Convergence Estimates for Baskakov-Kantorovich Operators Involving Exponential Weights

Prerna Sharma

Department of Mathematics, Sardar Vallabh Bhai Patel
University of Agriculture and Technology, Meerut
(U.P.), India

Email: mprerna_anand@yahoo.com

Abstract: In this paper, we introduce a modified class of Baskakov–Kantorovich operators constructed using the μ integral and investigate their approximation properties in the framework of exponential weights. The proposed operators are expressed through a composition of modified Baskakov operators with suitable differential and integral operators. We derive explicit expressions for the moments and evaluate the operators on exponential test functions to establish the fundamental properties of the proposed construction. Furthermore, we investigate their pointwise convergence and establish a Voronovskaya-type asymptotic formula describing the limiting behaviour of the approximation error. Quantitative estimates for the rate of convergence are obtained by employing an appropriate exponentially weighted modulus of continuity. Finally, numerical experiments and graphical illustrations are presented to demonstrate the approximation behaviour of the proposed operators and to support the theoretical findings. The results contribute to the study of modified positive linear operators and their convergence properties in exponentially weighted function spaces.

Keywords: Baskakov-Kantorovich operators, Linear positive operators, uniform convergence, μ-Integral, Modulus of continuity.

Mathematics Subject Classification: 41A25, 41A30

1. Introduction

The study of approximation properties and convergence rates for positive linear operators remains a vibrant field of research. Some comprehensive convergence estimates have been compiled in [10]. In the last decade, the construction of positive linear operators preserving exponential functions has received significant attention, with its advantages documented extensively in the literature (see, e.g., [2, 4, 11-14]. In this paper, we extend these investigations to a modified class of Baskakov-Kantorovich operators designed to preserve exponential functions. We begin by recalling the classical Baskakov operators B_n, which for any $f \in C[0,\infty)$ are defined as:

$$(B_n f)(x) = \sum_{v=0}^{\infty} b_{n,v}(x) f\left(\frac{v}{n}\right), \tag{1}$$

where

$$b_{n,v}(x) = \binom{n+v-1}{v} \frac{x^v}{(1+x)^{n+v}},\ x \geq 0, n \in N.$$

The Baskakov-Kantorovich operators are defined by:

$$(P_n f)(x) = n \sum_{v=0}^{\infty} b_{n+1,v}(x) \int_{v/n}^{(v+1)/n} f(t)\mathrm{d}t \tag{2}$$

A similar representation was investigated by Abel et al. [1] in the context of estimating convergence rates for bounded variation functions via the Bézier variant of Baskakov-Kantorovich operators. Subsequently, Zhang and Zhu [16] introduced an alternative Kantorovich-type modification of the Baskakov operators, defined as:

$$\left(\hat{P}_n f\right)(x) = n \sum_{v=0}^{\infty} b_{n,v}(x) \int_{v/n}^{(v+1)/n} f(t)\mathrm{d}t.$$

Consider

$$F_n(x) = \int_0^x f_n(t)\mathrm{d}t, \qquad f_n(t) = f\left(\frac{(n-1)t}{n}\right).$$

In [16], the authors observed the following structural relationship connecting $\hat{P}_n$ with the classical Baskakov operators B_n:

$$\left(\hat{P}_n f\right)(x) = (D \circ B_{n-1} \circ F_n)(x).$$

To preserve the exponential test functions $e^{\mu x}$ and $e^{2\mu x}$, Ozsarac (Firat) and Acar [8] introduced a modification of the classical Baskakov operators B_n, defined as:

$$(G_n f)(x) = \sum_{v=0}^{\infty} e^{\mu x} b_{n,v}(a_n(x)) e^{-\mu v/n} f\left(\frac{v}{n}\right),$$

where

$$a_n(x) = \frac{e^{-\mu x/n} - 1}{1 - e^{\mu/n}}. \tag{3}$$

The connection between these operators and the standard Baskakov operators B_n is given by the relation:

$$(G_n)f(x) = e^{\mu x}\left(B_n f_\mu\right)\left(a_n(x)\right),\ \ f_\mu = \frac{f}{e^\mu}.$$

The concepts of μ-differential and μ-integral were formulated by Paltanea [9] on $[0,1]$as $D_\mu: C^1[0,1] \to C[0,1]$ and $I_\mu: C[0,1] \to C^1[0,1]$, respectively, in the study of modified Bernstein-Kantorovich operators. These mappings extend to the domain $[0,\infty)$ as follows:

$$D_\mu(f,x) = f'(x) - \mu f(x), f \in C^1[0,\infty), x \in [0,\infty)$$

and $I_\mu(f,x) = e^{\mu x}\int_0^x e^{-\mu} f(t)\mathrm{d}t, f \in C[0,\infty), x \in [0,\infty).$

Motivated by the results in Aral et al. [5], we introduce the following modification of the standard Baskakov–Kantorovich operators (2) for $f \in L_1[0,\infty)$:

$$\left(\tilde{P}_n f\right)(x) = a_n'(x) n e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \int_{v/n}^{(v+1)/n} e^{-\mu t} f(t)\mathrm{d}t, \tag{4}$$

here $b_{n,v}$ denotes the Baskakov basis from (1) and $a_n(x)$ is defined in (3). These operators map to the operators G_n via the relation:

$$\left(\tilde{P}_n f\right)(x) = D_\mu \circ G_n \circ I_\mu(f)(x).$$

To establish the above relation, we consider $I_\mu = e^{\mu x} F_\mu, \quad F_\mu = \int_0^x e^{-\mu t} f(t)\mathrm{d}t$ and apply the identity

$$\left(b_{n,v}(a_n(x))\right)' = n a_n'(x)\left[b_{n+1,v-1}(a_n(x)) - b_{n+1,v}(a_n(x))\right].$$

We have:

$$D_\mu \circ G_n \circ I_\mu(f)(x) = e^{\mu x}\sum_{v=0}^{\infty} b_{n,v}'(a_n(x)) F_\mu\left(\frac{v}{n}\right)$$

$$= e^{\mu x} n a_n'(x)\sum_{v=0}^{\infty}\left[b_{n+1,v-1}(a_n(x)) - b_{n+1,v}(a_n(x))\right] F_\mu\left(\frac{v}{n}\right)$$

$$= e^{\mu x} n a_n'(x)\sum_{v=0}^{\infty} b_{n+1,v} a_n(x)\left(F_\mu\left(\frac{v+1}{n}\right) - F_\mu\left(\frac{v}{n}\right)\right)$$

$$= e^{\mu x} n a_n'(x)\sum_{v=0}^{\infty} b_{n+1,v} a_n(x)\int_{v/n}^{(v+1)/n} e^{-\mu t} f(t)\mathrm{d}t.$$

2. Moment Analysis:

This section presents the fundamental moment calculations and auxiliary evaluations required for establishing the main convergence theorems.

