Empirical Evaluation and Diagnostic Robustness of Grünwald–Letnikov Fractional Derivatives in Macroeconomic Modeling: Evidence from Senegal's GDP

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Abstract

This study investigates the application of a discrete Grünwald–Letnikov (GL) fractional regression framework to the modelling of Senegal's gross domestic product (GDP). The empirical analysis uses annual macroeconomic data covering the period 1990-2023, with 33 complete observations retained after constructing the cumulative transformation of gross capital formation and the first difference of the GDP deflator. The proposed GL model assigns a variable-specific fractional order to each of the seven transformed explanatory variables. Its in-sample performance is compared with a transformed Ordinary Least Squares (OLS) benchmark estimated on the same observations and using the same seven regressors. The GL model achieves lower in-sample MSE, RMSE, and MAD, as well as lower AICc and BIC values than the OLS benchmark. Sensitivity analysis under alternative optimization bounds shows that the overall in-sample fit remains stable and improves modestly as the admissible interval is expanded, although several estimated fractional orders change substantially and some move outside the initial $[-1, 1]$ interval. Residual diagnostics

indicate approximately normal and homoscedastic residuals, but reveal statistically significant first-order serial correlation. Therefore, the GL model improves in-sample fit and information criteria relative to the benchmark, while its residual dependence indicates that further dynamic or autocorrelation-aware specifications are required.

1 Introduction

Economic growth modelling remains a central topic in applied economics. Traditional regression approaches generally assume integer-order relationships between economic variables and output indicators. However, macroeconomic variables may exhibit persistence, cumulative effects, and dependence on their historical trajectories, which cannot always be represented adequately by conventional contemporaneous regressors. Such characteristics are particularly relevant in macroeconomic systems, where historical observations may influence current economic behaviour.

Classical integer-order models commonly rely on standard levels, cumulative sums, or first differences. Fractional calculus extends differentiation and integration to arbitrary real orders and provides a flexible and mathematically rigorous framework for representing history-dependent and long-memory effects [1, 2]. More generally, fractional operators allow the degree of accumulation or differencing to be adjusted continuously rather than restricted to integer values, making them potentially suitable for modelling persistent macroeconomic relationships [3, 4, 5]. In particular, the Grünwald–Letnikov (GL) formulation is well suited to discrete observations because its finite-sample representation is based on weighted historical observations.

Recent studies have applied fractional-calculus methods to economic-growth and GDP modelling. For example, empirical investigations of macroeconomic indicators have been conducted in Malaysia [6], while recent fractional modelling frameworks have also been developed for economic applications in Serbia [7]. Motivated by these developments, the present study develops a discrete Grünwald–Letnikov fractional regression framework for modelling Senegal's GDP using seven key economic indicators.

The empirical analysis compares the proposed GL model with a transformed Ordinary Least Squares (OLS) benchmark. In both models, gross capital formation is represented by its cumulative sum, while inflation is represented by the first difference of the GDP deflator. Consequently, both spec-

ifications use the same seven transformed explanatory variables and the same 33 complete annual observations. The main distinction is that the GL model estimates a variable-specific fractional order for each regressor, whereas the OLS benchmark uses the corresponding integer-order transformed variables.

The analysis has three main objectives. First, it evaluates whether the GL transformation improves in-sample fit relative to the transformed OLS benchmark. Second, it compares the two models using MSE, RMSE, MAD, R^2, AICc, and BIC. Third, it examines the sensitivity of the estimated fractional orders to alternative optimization bounds and assesses the residual properties of the fitted models, with particular attention to normality, heteroscedasticity, and serial correlation.

2 Methodology

The empirical framework consists of two related regression specifications. The first is a transformed OLS benchmark that uses a cumulative operator for gross capital formation and a first-difference operator for inflation. The second is a Grünwald–Letnikov fractional regression model that applies a variable-specific fractional transformation to the same seven regressors.

2.1 Data Specification and Variable Definitions

The empirical analysis uses annual macroeconomic data for Senegal covering the period 1990–2023. All variables were obtained from the World Development Indicators (WDI) database of the World Bank. Gross domestic product (GDP) is used as the dependent variable y_t.

The initial dataset contains 34 annual observations. Because the first difference of the inflation variable is undefined for the first observation, one observation is lost. After removing incomplete observations, the common estimation sample contains 33 observations. This same sample is used for both the transformed OLS model and the GL fractional model.

