Global Conference on Mathematics and Applications (GCMA 2026)

Bivariate Survival Dependence Modeling via Conditional Copula L-Moments

An Empirical Application to the Benchmark Diabetic Retinopathy Study (DRS)

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Roadmap of the Presentation

15-minute scientific briefing

Part I: Foundations & Methodology

- **Clinical Context & Hurdle:** Paired organ survival & failure of Likelihood under right-censoring.
- **The CCLM Framework:** Truncated Kendall distribution and shifted Legendre L-moments.
- **Estimation Algorithm:** 4-step semi-parametric procedure bypassing copula densities.

Part II: Application & Insights

- **The DRS Clinical Cohort:** $n = 197$ patients under severe censoring asymmetry (72.6%).
- **Empirical Findings:** Laser efficacy, copula contours, and parameter estimation.
- **Why CCLM Wins:** 11.6% variance reduction in multi-parameter models (BB1).
- **Takeaways & Perspectives:** Clinical translation and future work.

Clinical Motivation & The Right-Censoring Hurdle

Bilateral ophthalmological study meets the limits of standard likelihood

Clinical Setting: Paired Lifetimes (DRS)

- **Bilateral Design ($n = 197$):** Each patient provides:
  - T_1: Time to severe visual loss in **laser-treated eye**.
  - T_2: Time to severe visual loss in **untreated control eye**.
- **Systemic Biological Concordance:** Both eyes share identical hyperglycemia, genetic factors, and microvascular pathology ($\tau > 0$).
- **Severe Incomplete Follow-up:** Observation routinely halts before both eyes lose sight $\Longrightarrow$ **bivariate right-censoring**.

Why Pseudo-Likelihood (PML) Fails

Observables: $X_j = \min(T_j, C_j)$, $\delta_j = 1_{\{T_j \leq C_j\}}$.

- Likelihood splits into **4 configurations** with complex mixed copula derivatives.
- High censoring rates produce **flat likelihood surfaces** and **singular Hessians**.
- Boundary instability in 2-parameter models (BB1).

The Solution: A purely density-free moment approach bypassing likelihood optimization entirely!

Theoretical Foundation: Censored Kendall Variable & CCLM

Bypassing copula densities via univariate conditioning and Legendre polynomials

Double-Censored Kendall Variable

Survival copula: $S(t_1, t_2) = \widetilde{C}_\theta\big(S_1(t_1), S_2(t_2)\big)$.

- Define the Kendall variable: $V = \widetilde{C}_\theta(S_1(T_1), S_2(T_2))$.
- Conditioning on $\{T_1 > c_1, T_2 > c_2\}$ restricts support:

$$V \in [0, \widetilde{c}], \quad \widetilde{c} := \widetilde{C}_\theta(S_1(c_1), S_2(c_2))$$

- Reduces 2D censoring to a **univariate truncated variable** V!
- Conditional CDF $K_C^{(c)}$ and density $k_C^{(c)}$ depend **only on Archimedean generator** φ_θ [Idiou et al., 2023].

Conditional Copula L-Moments (CCLM)

Shifted Legendre polynomials P_{k-1} weighting [Hosking, 1990; Bouti et al., 2026]:

$$\boldsymbol{\lambda_k^{(c)}(\theta) = \int_0^{\widetilde{c}} v\, P_{k-1}\big(K_C^{(c)}(v)\big)\, k_C^{(c)}(v)\, dv}$$

- **1st L-Moment** ($k = 1$): $\lambda_1^{(c)} = \int_0^{\widetilde{c}} v\, k_C^{(c)}(v)\, dv$
- **2nd L-Moment** ($k = 2$): $\lambda_2^{(c)} = \int_0^{\widetilde{c}} v\,[2K_C^{(c)}(v) - 1]\, k_C^{(c)}(v) dv$

Key Distinction: CCM squares $v^2 \implies$ inflates variance near $\widetilde{c}$.
CCLM uses bounded $2K - 1 \in [-1, 1] \implies$ robust!