Lemma 2.1: For any $n \in N$ and $x \in [0,\infty)$, the operator $\tilde{P}_n$ satisfies the following equalities:

$$\left(\tilde{P}_n e_0\right)(x) = e^{\mu x\frac{n-1}{n}} e^{\mu}\left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}$$

$$\left(\tilde{P}_n e^{\mu}\right)(x) = -\frac{\mu}{n}\frac{e^{\mu x\frac{n-1}{n}}}{1 - e^{\frac{\mu}{n}}}$$

$$\left(\tilde{P}_n e^{2\mu}\right)(x) = e^{2\mu x}$$

$$\left(\tilde{P}_n e^{3\mu}\right)(x) = \frac{1}{2} e^{\mu x \frac{n-1}{n}} \left(e^{\frac{\mu}{n}} + 1\right) \left(e^{\frac{\mu}{n} - \frac{\mu x}{n}} + e^{-\frac{\mu x}{n}} - e^{\frac{\mu}{n}}\right)^{-(n+1)},$$

$$\left(\tilde{P}_n e^{4\mu}\right)(x) = \frac{1}{3} e^{\mu x \frac{n-1}{n}} \left(e^{\frac{2\mu}{n}} + e^{\frac{\mu}{n}} + 1\right) \times$$

$$\left(e^{\frac{2\mu}{n} - \frac{\mu x}{n}} - e^{\frac{2\mu}{n}} + e^{\frac{\mu}{n} - \frac{\mu x}{n}} - e^{\frac{\mu}{n}} + e^{-\frac{\mu x}{n}}\right)^{-(n+1)}.$$

Proof: Applying definition (4) of the operator $\tilde{P}_n$ together with relation (3), we obtain:

$$\left(\tilde{P}_n e_0\right)(x) = a_n'(x) n e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \int_{v/n}^{(v+1)/n} e^{-\mu t}\, \mathrm{d}t$$

$$= -\frac{1}{\mu} a_n'(x) n e^{\mu x} \left\{ \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) e^{-\mu(v+1)/n} \right.$$

$$\left. - \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) e^{-\mu v/n} \right\}$$

$$= -\frac{1}{\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{-\mu}{n}} - 1\right) \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) e^{\frac{-\mu v}{n}}$$

$$= -\frac{1}{\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{-\mu}{n}} - 1\right) \left(1 + a_n(x) - e^{\frac{-\mu}{n}} a_n(x)\right)^{-(n+1)}$$

$$= e^{\mu x \frac{n-1}{n}} e^{\mu} \left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}.$$

Again using (3), we have:

$$\left(\tilde{P}_n e^{\mu}\right)(x) = a_n'(x) n e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \int_{v/n}^{(v+1)/n} e^{-\mu t} e^{\mu t}\, \mathrm{d}t$$

$$= a_n'(x) e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x))$$

$$= a_n'(x) e^{\mu x} = -\frac{\mu}{n} \left(\frac{e^{\mu x \frac{n-1}{n}}}{1 - e^{\frac{\mu}{n}}} \right).$$

Substituting the explicit expression for $a_n(x)$ and its corresponding derivative into the formulation, we obtain:

$$\left(\tilde{P}_n e^{2\mu}\right)(x) = a_n'(x) n e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \int_{v/n}^{(v+1)/n} e^{-\mu t} e^{2\mu} \, \mathrm{d}t$$

$$= \frac{1}{\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{\mu}{n}} - 1\right) \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) e^{\frac{\mu v}{n}}$$

$$= \frac{1}{\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{\mu}{n}} - 1\right) \left[\left(1 - e^{\frac{\mu}{n}}\right) a_n(x) + 1\right]^{-(n+1)} = e^{2\mu x}.$$

Following an identical approach, it follows that:

$$\left(\tilde{P}_n e^{3\mu}\right)(x) = a_n'(x) n e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \int_{v/n}^{(v+1)/n} e^{-\mu t} e^{3\mu t} \, \mathrm{d}t$$

$$= \frac{1}{2\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{2\mu}{n}} - 1\right) \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) e^{\frac{2\mu v}{n}}$$

$$= \frac{1}{2\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{2\mu}{n}} - 1\right) \left(1 + a_n(x) - e^{\frac{2\mu}{n}} a_n(x)\right)^{-(n+1)}$$

$$= \frac{1}{2} e^{\mu x \frac{n-1}{n}} \left(e^{\frac{\mu}{n}} + 1\right) \left(e^{\frac{\mu}{n} - \frac{\mu x}{n}} + e^{-\frac{\mu x}{n}} - e^{\frac{\mu}{n}}\right)^{-(n+1)}.$$

Hence,

$$\left(\tilde{P}_n e^{4\mu}\right)(x) = a_n'(x) n e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \int_{v/n}^{(v+1)/n} e^{-\mu t} e^{4\mu t} \, \mathrm{d}t$$

$$= \frac{1}{3\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{3\mu}{n}} - 1\right) \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) e^{\frac{3\mu v}{n}}$$

$$= \frac{1}{3\mu} a_n'(x) n e^{\mu x} \left(e^{\frac{3\mu}{n}} - 1\right) \left[\left(1 - e^{\frac{3\mu}{n}}\right) a_n(x) + 1\right]^{-(n+1)}$$

$$= \frac{1}{3} e^{\mu x \frac{n-1}{n}} \left(e^{\frac{2\mu}{n}} + e^{\frac{\mu}{n}} + 1\right) = \left(e^{\frac{2\mu}{n} - \frac{\mu x}{n}} - e^{\frac{2\mu}{n}} + e^{\frac{\mu}{n} - \frac{\mu x}{n}} - e^{\frac{\mu}{n}} + e^{-\frac{\mu x}{n}}\right)^{-(n+1)}.$$

Lemma 2.2: We have:

$$\left(\tilde{P}_n e^{At}\right)(x) = \frac{n}{A-\mu} a_n'(x) e^{\mu x} \left(e^{(A-\mu)/n} - 1\right) \left[1 + \left(1 - e^{(A-\mu)/n}\right) a_n(x)\right]^{-(n+1)} \qquad (5)$$

where $a_n(x)$ is defined by (3).

