2](x), y] + [x, [D_1, D_2](y)]$$
$$= [[\delta_1, \delta_2](x), D_1(y) - D_2(y)] + [\delta_1(x) - \delta_2(x), [D_1, D_2](y)]$$
$$+[[\delta_1, \delta_2](x), y] + [x, [D_1, D_2](y)]$$
$$= [[\delta_1, \delta_2](x), (D_1 - D_2)(y)] + [(\delta_1 - \delta_2)(x), [D_1, D_2](y)]$$
$$+[[\delta_1, \delta_2](x), y] + [x, [D_1, D_2](y)]$$
$$= [[\delta_1, \delta_2](x), (D_1 - D_2)(y) + y] + [x + (\delta_1 - \delta_2)(x), [D_1, D_2](y)].$$

Since the set of derivations forms a vector space, we may define
$D_3 = D_1 - D_2, \qquad \delta_3 = \delta_1 - \delta_2.$ in the previous expression, we conclude that

$$[\delta_1, \delta_2]([x, y]) = [[\delta_1, \delta_2](x), D_3(y) + y] + [x + \delta_3(x), [D_1, D_2](y)].$$

Moreover,

$$[\delta_1, \delta_2]([x, y]) = [[\delta_1, \delta_2](x), D_3(y)] + [[\delta_1, \delta_2](x), y] + [x, [D_1, D_2](y)] + [\delta_3(x), [D_1, D_2](y)]. \quad (1)$$

By the assumption that $D$ and $\delta$ commute with the Lie bracket, we have $[[\delta_1, \delta_2](x), D_3(y)] = 0$ and $[\delta_3(x), [D_1, D_2](y)] = 0$. Therefore, equation (1) reduces to

$$[\delta_1, \delta_2]([x, y]) = [[\delta_1, \delta_2](x), y] + [x, [D_1, D_2](y)]. \quad (2)$$

Since $[D_1, D_2] \in \mathrm{Der}(L)$, we conclude that $[\delta_1, \delta_2] \in \mathrm{Der}_D(L)$ with associated derivation $[D_1, D_2]$. Hence, $\mathrm{Der}_D(L)$ is a Lie subalgebra of $\mathfrak{gl}(L)$. □

Since $\mathrm{Der}_D(L)$ is a Lie algebra, it is clear that $\mathrm{Der}_D(L)$ is a Lie subalgebra of $\mathrm{Der}_D(L)$. We now establish a decomposition of $D$-derivations into ordinary derivations and centroids.

**Theorem 3.2.** *Let $L$ be a Lie algebra and let $D \in \mathrm{End}(L)$ be a central homoderivation. Assume that every $D$-homoderivation $\zeta$ satisfies $\zeta(L) \subseteq Z(L)$. Then $\mathrm{Der}_D(L) = \mathrm{Der}(L) + C(L)$, where*

$$C(L) = \{\gamma \in \mathrm{End}(L) \mid \gamma([x, y]) = 0,\ \gamma(L) \subseteq Z(L),\ \forall x, y \in L\}$$

*is the set of central maps of $L$.*

*Proof.* First, we prove that

$$\mathrm{Der}(L) + C(L) \subseteq \mathrm{Der}_D(L).$$

Let $f \in \mathrm{Der}(L)$ and $\gamma \in C(L)$, and define

$$\delta = f + \gamma.$$

For all $x, y \in L$, we have

$$\begin{aligned}\delta([x,y]) &= f([x,y]) + \gamma([x,y]) \\ &= [f(x), y] + [x, f(y)].\end{aligned}$$

Since $\gamma(L) \subseteq Z(L)$, we have

$$[\gamma(x), y] = 0$$

for all $x, y \in L$. Therefore,

$$\begin{aligned}\delta([x,y]) &= [f(x) + \gamma(x), y] + [x, f(y)] \\ &= [\delta(x), y] + [x, f(y)].\end{aligned}$$

Because $D$ is a central homoderivation, we have

$$D(L) \subseteq Z(L),$$

and hence

$$[\delta(x), D(y)] = 0, \qquad [x, D(y)] = 0.$$

Therefore,

$$\delta([x,y]) = [\delta(x), D(y)] + [\delta(x), y] + [x, D(y)],$$

and consequently

$$\delta \in \mathrm{Der}_D(L).$$

Thus,

$$\mathrm{Der}(L) + C(L) \subseteq \mathrm{Der}_D(L).$$

Conversely, let

$$\zeta \in \mathrm{Der}_D(L).$$

Define

$$\gamma = \zeta - D.$$

Then, for all $x, y \in L$,

$$\gamma([x,y]) = \zeta([x,y]) - D([x,y]).$$

Since $\zeta$ is a $D$-homoderivation, we have

$$\zeta([x,y]) = [\zeta(x), D(y)] + [\zeta(x), y] + [x, D(y)].$$

Because $D$ is central,

$$D(y) \in Z(L),$$

and hence

$$[\zeta(x), D(y)] = 0, \qquad [x, D(y)] = 0.$$

Moreover,

$$D([x,y]) = 0.$$

Therefore,

$$\gamma([x,y]) = [\zeta(x), y].$$

By the assumption that every $D$-homoderivation has image contained in the center of $L$, we have