Semi-Parametric Algorithm: 4-Step Estimation Workflow

A transparent, reproducible pipeline from raw lifetimes to copula parameters

Step 1: Marginal Survival Estimation (Kaplan–Meier)

Compute univariate KM estimators $\widehat{S}_{\mathbf{1}}(t), \widehat{S}_{\mathbf{2}}(t)$ and pseudo-observations $\widehat{\boldsymbol{U}}_i = \big(\widehat{S}_{\mathbf{1}}(X_{\mathbf{1}i}), \widehat{S}_{\mathbf{2}}(X_{\mathbf{2}i})\big)$.

Step 2: Censoring-Adapted Empirical Copula $\widetilde{C}_n$

Construct $\widetilde{C}_n(u_{\mathbf{1}}, u_{\mathbf{2}}) = u_{\mathbf{1}} + u_{\mathbf{2}} - 1 + \frac{\mathbf{1}}{n}\sum_{i=\mathbf{1}}^{n} \frac{\delta_{\mathbf{1}i}\delta_{\mathbf{2}i}}{\widehat{C}(\widehat{S}_{\mathbf{1}i}, \widehat{S}_{\mathbf{2}i})} 1_{\{\mathbf{1}-F_{\mathbf{1}n} \geq u_{\mathbf{1}},\, \mathbf{1}-F_{\mathbf{2}n} \geq u_{\mathbf{2}}\}}$, then obtain $V_i = \widetilde{C}_n(\widehat{\boldsymbol{U}}_i)$.

Step 3: Sample Conditional L-Moments

Evaluate empirical counterparts: $\widehat{\lambda}_{\mathbf{1}}^{(c)} = \frac{\mathbf{1}}{n}\sum_{i=\mathbf{1}}^{n} V_i, \quad \widehat{\lambda}_{\mathbf{2}}^{(c)} = \frac{\mathbf{1}}{n}\sum_{i=\mathbf{1}}^{n} V_i \left[2\frac{\mathrm{rank}(V_i)}{n+\mathbf{1}} - 1\right]$.

Step 4: Root-Finding System Resolution

Solve the estimating system $\boldsymbol{\lambda}_{\boldsymbol{k}}^{(\boldsymbol{c})}(\theta) = \widehat{\boldsymbol{\lambda}}_{\boldsymbol{k}}^{(\boldsymbol{c})}$ $(k = 1, \ldots, d)$ using bounded numerical root-finding.

The DRS Clinical Cohort: Heavy Bivariate Right-Censoring

Benchmark dataset ($n = 197$ patients) exhibiting severe censoring asymmetry

Clinical Cohort Summary ($n = 197$, Kalbfleisch & Prentice, 2002)

Characteristic	Treated	Control	Bivariate
Total Patients (n)	197	197	197
Observed Events ($\delta = 1$)	54 (27.4%)	101 (51.3%)	–
Censored Units ($\delta = 0$)	**143** (72.6%)	**96** (48.7%)	–
Both Uncensored $(1, 1)$	–	–	38 (19.3%)
Both Censored $(0, 0)$	–	–	**80** (40.6%)
Discordant $(1, 0)/(0, 1)$	–	–	**79** (40.1%)
Median Follow-up (Mo.)	42.5	35.8	39.2

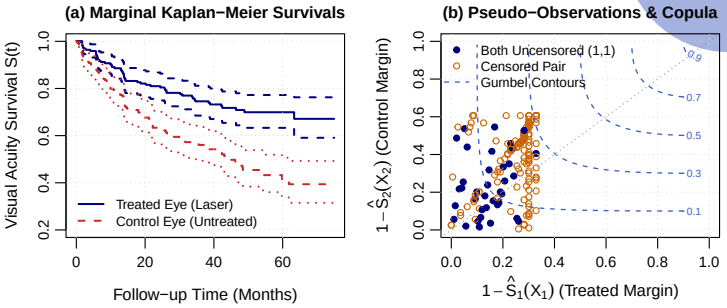
Censoring Challenge

- **Only** 19.3% **complete pairs!**
- 80.7% **of pairs** have ≥ 1 censored eye.
- Treated eyes underwent 72.6% **censoring** due to laser sight preservation.