Proof: Following (4), we have:

$$\left(\tilde{P}_n e^{At}\right)(x) = a_n'(x) n e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \int_{v/n}^{(v+1)/n} e^{-\mu t} e^{At}\, \mathrm{d}t$$

$$= \frac{n}{A-\mu} a_n'(x) e^{\mu x} \sum_{v=0}^{\infty} b_{n+1,v}(a_n(x)) \left[e^{(A-\mu)v/n}\left(e^{(A-\mu)/n} - 1\right)\right]$$

$$= \frac{n}{A-\mu} a_n'(x) e^{\mu x} \left(e^{\frac{A-\mu}{n}} - 1\right) \left[1 + \left(1 - e^{\frac{A-\mu}{n}}\right) a_n(x)\right]^{-(n+1)}.$$

Lemma 2.3: Let $\varphi_{\mu,x}^{v}(t) := (e^{\mu t} - e^{\mu x})^v, v = 0,1,2, \ldots$, then:

$$\left(\tilde{P}_n \varphi_{\mu,x}^{0}(t)\right)(x) = e^{\mu x \frac{n-1}{n}} e^{\mu} \left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}$$

$$\left(\tilde{P}_n \varphi_{\mu,x}^{1}(t)\right)(x) = \left(\tilde{P}_n (e^{\mu t} - e^{\mu x})\right)(x)$$

$$= -\frac{\mu}{n}\left(\frac{e^{\mu x \frac{n-1}{n}}}{1 - e^{\frac{\mu}{n}}}\right) - e^{\mu x} e^{\mu x\left(\frac{n-1}{n}\right)} e^{\mu} \left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}$$

$$= e^{\mu x\left(\frac{n-1}{n}\right)} \left(-\frac{\mu}{n\left(1 - e^{\frac{\mu}{n}}\right)} - e^{\mu(x+1)} \left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}\right).$$

And

$$\left(\tilde{P}_n\left(\varphi_{\mu,x}^{2}(t)\right)(x) = \left(\tilde{P}_n((e^{\mu t} - e^{\mu x})^2)\right)(x)\right.$$

$$= e^{2\mu x} + 2e^{\mu x} \frac{\mu}{n}\left(\frac{e^{\mu x \frac{n-1}{n}}}{1 - e^{\frac{\mu}{n}}}\right) + e^{2\mu x} e^{\mu x \frac{n-1}{n}} e^{\mu} \left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}$$

$$= e^{\mu x}\left(e^{\mu x} + 2\frac{\mu}{n}\left(\frac{e^{\mu x \frac{n-1}{n}}}{1 - e^{\frac{\mu}{n}}}\right) + e^{\mu x} e^{\mu x\left(\frac{n-1}{n}\right)} e^{\mu} \left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}\right).$$

Hence, we find:

$$\lim_{n\to\infty} n\left(\left(\tilde{P}_n e_0\right)(x) - 1\right) = \mu(\mu x(x+1) - 2x - 1),$$

$$\lim_{n\to\infty} n\left(\left(\tilde{P}_n e^{\mu}\right)(x) - e^{\mu x}\right) = -\mu e^{\mu x}\left(x + \frac{1}{2}\right).$$

$$\lim_{n\to\infty} n\left(\left(\tilde{P}_n e^{2\mu}\right)(x) - e^{2\mu x}\right) = 0. \tag{6}$$

By virtue of the limits derived above, we gain:

$$\begin{aligned}\lim_{n\to\infty} n\left(\tilde{P}_n \varphi^1_{\mu,x}(t)\right)(x) = &\lim_{n\to\infty} n\left(\left(\tilde{P}_n e^{\mu}\right)(x) - e^{\mu x}\right)\\ &-\lim_{n\to\infty} n e^{\mu x}\left(\left(\tilde{P}_n e_0\right)(x) - 1\right)\\ &= e^{\mu x}\mu\left[x + \tfrac{1}{2} - \mu x(x+1)\right] \end{aligned} \tag{7}$$

$$\begin{aligned}\lim_{n\to\infty} n\left(\tilde{P}_n \varphi^2_{\mu,x}(t)\right)(x) = &\lim_{n\to\infty} n\left(\left(\tilde{P}_n e^{2\mu}\right)(x)\right.\\ &-2e^{\mu x}\left(\tilde{P}_n e^{\mu}\right)(x) + e^{2\mu}\ \left.\left(\tilde{P}_n e_0\right)(x)\right)\\ &= \lim_{n\to\infty} n\left(\left(\tilde{P}_n e^{2\mu}\right)(x) - e^{2\mu x}\right)\\ &-2\lim_{n\to\infty} n e^{\mu x}\left(\left(\tilde{P}_n e^{\mu}\right)(x) - e^{\mu x}\right)\\ &+\lim_{n\to\infty} n e^{2\mu x}\left(\left(\tilde{P}_n e_0\right)(x) - 1\right)\\ &= e^{2\mu}\ \mu^2 x(x+1) \end{aligned} \tag{8}$$

$$\lim_{n\to\infty} n^2\left(\tilde{P}_n \varphi^4_{\mu,x}(t)\right)(x) = 3e^{4\mu x}x^2(1+x)^2 \tag{9}$$

Lemma 2.4: (Higher-order moments). Let $\tilde{P}_n$ be the modified Baskakov–Kantorovich operator defined by equation (4), and let $F_n(A,x) = \tilde{P}_n(e^{At},x)$ be its exponential generating function. Suppose that F_n is sufficiently differentiable with respect to A in a neighbourhood of $\mathrm{A} = 0$. Then, for $r = 0,1,2$ ...

$$\tilde{P}_n(t^r,x) = \frac{\partial^r}{\partial A^r}F_n(A,x) \text{ at } \mathrm{A} = 0.$$

In particular, the fourth raw moment is given by

$$\tilde{P}_n(t^4,x) = F_n(A,x)(l_1^4 + 6l_1^4 l_2 + 3l_2^2 + 4l_1 l_3 + l_4).$$

where $l_j = \frac{\partial^j}{\partial A^j} log F_n(A,x)$ at $\mathrm{A} = 0$, provided the logarithm is defined near $\mathrm{A} = 0$.

Proof. The result follows by differentiating the exponential generating function with respect to A, and evaluating the resulting expressions at $A = 0$. The fourth-order formula follows from repeated differentiation of $F_n = e^{(logF_n)}$The corresponding central moments are obtained by expanding $(t - x)^r$and applying the linearity of $\tilde{P}_n$.