The explanatory vector is defined as

$$X_t = [x_{1,t}, x_{2,t}, x_{3,t}, I(x_{4,t}), x_{5,t}, x_{6,t}, D(x_{7,t})]^\top,$$

where the transformed variables are defined as follows:

- x_1: Land area, measured in square kilometres;

- x_2: Arable land, expressed as a percentage of total land area;
- x_3: Total population;
- $I(x_4)$: Cumulative gross capital formation, measured in constant 2015 US dollars;
- x_5: Exports of goods and services, measured in constant 2015 US dollars;
- x_6: General government final consumption expenditure, measured in constant 2015 US dollars;
- $D(x_7)$: First difference of inflation, where inflation is measured by the annual percentage change in the GDP deflator.

The cumulative and differenced transformations are computed as

$$I(x_{4,t}) = \sum_{s=1}^{t} x_{4,s}, \tag{1}$$

and

$$D(x_{7,t}) = x_{7,t} - x_{7,t-1}. \tag{2}$$

2.2 Transformed OLS

The transformed OLS benchmark is specified as

$$GDP_t = \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 I(x_{4,t}) + \beta_5 x_{5,t} + \beta_6 x_{6,t} + \beta_7 D(x_{7,t}) + \varepsilon_t. \tag{3}$$

The model is estimated without an intercept, consistently with the implementation in R:

$$\widehat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2. \tag{4}$$

The OLS benchmark contains seven estimated regression coefficients.

2.3 Grünwald–Letnikov Fractional Operator

For a discrete time series x_t, the Grünwald–Letnikov fractional operator of order α with time step $h = 1$ is implemented as

$$D^{\alpha}_{GL} x_t = \frac{1}{h^{\alpha}} \sum_{j=0}^{t-1} w_j(\alpha) x_{t-j}, \tag{5}$$

where the weights are computed recursively according to

$$w_0(\alpha) = 1, \tag{6}$$

and

$$w_j(\alpha) = \left(1 - \frac{\alpha + 1}{j}\right) w_{j-1}(\alpha), \qquad j \geq 1. \tag{7}$$

In the implementation, the fractional transformation is applied to each of the seven transformed regressors in the common design matrix. Thus, the GL model does not use the original variables $x_1, \ldots, x_7$ directly; it uses

$$x_1,\ x_2,\ x_3,\ I(x_4),\ x_5,\ x_6,\ D(x_7).$$

2.4 GL Fractional Regression Model

The fractional regression model is given by

$$GDP_t = \sum_{k=1}^{7} c_k D^{\alpha_k}_{GL}(X_{k,t}) + \varepsilon_t, \tag{8}$$

where $X_{k,t}$ denotes the k-th transformed explanatory variable, c_k is its regression coefficient, and α_k is its variable-specific fractional order.

For a fixed vector

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_7),$$

the coefficient vector

$$\mathbf{c} = (c_1, \ldots, c_7)^{\top}$$

2.5 Evaluation Criteria

The models are compared using in-sample measures because no out-of-sample forecasting experiment is performed. The residual for observation t is defined as

$$e_t = y_t - \widehat{y}_t.$$

The mean squared error is

$$MSE = \frac{1}{n}\sum_{t=1}^{n} e_t^2, \tag{9}$$

and the root mean squared error is

$$RMSE = \sqrt{MSE}. \tag{10}$$

The mean absolute deviation is

$$MAD = \frac{1}{n}\sum_{t=1}^{n} |e_t|. \tag{11}$$

The coefficient of determination is calculated as

$$R^2 = 1 - \frac{\sum_{t=1}^{n} e_t^2}{\sum_{t=1}^{n}(y_t - \bar{y})^2}. \tag{12}$$

For a model with m estimated parameters, the corrected Akaike information criterion is computed as

$$AICc = n \ln\left(\frac{RSS}{n}\right) + 2m + \frac{2m(m+1)}{n-m-1}, \tag{13}$$

where

$$RSS = \sum_{t=1}^{n} e_t^2.$$

The Bayesian information criterion is calculated as

$$BIC = n \ln\left(\frac{RSS}{n}\right) + m \ln(n). \tag{14}$$

For the OLS model, $m = 7$, whereas for the GL model, $m = 14$.

3 Estimated Results

3.1 Fractional-Order Estimates

The main GL model was estimated over the interval $[-1, 1]$. The estimated fractional orders are reported in Table 1.

Table 1: Estimated fractional orders and GL coefficients

Variable	**Estimated order** α_k	**Coefficient** c_k
x_1	0.6161	-7.0954×10^{4}
x_2	-0.5702	3.4111×10^{2}
x_3	0.4697	2.2429×10^{3}
$I(x_4)$	1.0000	6.5546×10^{-1}
x_5	0.0602	3.9022×10^{-1}
x_6	0.8867	4.3621×10^{-1}
$D(x_7)$	-0.8836	-2.4480×10^{-1}

The estimates show heterogeneous fractional behaviour across the regressors. The estimated orders for x_2 and $D(x_7)$ are negative, whereas the orders for x_1, x_3, $I(x_4)$, x_5, and x_6 are positive. The order associated with $I(x_4)$ reaches the upper boundary of the initial interval, suggesting that the imposed bound may be restrictive for this variable. Accordingly, this estimate should not be interpreted as evidence that the true fractional order is exactly equal to one.