Clinical Question: Does inter-eye dependence persist despite heavy censoring and severe discordance?

Empirical Survival & Copula Concordance (Visual Evidence)

Jointly assessing laser treatment efficacy and inter-eye dependence



Panel (a): Laser Sight Preservation

At 5 Years (60 Mo): Treated $\approx$ 71% vs. Control $\approx$ 46% ($p < 0.001$).
→ Distinct 95% confidence bands prove substantial laser protective effect.

Panel (b): Inter-Eye Concordance

Strong Dependence: Diagonal clustering, Gumbel $\widehat{\theta} = 2.47$.
Extreme Upper Tail: $\lambda_U \approx 0.68$ (Vision loss in one eye signals high fellow risk).

Dependence Estimates Across Archimedean Copulas ($n = 197$)

Benchmarking proposed CCLM against conventional CCM and goodness-of-fit

Copula	Method	Estimate ($\widehat{\theta}$)	SE	95% Bootstrap CI	Kendall's $\widehat{\tau}$	GoF (S_n)
Gumbel	**CCLM**	$\widehat{\theta} = 2.4705$	0.5333	[1.3765, 3.3947]	0.5952	0.2362
	CCM	$\widehat{\theta} = 2.4705$	0.5333	[1.3765, 3.3947]	0.5952	0.2362
Clayton	**CCLM**	$\widehat{\theta} = 1.9321$	0.7106	[0.6173, 3.4484]	0.4914	0.2204
	CCM	$\widehat{\theta} = 1.9321$	0.7106	[0.6173, 3.4484]	0.4914	0.2204
BB1	**CCLM**	$(\widehat{\alpha}, \widehat{\beta}) = (0.010^*, 2.492)$	(0.082, 0.541)	$\beta \in [1.381, 3.420]$	0.6007	0.2359
(2-Param.)	**CCM**	$(\widehat{\alpha}, \widehat{\beta}) = (0.010^*, 2.586)$	(0.095, 0.612)	$\beta \in [1.415, 3.824]$	0.6153	0.2371

Kendall's τ: Strong positive inter-eye correlation ($\widehat{\tau} \approx 0.50 - 0.60$). Gumbel 95% bootstrap CI for τ is [0.273, 0.705].

Key Advantage in BB1: CCLM reduces SE($\widehat{\beta}$) from 0.612 to 0.541 (11.6% **reduction**), yields tighter CI and superior GoF ($S_n = 0.2359$).

Methodological Breakthrough: Why CCLM Beats CCM

The fundamental mathematics behind variance reduction

CCM Quadratic Moment: $m_2^{(c)}$

Conventional moment:

$$m_2^{(c)} = \int_0^{\widetilde{c}} \boldsymbol{v}^2 \, k_C^{(c)}(v)\, dv$$

- Quadratic term v^2 concentrates excessive weight near $\widetilde{c}$.
- Under heavy censoring (72.6%), observations near $\widetilde{c}$ are sparse $\Longrightarrow$ **severe variance inflation**.
- $\mathrm{SE}(\widehat{\beta}) = 0.612$, wider bootstrap interval.

CCLM Shifted Legendre: $\lambda_2^{(c)}$

Proposed L-moment:

$$\lambda_2^{(c)} = \int_0^{\widetilde{c}} v \left[2\boldsymbol{K_C^{(c)}}(\boldsymbol{v}) - 1\right] k_C^{(c)}(v)\, dv$$

- $P_1(u) = 2u - 1 \in [-1, 1]$ is strictly **bounded and orthogonal**.
- Dampens extreme variations in censored tails.
- Produces a tighter standard error ($\mathrm{SE} = 0.541$) and better fit ($S_n = 0.2359$).