3. Quantitative Estimates and Voronovskaya-Type Theorems

In this section, we analyze the rate of uniform convergence for the operators $\tilde{P}_n$ on $[0, \infty)$. We employ the unweighted Chebyshev norm on $C^*[0, \infty)$, defined as the subspace of continuous functions $C[0, \infty)$ for which $\lim_{x\to\infty} f(x)$ exists and is finite. Holhos [7] established a quantitative estimate on the space $C^*[0, \infty)$, extending the work of Boyanov and Veselinov [6] regarding the uniform convergence of sequences of positive linear operators. Subsequently, this framework was applied to Baskakov operators in [15]. The following theorem provides the pointwise convergence rate for $\tilde{P}_n$:

Theorem 3.1: Let $\tilde{P}_n: C^*[0, \infty) \to C^*[0, \infty)$ be a sequence of positive linear operators. Then, for every $x \in [0, \infty)$, the following conditions hold:

$$\lim_{n\to\infty} \left(\tilde{P}_n e^{-kt}\right)(x) = e^{-kx}, k = 0,1,2 \text{ at the point } x \in [0, \infty), \text{ precisely when}$$

$$\lim_{n\to\infty} \left(\tilde{P}_n f\right)(x) = f(x), \forall\, f \in C^*[0, \infty).$$

Proof: Applying (5) and performing the necessary symbolic computations, we obtain:

$$\left(\tilde{P}_n e^{At}\right)(x) = e^{Ax} + \frac{1}{2n} e^{Ax}[A - 2\mu + 2Ax + A^2x - 4\mu x - 3A\mu x$$

$$+2\mu^2 x + A^2x^2 - 3A\mu x^2 + 2\mu^2 x^2] + O(n^{-2}).$$

Applying

$$\left(\tilde{P}_n e_0\right)(x) = 1 + \frac{1}{n}(-\mu - 2\mu x + \mu^2 x + \mu^2 x^2) + O(n^{-2}),$$

$$\left(\tilde{P}_n e^{-t}\right)(x) = e^{-x} + \frac{1}{2n}[e^{-x}(1 + 2\mu)(-1 - 2x + x + \mu x + x^2 + \mu x^2)] + O(n^{-2}),$$

$$\left(\tilde{P}_n e^{-2t}\right)(x) = e^{-2x} + \frac{1}{n} e^{-2x}\left((1 + \mu)(-1 + \mu x + 2x^2 + \mu x^2)\right) + O(n^{-2}).$$

Hence, all requirements of the theorem are met, which completes the proof of the pointwise convergence of $\tilde{P}_n$.

Theorem 3.2 (Voronovskaya-type result):

Let $\mu \neq 0, x \geq 0$ and let f be twice differentiable at $x \in [0,\infty)$, with the required growth and integrability conditions for the operator $\tilde{P}_n$. Suppose that the asymptotic moment relations established in Lemma 2.3 hold, then:

$$\lim_{n\to\infty} n\left(\left(\tilde{P}_n f\right)(x) - f(x)\right) = \mu[\mu x(x+1) - 2x - 1]f(x)$$

$$+\left[x + \frac{1}{2} - \frac{3\mu x(x+1)}{2}\right]f'(x) + \frac{x(x+1)}{2}f''(x).$$

Proof: Taking

$$\varphi_{\mu,x}(t) = e^{\mu t} - e^{\mu x}.$$

Using Taylor's formula in the variable $e^{\mu t}$, we obtain:

$$f(t) = f(x) + \frac{f'(x)}{\mu e^{\mu x}}\varphi_{\mu,x}(t) + \frac{f''(x) - \mu f'(x)}{2\mu^2 e^{2\mu x}}\varphi_{\mu,x}^2(t) + h_x(t)\varphi_{\mu,x}^2(t).$$

where $h_x(t) \to 0$ as $t \to x$.

Applying the positive linear operator $\tilde{P}_n$and evaluating at $x \in [0,\infty)$, we get:

$$\left(\tilde{P}_n f\right)(x) - f(x) = f(x)\left[\left(\tilde{P}_n 1\right)(x) - 1\right] + \frac{f'(x)}{\mu e^{\mu x}}(\tilde{P}_n\varphi_{\mu,x})(x)$$

$$+\frac{f''(x) - \mu f'(x)}{2\mu^2 e^{2\mu}}(\tilde{P}_n\varphi_{\mu,x}^2)(x) + \left[\tilde{P}_n[h_x\varphi_{\mu,x}^2]\right](x).$$

Multiplying n and taking the limit, the first three terms are evaluated using the corrected moment relations:

$$\lim_{n\to\infty} n\left[\left(\tilde{P}_n 1\right)(x) - 1\right] = \mu[\mu x(x+1) - 2x - 1].$$

$$\lim_{n\to\infty} n(\tilde{P}_n\varphi_{\mu,x})(x) = \mu e^{\mu x}\left[\left(x + \frac{1}{2}\right) - \mu x(x+1\right].$$

And

$$\lim_{n\to\infty} n(\tilde{P}_n\varphi_{\mu,x}^2)(x) = \mu^2 e^{2\mu x}x(x+1).$$

It remains to show that

$$\lim_{n\to\infty} n(\tilde{P}_n[h_x\varphi_{\mu,x}^2])(x) = 0.$$

Let $\varepsilon > 0$. Since $h_x(t) \to 0$ as $t \to x$, there exist $\delta > 0$ such that:

$|h_x(t)| \leq \varepsilon$, whenever $\left|\varphi_{\mu,x}(t)\right| < \delta$.

Under the assumed global growth conditions, we choose a constant $C_\delta > 0$, such that

$$|h_x(t)|\left|\varphi_{\mu,x}(t)\right|^2 \leq \varepsilon\left|\varphi_{\mu,x}(t)\right|^2 + C_\delta\left|\varphi_{\mu,x}(t)\right|^4.$$

Positivity of $\tilde{P}_n$, therefore gives:

$$n\left|(\tilde{P}_n[h_x\varphi_{\mu,x}^2])(x)\right| \leq \varepsilon n(\tilde{P}_n\varphi_{\mu,x}^2)(x) + C_\delta n(\tilde{P}_n\varphi_{\mu,x}^4)(x).$$

The second-central-moment limit shows that the first term on the right is bounded by $C\varepsilon$ (constant times ε). Moreover, the fourth-central-moment estimate

$$n^2\left(\tilde{P}_n\varphi_{\mu,x}^4\right)(x) = O(1),$$

this implies that

$$\lim_{n\to\infty} n\left(\tilde{P}_n\varphi_{\mu,x}^4\right)(x) = 0.$$

Consequently,

$$\lim_{n\to\infty} \sup n\left|(\tilde{P}_n[h_x\varphi_{\mu,x}^2])(x)\right| \le C_x\varepsilon.$$

for a constant C_x independent of ε. Letting $\varepsilon \to 0$, we conclude that the remainder tends to zero. Combining the limits of the first three terms yields:

$$\lim_{n\to\infty} n\left(\left(\tilde{P}_n f\right)(x) - f(x)\right) = \mu[\mu x(x+1) - 2x - 1]f(x)$$

$$+\left[x + \frac{1}{2} - \mu x(x+1) - \frac{\mu}{2}x(x+1)\right]f'(x) + \frac{x(x+1)}{2}f''(x).$$

which simplifies to the asserted formula. This completes the proof.

