Necessary and Sufficient conditions for stability via resolvent and Lyapunov functionals of Integro-dynamic equations on time scales

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Abstract

We obtain necessary and sufficient conditions for the uniform stability of a system of Volterra integro-dynamical equations on time scales. Our work will rely on the notion of the resolvent and Lyapunov functionals. The results of this work provide improvements for its counterparts in particular time scales. The theory is illustrated with several examples.

Outline

- Resolvent and derivation of Variations of Parameters formula
- Necessary and Sufficient conditions for Uniform Stability via the Resolvent
- Applications and Comparison

The work of the above three bullets was a joint work with Dr. Murat Adivar and published under

Necessary and sufficient conditions for uniform stability of Volterra integro-dynamic equations using new resolvent equation" Analele Stiintifice ale Universitatii
Ovidius Constanta, Seria Matematica, Vol. 21(3), 2014, 17–32.

- Necessary and Sufficient conditions for Uniform Stability via Lyapunov Functionals
- Applications and Comparison
- Open Problem

This work was published as

Youssef N. Raffoul, Necessary and sufficient conditions for stability of Volterra integro-dynamic equation on time scales. Arch. Math. (Brno), 52(1):21–33, 2016.

We consider the system of Volterra integro-dynamic equation

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}, \qquad (1)$$

where A is and $n \times n$ matrix function that is continuous on $[t_0, \infty)_{\mathbb{T}}$, B is an $n \times n$ matrix function that is continuous on

$$\Omega := \{(t,u) \in \mathbb{T} \times \mathbb{T} : \; t_0 \le u \le t < \infty\},$$

and $\mathbb{T}$ is a time scales that is unbounded above with $0 \in \mathbb{T}$. We adopt the notation

$$[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}.$$

Resolvent

Definition (Adivar-Raffoul)

The resolvent matrix solution $R(t,s)$ of (1) is the unique solution of

$$R^{\Delta_s}(t,s) = -R(t,\sigma(s))A(s) - \int_{\sigma(s)}^{t} R(t,\sigma(u))B(u,s)\Delta u, \qquad (2)$$

With $R(t,t) = I$, where I is the $n \times n$ identity matrix.

Our variation of parameters formula depends on an initial function φ and therefore we state the following definition.

Definition

Let $\varphi(t)$ be a given bounded and initial function. We say $x(t,\tau_0,\varphi)$ is a solution of (1) if $x(t) = \varphi(t)$ for $t_0 \le t \le \tau_0$ and $x(t,\tau_0,\varphi)$ satisfies (1) for $t \ge \tau_0$.

In the next theorem we derive the solution of (1) in term of the resolvent. The obtained solution is referred to as the new variation of parameters.

Theorem

(New Variation of parameters) Let φ be a given bounded and continuous initial function defined on $t_0 \leq t \leq \tau_0$. $x(t)$ is a solution of (1) if and only if

$$x(t) = R(t, \tau_0)\varphi(\tau_0) + \int_{\tau_0}^{t} R(t, \sigma(s)) \int_{t_0}^{\tau_0} B(s, u)\varphi(u)\Delta u \Delta s \quad (3)$$

Proof.

Note that

$$[R(t,s)x(s)]^{\Delta} = R(t,\sigma(s))x^{\Delta}(s) + R^{\Delta_s}(t,s)x(s). \qquad (4)$$

An integration of (4) from τ_0 to t gives

$$R(t,t)x(t) - R(t,\tau_0)\varphi(\tau_0) = \int_{\tau_0}^{t} [R(t,\sigma(s))x^{\Delta}(s) + R^{\Delta_s}(t,s)x(s)]\Delta s. \qquad (5)$$

Hence, (5) implies that

$$x(t) = R(t,\tau_0)\varphi(\tau_0) + \int_{\tau_0}^{t} [R(t,\sigma(u))x^{\Delta}(u) + R^{\Delta_u}(t,u)x(u)]\Delta u. \qquad (6)$$

Using (1) into (6) yields $\square$

Proof.

$$x(t) = R(t,\tau_0)\varphi(\tau_0) + \int_{\tau_0}^{t} R(t,\sigma(u))A(u)x(u)\Delta u \tag{7}$$
$$+ \int_{\tau_0}^{t} R(t,\sigma(u)) \int_{t_0}^{u} B(u,s)x(s)\Delta s\Delta u + \int_{\tau_0}^{t} R^{\Delta_u}(t,u)x(u)\Delta u.$$

Next we consider the third term on the right side of (7). That is

$$\begin{aligned} &\int_{\tau_0}^{t} R(t,\sigma(u)) \int_{t_0}^{u} B(u,s)x(u)\Delta s\Delta u \\ = &\int_{\tau_0}^{t} R(t,\sigma(u)) \int_{t_0}^{\tau_0} B(u,s)\varphi(s)\Delta s\Delta u \\ + &\int_{\tau_0}^{t} R(t,\sigma(u)) \int_{\tau_0}^{u} B(u,s)x(s)\Delta s\Delta u. \end{aligned}$$

By changing the limits of integration, we get

Proof-Continued

Proof.

$$\int_{\tau_0}^{t} R(t,\sigma(u)) \int_{\tau_0}^{u} B(u,s)x(s)\Delta s\Delta u$$
$$= \int_{\tau_0}^{t} \left\{ \int_{\sigma(s)}^{t} R(t,\sigma(u))B(u,s)\Delta u \right\} x(s)\Delta s.$$

$$x(t) = R(t,\tau_0)\varphi(\tau_0) + \int_{\tau_0}^{t} R(t,\sigma(u)) \int_{t_0}^{\tau_0} B(u,s)\varphi(s)\Delta s\Delta u$$
$$+ \int_{\tau_0}^{t} \Big[R(t,\sigma(s))A(s)$$
$$+ \int_{\sigma(s)}^{t} R(t,\sigma(u))B(u,s)\Delta u + R^{\Delta_s}(t,s) \Big] x(s)\Delta s.$$

Now the third term on the right is zero due to (2). Interchange s with u to get (3). This completes the proof.