Clinical Insights & Biostatistical Takeaways

Translating mathematical modeling into ophthalmological and statistical practice

For Clinical Practice

- **Laser Efficacy Confirmed:** Photocoagulation significantly delays vision loss ($\approx 71\%$ vs 46% at 5 years).
- **High Fellow-Eye Risk ($\tau \approx 0.60$):** Sight loss in one eye is an urgent warning sign for rapid contralateral deterioration.
- **Proactive Bilateral Care:** Mandates routine bilateral monitoring even when symptoms appear in only one eye.

For Biostatistical Practice

- **Zero Likelihood Singularities:** Completely bypasses non-invertible Hessian matrices and boundary collapse of PML.
- **Heavy Censoring Resilience:** Functions smoothly even when $> 70\%$ of records are censored.
- **Plug-and-Play Implementation:** Computationally lightweight via standard 1D/2D root-finders.

Conclusions & Ongoing Research Horizons

Summary of contributions

Core Takeaway: CCLM establishes a robust, density-free, L-moment copula estimation framework under bivariate right-censoring, empirically validated on the benchmark DRS clinical trial.

Key Contributions

- Exact equivalence with CCM in 1-parameter models ($\widehat{\theta} = 2.47$ for Gumbel, 1.93 for Clayton).
- 11.6% standard error reduction in 2-parameter BB1 copula.
- Solves the multi-parameter instability of conventional product moments under heavy right-censoring (72.6%).

Ongoing Horizons

- Extension to **interval-censored** & **doubly-censored** bivariate lifetimes.
- Incorporating clinical covariates (semi-parametric regression with copula L-moments).
- Scaling to high-dimensional vine copula models ($d > 2$).

Key References

Foundational literature on Copula L-Moments and Censored Inference

Foundational Literature on Copula L-Moments & Censored Inference

1. **Bouti, B., Idiou, N., & Benatia, F. (2026).** A semi-parametric L-moments estimation method for copulas under right-censoring. *Gulf Journal of Mathematics*, 22(2): 1–18.

2. **Idiou, N., Benatia, F., & Brahimi, B. (2023).** A semi-parametric estimation of copula models based on moments method under right censoring. *TWMS Journal of Applied and Engineering Mathematics*, 13(3): 850–865.

3. **Hosking, J. R. M. (1990).** L-moments: Analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society: Series B*, 52(1): 105–124.

4. **Benatia, F., Brahimi, B., & Necir, A. (2011).** A semiparametric estimation procedure for multi-parameter Archimedean copulas based on the L-moments method. *Afrika Statistika*, 6(1): 335–345.



Thank You for Your Attention!

Questions & Discussion are Most Welcome

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Backup: Analytical Forms of $K_C^{(c)}$ and $k_C^{(c)}$

Exact derivations from Archimedean generators (Idiou et al., 2023; Bouti et al., 2026)

Theorem: Conditional Distribution and Density Functions

For an Archimedean survival copula $\widetilde{C}_{\boldsymbol{\theta}}$ with generator $\varphi_{\boldsymbol{\theta}}$, the conditional distribution of $V = \widetilde{C}_{\boldsymbol{\theta}}(S_1(T_1), S_2(T_2))$ given $T_1 > c_1, T_2 > c_2$ is:

$$\mathrm{K}_{\mathrm{C}}^{(\mathrm{c})}(\mathrm{v} \mid \mathrm{c}_1, \mathrm{c}_2) = \frac{1}{\widetilde{\mathrm{c}}} \left[\mathrm{v} - \frac{\varphi_{\boldsymbol{\theta}}(\mathrm{v}) - \varphi_{\boldsymbol{\theta}}(\widetilde{\mathrm{c}})}{\varphi'_{\boldsymbol{\theta}}(\mathrm{v})} \right], \quad 0 \leq v \leq \widetilde{c}$$

and the corresponding conditional probability density function is:

$$\mathrm{k}_{\mathrm{C}}^{(\mathrm{c})}(\mathrm{v} \mid \mathrm{c}_1, \mathrm{c}_2) = \frac{1}{\widetilde{\mathrm{c}}} \frac{\left[\varphi_{\boldsymbol{\theta}}(\mathrm{v}) - \varphi_{\boldsymbol{\theta}}(\widetilde{\mathrm{c}})\right] \varphi''_{\boldsymbol{\theta}}(\mathrm{v})}{\left[\varphi'_{\boldsymbol{\theta}}(\mathrm{v})\right]^2}, \quad 0 \leq v \leq \widetilde{c}$$

where $\widetilde{c} := \widetilde{C}_{\boldsymbol{\theta}}(S_1(c_1), S_2(c_2))$.

Significance: Computable entirely from univariate derivatives φ' and φ''. Requires no two-dimensional integration of the copula density $c_{\boldsymbol{\theta}}(u_1, u_2)$, eliminating Hessian singularities.

Backup: Empirical Formulas & Rank-Based Weights

Sample estimators under right censoring

Censoring-Adapted Empirical Copula

Inverse-probability weighted empirical copula:

$$\widetilde{C}_n(\boldsymbol{u}) = u_1 + u_2 - 1 + \frac{1}{n}\sum_{i=1}^{n} W_i 1_{\{\widehat{S}_{1i} \geq u_1, \widehat{S}_{2i} \geq u_2\}}$$

where weights account for joint censoring:

$$W_i = \frac{\delta_{1i}\delta_{2i}}{\widetilde{C}(\widehat{S}_{1i}, \widehat{S}_{2i})}$$

Sample Kendall values: $V_i = \widetilde{C}_n(\widehat{S}_{1i}, \widehat{S}_{2i})$.

Sample L-Moments Formulation

Empirical sample L-moments:

$$\widehat{\lambda}_1^{(c)} = \frac{1}{n}\sum_{i=1}^{n} V_i$$

$$\widehat{\lambda}_2^{(c)} = \frac{1}{n}\sum_{i=1}^{n} V_i \left[2\,\frac{\mathrm{rank}(V_i)}{n+1} - 1\right]$$

- Rank transformation $\frac{\mathrm{rank}(V_i)}{n+1} \in (0,1)$ approximates empirical CDF $K_n(V_i)$.
- Guarantees exact $[-1, 1]$ bounds.

Backup: Goodness-of-Fit & Bootstrap Resampling

Model selection and inference procedure

Cramér–von Mises Statistic (S_n)

Evaluated on pseudo-observations:

$$S_n = \sum_{i=1}^{n} \left\{ \widetilde{C}_n(\widehat{\boldsymbol{U}}_i) - \widetilde{C}_{\widehat{\boldsymbol{\theta}}}(\widehat{\boldsymbol{U}}_i) \right\}^2$$

- Measures distance between empirical copula $\widetilde{C}_n$ and parametric copula $\widetilde{C}_{\widehat{\boldsymbol{\theta}}}$.
- **Lowest value indicates best fit:**
  - Clayton: $S_n = 0.2204$ (Best fit)
  - BB1 (CCLM): $S_n = 0.2359$
  - Gumbel: $S_n = 0.2362$

Bootstrap Inference Scheme

Nonparametric Paired Bootstrap ($B = 1000$):

1. Resample $n = 197$ paired quadruplets $(X^*_{\mathbf{1}i}, \delta^*_{\mathbf{1}i}, X^*_{\mathbf{2}i}, \delta^*_{\mathbf{2}i})$ with replacement.
2. Re-estimate marginals $\widehat{S}^*_{\mathbf{1}}, \widehat{S}^*_{\mathbf{2}}$ and empirical copula $\widetilde{C}^*_n$.
3. Re-solve estimating equations: $\boldsymbol{\lambda}^{(c)}_k(\boldsymbol{\theta}^*) = \widehat{\boldsymbol{\lambda}}^{(c)*}_k$.
4. Standard errors: sample standard deviation across $B = 1000$ bootstrap replicates.
5. 95% CI: empirical 2.5% and 97.5% quantiles.