4. Convergence Rate of the operators

For a fixed $\mu > 0$, let V_μ denotes the exponentially weighted space of all real-valued functions f defined on R^+satisfying $|f(x)| \le Ke^{\mu x},\ x \in R^+$ for some constant $K > 0$. Let M_μ be the subspace of V_μ consisting of continuous functions. Endowed with the norm

$$\|f\|_\mu = \sup_{x\in\ge 0} |f(x)|e^{-\mu x}.$$

the space $V_\mu$becomes a normed linear space.

Let $M_\mu^m(R^+)$ denotes the subspace of $M_\mu(R^+)$ consisting of all functions f satisfying

$\lim_{x\to\infty} \frac{|f(x)|}{e^{\mu x}} = m$, where m is a positive constant.

Our objective is to determine the rate of convergence by employing an appropriate weighted modulus of continuity. To this end, we consider a weighted modulus of continuity for functions $f \in M_\mu^m(R^+)$, which possesses properties analogous to those of the classical modulus of continuity on compact intervals. The exponentially weighted modulus of continuity provides an effective tool for estimating the rate of convergence of sequences of positive linear operators in exponential weighted spaces. Accordingly, we employ the following weighted modulus of continuity:

$$\tilde{\omega}(f;\delta) = \sup_{|t-x|\le\delta, x\ge 0} \frac{|f(t) - f(x)|}{e^{\mu t} + e^{\mu x}}, \delta \ge 0$$

for $f \in M_\mu^m(R^+)$. The weighted modulus of continuity $\widetilde{\omega}(f;\cdot)$ satisfies the following basic properties (see [3]).

Postulate 4.1: Given $f \in M_\mu^m(R^+)$, $\lim_{\delta\to 0}\widetilde{\omega}(f;\delta) = 0$.

Postulate 4.2: For any $\lambda > 0$ and $f \in M_\mu^m(R^+)$, we have:

$$\widetilde{\omega}(f;\lambda\delta) \le 2(1+\lambda)\widetilde{\omega}(f;\delta). \tag{10}$$

Postulate 4.3: We have:

$$\sup_{x\in R^+}\frac{\left(\tilde{P}_n\varphi_{\mu,x}^2(t)\right)(x)}{e^{2\mu x}} \le 1 - \frac{2\mu}{n}\frac{1}{e^{\frac{\mu}{n}}-1} + e^{-\frac{\mu}{n}}.$$

Proof: By definition, we have:

$$\sup_{x\in R^+}\frac{\left(\tilde{P}_n\varphi_{\mu,x}^2(t)\right)(x)}{e^{2\mu x}} = \sup_{x\in R^+}\left(1 + \frac{2\mu}{n}\frac{e^{\mu x\frac{n-1}{n}}}{1-e^{\frac{\mu}{n}}} + e^{\mu x\frac{n-1}{n}}e^{\mu}\left(1 - e^{-\frac{\mu x}{n}} + e^{\frac{\mu}{n}}\right)^{-(n+1)}\right)$$

$$= \sup_{x\in R^+}\left(1 - \frac{2\mu}{n}\frac{e^{-\frac{\mu x}{n}-\mu x}}{e^{\frac{\mu}{n}}-1} + e^{2\mu x}e^{\mu}\left(e^{\frac{\mu x}{n}} - 1 + e^{\frac{\mu}{n}}e^{\frac{\mu x}{n}}\right)^{-(n+1)}\right) \le 1 - \frac{2\mu}{n}\frac{1}{e^{\frac{\mu}{n}}-1} + e^{-\frac{\mu}{n}}.$$

Remark 4.4: On simplifying, we have:

$$\lim_{n\to\infty}\left(1 - \frac{2\mu}{n}\frac{1}{e^{\frac{\mu}{n}}-1} + e^{-\frac{\mu}{n}}\right) = 0.$$

Theorem 4.5: For $0 < 2a < b$ and $f \in M_{\rho_a}^m(R^+)$, we have:

$$\left\|\left(\tilde{P}_n f\right) - f\right\|_{3\mu} \le 8\widetilde{\omega}(f;\sqrt{u_n})\left[e^{-\frac{\mu}{n}} + \frac{2\mu}{n}\frac{1}{e^{\frac{\mu}{n}}-1} + 1 + \sqrt{e^{-\frac{\mu}{n}}}\right],$$

$$\text{where } u_n = \frac{1}{\mu^2}\left(1 - \frac{2\mu}{n}\frac{1}{e^{\frac{\mu}{n}}-1} + e^{-\frac{\mu}{n}}\right).$$

Proof: Using the definition of the modulus of smoothness $\widetilde{\omega}(f;\delta)$, together with property (10), for any $\lambda > 0$, we choose:

$$\lambda = \frac{|t-x|}{\delta} \quad \delta > 0.$$

Then, we obtain:

$$|f(t) - f(x)| \le (e^{\mu t} + e^{\mu x})\widetilde{\omega}(f;|t-x|)$$

$$\leq 2(e^{\mu t}+e^{\mu x})\left(1+\frac{|t-x|}{\delta}\right)\widetilde{\omega}(f;\delta).$$

By the mean value theorem, we obtain:

$$|f(t)-f(x)|\leq 2(e^{\mu t}+e^{\mu x)}\left(1+\frac{|e^{\mu t}-e^{\mu x}|}{\delta\mu}\right)\widetilde{\omega}(f;\delta).$$