The economic interpretation of the signs of the estimated orders should be made cautiously. A negative fractional order indicates that the numerical GL transformation behaves in an integration-like direction within the implemented operator, while a positive order indicates a differentiation-like transformation. These signs do not, by themselves, establish a structural causal mechanism or prove a specific economic memory process.

3.2 In-Sample Model Comparison

Both models are estimated using the same 33 observations and the same seven transformed explanatory variables. The difference between them lies in the transformation applied to these variables: the OLS model uses the cumulative

and first-difference transformations, whereas the GL model applies variable-specific fractional operators to the same transformed regressors.

Table 2: In-sample comparison of the transformed OLS and GL models

Model	**MSE**	**RMSE**	**MAD**	R^2	**AICc**	**BIC**
Transformed OLS	8.650×10^{16}	2.941×10^{8}	2.393×10^{8}	0.9971	1305.444	1311.440
GL fractional model	2.625×10^{16}	1.620×10^{8}	1.295×10^{8}	0.9991	1298.946	1296.564

The GL model produces lower in-sample error measures than the transformed OLS benchmark. Specifically, its MSE decreases from 8.650×10^{16} for OLS to 2.625×10^{16} for GL. Similarly, RMSE decreases from approximately 294.11 million to 162.02 million, while MAD decreases from approximately 239.35 million to 129.54 million.

The GL model also achieves a higher R^2, increasing from 0.9971 for OLS to 0.9991. Its AICc and BIC values are lower than those of the OLS benchmark, despite the fact that the GL model estimates 14 parameters compared with 7 for OLS. These results indicate a better in-sample fit and a more favourable information-criterion balance for the GL specification. However, because the evaluation is conducted in-sample, these findings should not be interpreted as evidence of superior out-of-sample forecasting accuracy.

4 Residual Diagnostics

Residual diagnostics were conducted separately for the transformed OLS and GL models. The reported diagnostics for the GL model are based on the residuals generated by the fitted fractional specification.

4.1 Normality of Residuals

The Jarque–Bera test for the GL residuals produced a test statistic of

$$JB = 1.1028,$$

with a p-value of 0.5762. Since the p-value exceeds the 5% significance level, the null hypothesis of normally distributed residuals is not rejected.

Thus, the GL residuals do not exhibit statistically significant evidence of non-normality according to this test.

For the transformed OLS model, the Jarque–Bera statistic was 1.0235, with a p-value of 0.5995. The same conclusion applies to the OLS residuals.

4.2 Heteroscedasticity

The studentized Breusch–Pagan test for the GL residuals yielded

$$BP = 8.2887,$$

with 6 degrees of freedom and a p-value of 0.2177. Therefore, the null hypothesis of constant variance is not rejected at the 5% significance level. The test does not provide evidence of heteroscedasticity in the GL residuals.

For the transformed OLS model, the Breusch–Pagan statistic was 3.706, with a p-value of 0.7164. This result likewise does not indicate statistically significant heteroscedasticity.

4.3 Serial Correlation

The results for serial dependence differ substantially from those for normality and variance stability. For the GL model, the Breusch–Godfrey test of first-order serial correlation produced

$$LM = 5.8591, \qquad p = 0.0155.$$

At the 5% significance level, the null hypothesis of no first-order serial correlation is rejected. The Durbin–Watson test led to the same conclusion, with

$$DW = 1.1827, \qquad p = 0.001543.$$

These results indicate positive serial correlation in the GL residuals.

The transformed OLS model also exhibits significant residual autocorrelation. The Breusch–Godfrey test yielded

$$LM = 15.764, \qquad p = 7.174 \times 10^{-5},$$

while the Durbin–Watson statistic was

$$DW = 0.62886, \qquad p = 6.03 \times 10^{-8}.$$

Therefore, the residual diagnostics do not support the claim that the fitted models have statistically independent errors. The GL model reduces the strength of the detected serial dependence relative to OLS, as reflected by its higher Durbin–Watson statistic and lower Breusch–Godfrey statistic, but it does not eliminate the problem.