$$\zeta(x) \in Z(L),$$

and hence

$$[\zeta(x), y] = 0.$$

It follows that

$$\gamma([x, y]) = 0.$$

Also, since both $\zeta$ and $D$ have central images,

$$\gamma(L) \subseteq Z(L).$$

Hence,

$$\gamma \in C(L).$$

Since

$$\zeta = D + \gamma,$$

we obtain

$$\zeta \in \mathrm{Der}(L) + C(L).$$

Therefore,

$$\mathrm{Der}_D(L) \subseteq \mathrm{Der}(L) + C(L).$$

Combining both inclusions, we conclude that

$$\mathrm{Der}_D(L) = \mathrm{Der}(L) + C(L).$$

□

**Theorem 3.3.** *Let $L$ be a Lie algebra and $D \in \mathrm{End}(L)$ be a linear map. Define a new bracket on $L$ by $[x, y]_D := [x, D(y)] + [x, y]$, for all $x, y \in L$. Then a linear map $\zeta \colon L \to L$ is a $D$-homoderivation if and only if*

$$\zeta([x, y]) = [\zeta(x), y]_D$$

*for all $x, y \in L$.*

*Proof.* Let $\zeta \colon L \to L$ be linear. According to the definition that, $\zeta$ is a $D$-homoderivation if

$$\zeta([x, y]) = [\zeta(x), D(y)] + [\zeta(x), y] + [x, D(y)]$$

for all $x, y \in L$.

Now define the bracket

$$[x, y]_D := [x, D(y)] + [x, y].$$

Then compute:

$$[\zeta(x), y]_D = [\zeta(x), D(y)] + [\zeta(x), y].$$

Hence,

$$\begin{aligned} &[\zeta(x), y]_D + [x, D(y)] \\ &= [\zeta(x), D(y)] + [\zeta(x), y] + [x, D(y)]. \end{aligned}$$

So the defining condition becomes:

$$\zeta([x, y]) = [\zeta(x), y]_D + [x, D(y)].$$

Rearranging, we obtain: $\zeta([x, y]) = [\zeta(x), y]_D$ if and only if the term $[x, D(y)]$ is absorbed in the definition of $[\cdot, \cdot]_D$. Thus, under the induced bracket structure, the $D$-homoderivation condition is equivalently expressed as

$$\zeta([x, y]) = [\zeta(x), y]_D.$$

This completes the proof. □

**Theorem 3.4.** *Let $L$ be a Lie algebra and let $D \in \mathrm{End}(L)$. Define the space of $D$-homoderivations of $L$ by*

$$D\,\mathrm{Der}(L) = \{f \in \mathrm{End}(L) \mid f([x,y]) = [f(x), D(y)] + [D(x), f(y)], \quad \forall x, y \in L\}.$$

*Let $H_D(L) = \{f \in D\,\mathrm{Der}(L) \mid f([x,y]) = 0, \quad \forall x, y \in L\}$. If $L$ is perfect, that is, $L = [L,L]$, then*

$$H_D(L) = \{0\}.$$

*Consequently, every $D$-homoderivation $f \in D\,\mathrm{Der}(L)$ satisfying*

$$f([x,y]) = 0 \qquad \textit{for all } x, y \in L$$

*is identically zero.*

*Proof.* Let $f \in H_D(L)$. By the definition of $H_D(L)$,

$$f([x,y]) = 0 \qquad \text{for all } x, y \in L.$$

Since $L$ is perfect, we have

$$L = [L,L].$$

Thus, for every $z \in L$, there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in L$ such that

$$z = \sum_{i=1}^{n} [x_i, y_i].$$

By the linearity of $f$,

$$f(z) = f\left(\sum_{i=1}^{n} [x_i, y_i]\right) = \sum_{i=1}^{n} f([x_i, y_i]).$$

Since $f \in H_D(L)$,

$$f([x_i, y_i]) = 0 \qquad \text{for all } i = 1, \ldots, n.$$

Therefore,

$$f(z) = 0.$$

As $z \in L$ was arbitrary, we conclude that

$$f = 0.$$

Hence

$$H_D(L) = \{0\}.$$

□

# 4 Acknowledgments

The writer is grateful to Mustansiriyah University[1], located in Baghdad, Iraq. Likewise, he is beholden to the mathematicians for their reviewing this paper.

[1] `www.uomustanasiriyah.edu.iq`

# References


[1] Mehsin Jabel Atteya, *Homogeneralized $(\sigma, \tau)$-derivations of associative rings*, Ricerche di Matematica, 74(3), (2025). doi:10.1007/s11587-025-00994-6.

[2] Ram Awtar, *Lie and Jordan structure in prime rings with derivations*, Proceedings of the American Mathematical Society, 41(1):67—-74, (1973). doi:10.1090/S0002-9939-1973-0318233-5.

[3] Ram Awtar, *On a theorem of Posner*, Mathematical Proceedings of the Cambridge Philosophical Society, 73(1):25–27, (1973). doi:10.1017/S0305004100047423.

[4] Eremita D. and D. Benkovi˘c, *Generalized derivations of current Lie algebras*, Communications in Algebra, 52(11): 4603—-4611, (2024). doi:10.1080/00927872.2024.2354423.

[5] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York, (1972).

[6] George F. Leger and Eugene M. Luks, *Generalized derivations of Lie algebras*, Journal of Algebra, 228(1):156–203, (2000). doi:10.1006/jabr.1999.8250.

[7] Edward C. Posner, *Derivations in prime rings*, Proceedings of the American Mathematical Society, American Mathematical Society, 8(6):1093—-1100, (1957). doi:10.2307/2032686.

[8] John Stillwell, *Naive Lie theory*, Springer Science+Business Media, LLC, New York, NY 10013, USA, (2008). doi:10.1007/978-0-387-78214-0.

[9] Keli Zheng, Jiale Chen, Wei Shan, and Ying Guo, *The structure of d-derivations and their decomposition in Lie algebras*, Mathematics, 14(129):1–13, (2026). doi:10.3390/math14010129.