By the Cauchy–Schwarz inequality, for every $x\in R^{+}$, we have:

$$\left|\left(\tilde{P}_n f\right)(x)-f(x)\right|\leq \tilde{P}_n(|f(t)-f(x)|;x)$$

$$\leq \widetilde{2\omega}(f;\delta)\left(\widetilde{P}_n(e^{\mu t}+e^{\mu x})\left(1+\frac{|e^{\mu t}-e^{\mu x}|}{\delta\mu}\right)\right)(x)$$

$$=\widetilde{2\omega}(f;\delta)\left((\widetilde{P}_n e^{\mu})(x)+\frac{1}{\mu\delta}\sqrt{\left(\widetilde{P}_n e^{2\mu}\right)(x)}\sqrt{\left(\widetilde{P}_n\varphi^2_{\mu,x}(t)\right)(x)}+e^{\mu x}\left(\tilde{P}_n e_0\right)(x)\right.$$
$$\left.+\frac{e^{\mu x}}{\delta\mu}\sqrt{(\tilde{P}_n e_0)(x)}\sqrt{(\widetilde{P}_n\varphi^2_{\mu,x}(t))(x)}\right)$$

$$\leq \widetilde{2\omega}(f;\delta)\left(e^{\mu x}\left(\tilde{P}_n e_0\right)(x)+\left(\widetilde{P}_n e^{\mu}\right)(x)\right)(x)$$

$$+\frac{e^{\mu x}}{\delta\mu}\left[1+\sqrt{(\tilde{P}_n e_0)(x)}\right]\sqrt{\left(\widetilde{P}_n\varphi^2_{\mu,x}(t)\right)(x)}.$$

Applying Lemma 2.3 and postulate 4.3, we obtain:

$$\frac{\left|\left(\tilde{P}_n f\right)(x)-f(x)\right|}{e^{3\mu x}}\leq$$

$$\widetilde{2\omega}(f;\delta)\left(\frac{\left(\tilde{P}_n e_0\right)(x)}{e^{2\mu x}}+\frac{\left(\widetilde{P}_n e^{\mu}\right)(x)}{e^{3\mu}}+\frac{1}{\mu\delta}\sqrt{\frac{\left(\widetilde{P}_n\varphi^2_{\mu,x}(t)\right)(x)}{e^{2\mu x}}}\left[1+\sqrt{\frac{(\tilde{P}_n e_0)(x)}{e^{2\mu x}}}\right]\right).$$

Hence,

$$\sup_{x\in R^{+}}\frac{\left|\left(\tilde{P}_n f\right)(x)-f(x)\right|}{e^{3\mu x}}$$

$$\leq \widetilde{2\omega}(f;\delta)\left[e^{-\frac{\mu}{n}}+\frac{2\mu}{n}\frac{1}{\frac{\mu}{e^{n}}-1}+\frac{1}{\mu\delta}\sqrt{\left(1-\frac{2\mu}{n}\frac{1}{\frac{\mu}{e^{n}}-1}+e^{-\frac{\mu}{n}}\right)}\left(1+\sqrt{e^{-\frac{\mu}{n}}}\right)\right].$$

Choosing, $\mu\delta = \sqrt{\left(1 - \frac{2\mu}{n}\frac{1}{\frac{\mu}{e^{n}}-1} + e^{-\frac{\mu}{n}}\right)}$, we get the conclusion.

Theorem 4.6: If $x^{-1}\left(F_{\mu}(x+\left(\frac{1}{n}\right) - F_{\mu}(x)\right)$, is decreasing function on $(0\,,\infty)$, then $\left(a_n(x)a_n'(x)\right)^{-1}\widetilde{P}_{n,e^{\mu}}f(x)$ is also decreasing on $(0\,,\infty)$, where $\widetilde{P}_{n,e^{\mu}}f(x) = \left(\frac{\widetilde{P}_n f}{e^{\mu}}\right)(x)$.

Proof: Applying (4), we have:

$$\widetilde{P}_{n,e^{\mu}}f(x) = a_n'(x)n\sum_{v=0}^{\infty} b_{n+1,v}(a_n(x))\int_{v/n}^{(v+1)/n} f_{e^{\mu}}\,(\mathrm{t})dt$$

$$= a_n'(x)n\frac{1}{(1+a_n(x))^{n+1}}\int_0^{1/n} f_{e^{\mu}}\,(\mathrm{t})\mathrm{dt}$$

$$+a_n'(x)n\sum_{v=1}^{\infty}\binom{n+v}{v}\frac{(a_n(x))^v}{\left(1+a_n(x)\right)^{n+v+1}}\int_{\frac{v}{n}}^{\frac{v+1}{n}} f_{e^{\mu}}(\mathrm{t})dt,$$

where, $f_{e^{\mu}}(\mathrm{t}) = \frac{f}{e^{\mu}}(t)$.

Multiplying above equality by $\left(a_n(x)a_n'(x)\right)^{-1}$, on both the sides, we reach:

$$\left(a_n(x)a_n'(x)\right)^{-1}\widetilde{P}_{n,e^{\mu}}f(x) = n\frac{1}{a_n(x)(1+a_n(x))^{n+1}}\int_0^{1/n} f_{e^{\mu}}(\mathrm{t})dt$$

$$+n\sum_{v=1}^{\infty}\binom{n+v}{v}\frac{(a_n(x))^{v-1}}{(1+a_n(x))^{n+v+1}}\int_{\frac{v}{n}}^{\frac{v+1}{n}} f_{e^{\mu}}(t)dt.$$

Upon differentiating both sides of the above equation with respect to x and invoking the hypothesis, we arrive at:

$$\left(\left(a_n(x)a_n'(x)\right)^{-1}\widetilde{P}_{n,e^{\mu}}f(x)\right)' = -n\frac{a_n{}'(x)}{(a_n(x))^2\left(1+a_n(x)\right)^{n+1}}\int_0^{\frac{1}{n}} f_{e^{\mu}}(t)dt$$

$$-n(n+1)\frac{a_n{}'(x)}{a_n(x)\left(1+a_n(x)\right)^{n+2}}\int_0^{1/n} f_{e^{\mu}}(\mathrm{t})dt$$

$$+(n+1)\sum_{v=1}^{\infty}\binom{n+v+1}{v}v\frac{a_n{}'(x)(a_n(x))^{v-1}}{\left(1+a_n(x)\right)^{n+v+2}}\times\left\{\frac{n}{v+1}\int_{\frac{v+1}{n}}^{\frac{v+2}{n}} f_{e^{\mu}}(\mathrm{t})dt - n/v\int_{\frac{v}{n}}^{\frac{v+1}{n}} f_{e^{\mu}}(\mathrm{t})dt\right\}$$

$$\leq 0.$$

Theorem 4.7: Let f be a continuous, non-negative function. Assume that

$$\frac{v}{v+1}\left(F_\mu\left(\frac{v+2}{n}\right)-F_\mu\left(\frac{v+1}{n}\right)\right)\geq F_\mu\left(\frac{v+1}{n}\right)-F_\mu\left(\frac{v}{n}\right)$$

holds for $v\in\mathbb{N}_0$. Then

$$\left(\left(a_n'(x)\right)^{-1}\widetilde{P}_{n,e^\mu}f(x)\right)'\geq\left(a_n(x)\right)^{-1}\widetilde{P}_{n,e^\mu}f(x).$$