Table 3: Residual diagnostic tests

Model and test	**Null hypothesis**	**Statistic**	p**-value**
OLS: Jarque–Bera	Normal residuals	1.0235	0.5995
OLS: Breusch–Pagan	Homoscedasticity	3.7060	0.7164
OLS: Breusch–Godfrey	No AR(1) correlation	15.7640	7.174×10^{-5}
OLS: Durbin–Watson	No positive autocorrelation	0.6289	6.03×10^{-8}
GL: Jarque–Bera	Normal residuals	1.1028	0.5762
GL: Breusch–Pagan	Homoscedasticity	8.2887	0.2177
GL: Breusch–Godfrey	No AR(1) correlation	5.8591	0.0155
GL: Durbin–Watson	No positive autocorrelation	1.1827	0.001543

Overall, the diagnostics suggest that both models have approximately normal and homoscedastic residuals, but neither model fully removes serial correlation. Consequently, the estimated fit improvements should be interpreted as in-sample improvements rather than as complete evidence of a statistically adequate dynamic specification.

5 Sensitivity Analysis Across Optimization Bounds

To evaluate the numerical sensitivity of the estimated fractional orders, the GL model was re-estimated under the intervals $[-1, 1]$, $[-1.5, 1.5]$, and $[-2, 2]$. All three estimations used the same 33 observations and the same seven transformed explanatory variables.

All three optimization runs returned convergence code 0, indicating successful termination of the L-BFGS-B optimization. Expanding the parameter bounds improves the in-sample fit: MSE decreases from 2.6250×10^{16} under $[-1, 1]$ to approximately 2.2638×10^{16} under $[-2, 2]$. The corresponding R^2

Table 4: Sensitivity analysis under alternative optimization bounds

Bounds	**MSE**	**RMSE**	**MAD**	R^2	**AICc**	**BIC**	**Code**
$[-1, 1]$	2.6250×10^{16}	1.6202×10^{8}	1.2954×10^{8}	0.9991271	1298.946	1296.564	0
$[-1.5, 1.5]$	2.2640×10^{16}	1.5046×10^{8}	1.1456×10^{8}	0.9992471	1294.063	1291.681	0
$[-2, 2]$	2.2638×10^{16}	1.5046×10^{8}	1.1443×10^{8}	0.9992472	1294.060	1291.678	0

increases from 0.9991271 to 0.9992472, while AICc decreases from 1298.946 to 1294.060.

The estimated fractional orders also change when the bounds are expanded. For example, the order associated with $I(x_4)$ increases from 1.0000 under $[-1, 1]$ to approximately 1.3334 under $[-2, 2]$. The order associated with x_6 increases from 0.8867 to approximately 1.5508, while the order associated with $D(x_7)$ changes from -0.8836 to approximately -1.2239.

These results indicate that the numerical fit is stable in the sense that all optimization runs converge and the fit improves smoothly as the bounds are expanded. However, the fractional-order estimates are not completely invariant to the selected bounds. In particular, some estimates reach or move beyond the initial interval $[-1, 1]$. Thus, the initial bounds are partly restrictive, and the estimated orders should be interpreted together with the sensitivity analysis rather than as bound-independent structural constants.

6 Conclusion

This study developed a discrete Grünwald–Letnikov fractional regression model for Senegal's GDP and compared it with a transformed OLS benchmark. Both models used the same seven transformed explanatory variables and the same 33 complete annual observations. The OLS benchmark used the cumulative transformation of gross capital formation and the first difference of the GDP deflator, while the GL model applied variable-specific fractional operators to these same regressors.

The GL model achieved better in-sample performance than the transformed OLS model. Its MSE, RMSE, and MAD were lower, its R^2 was higher, and both AICc and BIC were lower despite the larger number of estimated parameters. These results support the usefulness of the GL transformation for improving the in-sample representation of Senegal's GDP dynamics.

The sensitivity analysis showed that all optimization runs converged successfully. Expanding the admissible bounds from $[-1, 1]$ to $[-2, 2]$ produced modest improvements in the in-sample fit and information criteria. Nevertheless, several fractional orders changed noticeably across the bounds, indicating that the estimated orders are sensitive to the optimization domain. This finding calls for caution when assigning a structural economic interpretation to individual fractional orders.

The residual diagnostics provide a mixed assessment. The residuals of both models are approximately normal and do not show significant evidence of heteroscedasticity. However, both the Breusch–Godfrey and Durbin–Watson tests indicate statistically significant positive serial correlation, although the evidence is weaker for the GL model than for the OLS benchmark. Therefore, the present GL specification improves in-sample fit but does not fully capture the temporal dependence in the residuals.

Future research should extend the model by incorporating an explicit error-dynamics structure, such as autoregressive residual terms, generalized least squares, or HAC-based inference. Out-of-sample validation should also be conducted before making claims about forecasting performance.

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