Proof: According to theorem 4.6, we have:

$$\widetilde{P}_{n,e^\mu}f(x)=a_n'(x)n\sum_{v=0}^{\infty}b_{n+1,v}\big(a_n(x)\big)\int_{\frac{v}{n}}^{\frac{v+1}{n}}f_\mu(\mathrm{t})dt.$$

On simplifying, we get:

$$\left((a_n'(x))^{-1}\widetilde{P}_{n,e^\mu}f(x)\right)'=a_n'(x)n(n+1)\sum_{v=0}^{\infty}\binom{n+v+1}{v}\frac{(a_n(x))^v}{\big(1+a_n(x)\big)^{n+v+2}}$$

$$\times\left\{\int_{\frac{v+1}{n}}^{\frac{v+2}{n}}f_\mu(t)dt-\int_{\frac{v}{n}}^{\frac{v+1}{n}}f_\mu(t)dt\right\}.$$

Applying the hypothesis, we get:

$$(a_n(x))^{-1}\widetilde{P}_{n,e^\mu}f(x)=a_n'(x)n\sum_{v=1}^{\infty}\binom{n+v}{v}\frac{(a_n(x))^{v-1}}{\big(1+a_n(x)\big)^{n+v+1}}\int_{\frac{v}{n}}^{\frac{v+1}{n}}f_\mu(t)dt$$

$$=a_n'(x)n\sum_{v=0}^{\infty}\binom{n+v+1}{v+1}\frac{(a_n(x))^v}{\big(1+a_n(x)\big)^{n+v+2}}\int_{\frac{v+1}{n}}^{\frac{v+2}{n}}f_\mu(t)dt$$

$$=a_n'(x)n(n+1)\sum_{v=0}^{\infty}\binom{n+v+1}{v}\left(\frac{1}{v+1}\right)\frac{(a_n(x))^v}{\big(1+a_n(x)\big)^{n+v+2}}\int_{\frac{v+1}{n}}^{\frac{v+2}{n}}f_\mu(t)dt.$$

Hence, we obtain: $\left((a'_n(x))^{-1}\widetilde{P}_{n,e^\mu}f(x)\right)'-(a_n(x))^{-1}\widetilde{P}_{n,e^\mu}f(x)$

$$=a_n'(x)n(n+1)\sum_{v=0}^{\infty}\binom{n+v+1}{v}\frac{(a_n(x))^v}{\big(1+a_n(x)\big)^{n+v+2}}\left\{\int_{\frac{v+1}{n}}^{\frac{v+2}{n}}f_\mu(t)dt-\int_{\frac{v}{n}}^{\frac{v+1}{n}}f_\mu(t)dt\right\}$$

$$-a_n'(x)n(n+1)\sum_{v=0}^{\infty}\binom{n+v+1}{v}\left(\frac{1}{v+1}\right)\frac{(a_n(x))^v}{\left(1+a_n(x)\right)^{n+v+2}}\int_{\frac{v+1}{n}}^{\frac{v+2}{n}} f_{e^{\mu}}(t)dt$$

$$= a_n'(x)n(n+1)\sum_{v=0}^{\infty}\binom{n+v+1}{v}\frac{(a_n(x))^v}{\left(1+a_n(x)\right)^{n+v+2}}$$
$$\times\left\{\frac{v}{v+1}\int_{\frac{v+1}{n}}^{\frac{v+2}{n}} f_{\mu}(t)dt-\int_{\frac{v}{n}}^{\frac{v+1}{n}} f_{\mu}(t)dt\right\}\geq 0.$$

Hence the result.

5. Numerical Illustrations and Graphical Analysis

The theoretical results established in the preceding sections are complemented here by numerical experiments for the operator defined in (4). All computations were performed with $\mu = 1$ on the compact interval $[0, 2]$. The calculations use $n = 10, 20, 40, 80, 160$.

5.1: Approximation of a smooth test function:

As a first test, consider $f(x) = e^{-x}$, Figure 1 compares f with the numerical approximants for increasing values of n. The approximation improves systematically as n increases, and the curves become progressively closer to the target function throughout $[0, 2]$.

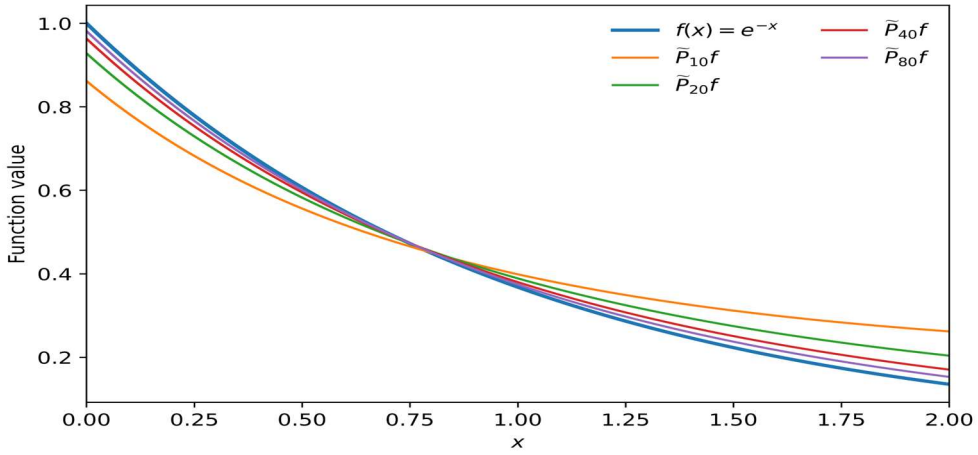


Figure 1. Approximation of $f(x) = e^{-x}$, by the proposed Baskakov–Kantorovich operator for several values of *n*.

5.2: Pointwise absolute error:

To quantify the visual behavior, we consider the pointwise error $E_n(x) = \left|\left(\tilde{P}_n f\right)(x) - f(x)\right|$. Figure 2 is plotted on a logarithmic vertical scale. The error decreases throughout the interval as n increases, providing a direct numerical illustration of convergence.

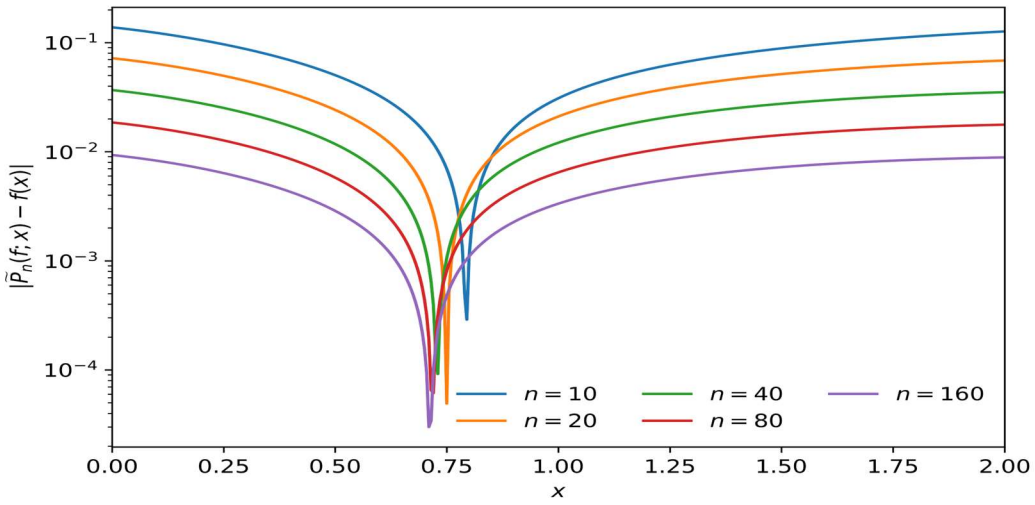


Figure 2. Pointwise absolute error for $f(x) = e^{-x}$.

5.3: Numerical checks of exponential test functions:

The exponential test functions were also examined because the construction of the modified operators is motivated by exponential-preservation properties. The maximum errors for 1, $e^{\mu x}$ and $e^{2\mu}$ are listed in Table 1. The numerical calculation gives machine-level accuracy for $e^{2\mu x}$, whereas the errors for 1 and $e^{\mu x}$ decay approximately at first order. These results should be read together with the exact moment identities in Lemma 2.1 and should not be interpreted as exact preservation of every exponential test function.

Table 1. Maximum errors for exponential test functions with $\mu = 1$.

Test Function	n	Maximum Error
$e^{0\mu x} = 1$	10	1.17641485e-01
$e^{0\mu x} = 1$	20	6.06052985e-02
$e^{0\mu x} = 1$	40	3.07691285e-02
$e^{0\mu x} = 1$	80	1.55038626e-02
$e^{0\mu x} = 1$	160	7.78206557e-03
$e^{\mu x}$	10	1.63685047e+00
$e^{\mu x}$	20	8.68916181e-01
$e^{\mu x}$	40	4.47861039e-01
$e^{\mu x}$	80	2.27383982e-01
$e^{\mu x}$	160	1.14568537e-01
$e^{2\mu x}$	10	6.24851282e-11
$e^{2\mu x}$	20	5.75539616e-13
$e^{2\mu x}$	40	1.21502808e-12
$e^{2\mu x}$	80	2.88480351e-12
$e^{2\mu x}$	160	3.16902060e-12

Acknowledgement

The author would like to express sincere gratitude to the anonymous referee for the careful review of the manuscript and for the valuable comments and constructive suggestions. The referee's observations have

helped improve the clarity, presentation and overall quality of the paper. The author is thankful for the referee's time, effort and insightful recommendations.

References

[1] Abel, U., Gupta, V., Ivan, M., On the rate of convergence of Baskakov–Kantorovich–Bézier operators for bounded variation functions, Rev. Anal. Numér. Théor. Approx. 31(2) (2002), 123–133.

[2] Acar, T., Aral, A., Rasa, I., Positive linear operators preserving τ and τ 2, Constr. Math. Anal. 2(3) (2019), 98–102.

[3] Acar, T., Aral, A., Cárdenas-Morales, D., Garrancho, P., Szász–Mirakyan type operators which fix exponentials, Results Math. 72 (2017), 1393–1404.

[4] Aral, A., Limmam, M.L., Ozsarac, F., Approximation properties of Sz.sz-Mirakyan–Kantorovich type operators, Math. Methods Appl. Sci. 42(16) (2019), 5233–5240.

[5] Aral, A., Otrocol, D., Rasa, I, On approximation by some Bernstein–Kantorovich exponential-type Polynomials, Periodica Mathematica Hungarica 79 (2019), 236-254.

[6] Boyanov, B.D., Veselinov, V.M., A note on the approximation of functions in an infinite interval by linear positive operators, Bull. Math. Soc. Sci. Math. Roum. 14(62) (1970), 9–13.

[7] Holho¸s, A., The rate of approximation of functions in an infinite interval by positive linear operators, Stud. Univ. Babes-Bolyai Math. 55(2) (2010), 133–142.

[8] Ozsarac, F., Acar, T., Reconstruction of Baskakov operators preserving some exponential functions, Math. Methods Appl. Sci. 42 (2019), 5124–5132.

[9] Paltanea, R., A note on Bernstein-Kantorovich operators, Bull. Univ. Transilv. Brasov Ser. III 6(55–2) (2013), 27–32.

[10] Sharma, P. Some Aspects of Approximation Theory Based on Various Positive Linear Operators, Eliva Press: Chişinău, Moldova, 2023.

[11] Sharma P., Sharma D., Approximation by Durrmeyer type operators using Polya distribution, Mathematical modeling for computer applications, Scrivenger Publishers (2024), 61-73.

[12] Sharma P., Study of Some Approximation Estimates Concerning Convergence of (p, q)-Variant of Linear Positive Operators, 2022 Proceedings of International E-Conference on Mathematical and Statistical Sciences: A Selçuk Meeting.

[13] Sharma P.M., Iterative combinations for Srivastava-Gupta operators, Asian European J. of Math., 14(7) (2021) 2150108, 10 pages.

[14] Sharma P.M., Abid M., Approximation by (p, q) Szász-Beta–Stancu operators, Arab. J. Math. 9 (2020) 191–200.

15. Yilmaz, O., Gupta, V., Aral, A., On Baskakov operators preserving exponential function, J. Numer. Anal. Approx. Theory 46(2) (2017), 150–161.

16. Zhang, C., Zhu, Z., Preservation properties of the Baskakov–Kantorovich operators. Comput. Math. Appl. 57(9) (2009), 1450–1455.