Set Up

For $x \in R^n$, $|x|$ denotes the Euclidean norm of x. For any $n \times n$ matrix A, define the norm of A by $|A| = \sup\{|Ax| : |x| \leq 1\}$. $C(X, Y)$ denotes the set of continuous functions $\phi : X \to Y$. Let $C(t)$ denote the set of continuous functions $\phi : [t_0, t]_{\mathbb{T}} \to R^n$ and $\|\phi\| = \sup\{|\phi(s)| : t_0 \leq s \leq t\}$. For each $\tau_{t_0} \in [t_0, t]_{\mathbb{T}}$ and $\phi \in C(\tau_0)$, there is a unique function $x : [t_0, t]_{\mathbb{T}} \to R^n$ which satisfies (1) on $[\tau_0, +\infty)$ with $x(s) = \phi(s)$ for $s \in [t_0, \tau_0]_{\mathbb{T}}$. Such a function $x(t)$ is called a solution of (1) through (τ_0, ϕ) and is denoted by $x(t, \tau_0, \phi)$.

Definition

The zero solution of (1) is stable if for each $\varepsilon > 0$ and each $\tau_0 \geq t_0$, there exists a $\delta = \delta(\varepsilon, \tau_0) > 0$ such that $[\phi \in C(\tau_0), : \|\phi\| < \delta]$ imply $|x(t, \tau_0, \phi)| < \varepsilon$ for all $t \geq \tau_0$. The zero solution of (1) is uniformly stable (US) if δ is independent of τ_0.

Theorem

The proof of the next theorem is long and tedious and can be found on Page 162 in the book by
Adivar-Raffoul, Stability, Periodicity and Boundedness in Functional Dynamical Systems on Time Scales, Springer, 2020.

Theorem

The zero solution of (1) is uniformly stable if and only if

$$\sup_{t\in[t_0,\infty)_{\mathbb{T}}} \left\{ |R(t,\tau_0)| + \int_{t_0}^{\tau_0} \left| \int_{\tau_0}^{t} R(t,\sigma(s))B(s,u)\Delta s \right| \Delta u \right\} < +\infty. \tag{8}$$

Corollary

Let $R(t,s)$ be the solution of (2) Then there exists a constant K such that $|R(t,s)| \leq K$ for $t \geq s \geq t_0$ if and only if

$$\sup_{t\in[t_0,\infty)_{\mathbb{T}}} \left| \int_s^t R(t,\sigma(u))\Big(A(u) + \int_{\sigma(s)}^u B(u,v)\Delta v\Big)\Delta u\right| < +\infty \quad (9)$$

Corollary

Suppose that

$$\sup_{t\in[\tau_0,\infty)_{\mathbb{T}}} \int_{t_0}^t |R(t,\sigma(s))|\Big(|A(s)| + \int_{t_0}^s |B(s,u)|\Delta u\Big)\Delta s < +\infty.$$

Then there exists a constant K such that $|R(t,s)| \leq K$ for $t \geq s \geq t_0$ and the zero solution of (1) is (US).

Δ derivative of $|x|$

In the continuous case one can easily find

$$\frac{d}{dt}|x(t)| = \frac{x(t)}{|x(t)|}x'(t)$$

By using the equation $x^2(t) = |x(t)|^2$ and the product rule in real case, we have

$$|x|^\Delta = \frac{x + x^\sigma}{|x| + |x^\sigma|}x^\Delta \text{ for } x \neq 0, \tag{10}$$

since the product rule is changed to $(fg)^\Delta = f^\Delta g^\sigma + fg^\Delta$ in time scale calculus. That is, the coefficient of x^Δ in (10) depends not only on the sign of $x(t)$ but also on that of $x^\sigma(t)$. Therefore, the equality $|x|^\Delta = \frac{x}{|x|}x^\Delta$ holds only if $xx^\sigma \geq 0$ and $x \neq 0$.

Let us keep this case distinct from the case $xx^{\sigma} < 0$ by separating the time scale $\mathbb{T}$ into two parts as follows

$$\mathbb{T}_{-} := \{s \in \mathbb{T} : x(s)\, x^{\sigma}(s) < 0\},$$
$$\mathbb{T}_{+} := \{s \in \mathbb{T} : x(s)\, x^{\sigma}(s) \geq 0\}.$$

Note that the set $\mathbb{T}_{-}$ consists only of right scattered points of $\mathbb{T}$. To see the relation between $|x|^{\Delta}$ and $\frac{x}{|x|}x^{\Delta}$ we state the following lemma and its proof can be found on Page 14 in the book by **Adivar-Raffoul, Stability, Periodicity and Boundedness in Functional Dynamical Systems on Time Scales, Springer, 2020.**

Lemma

Let $x \neq 0$ be Δ-differentiable. Then

$$|x|^{\Delta} = \frac{x}{|x|} x^{\Delta} \text{ for } t \in \mathbb{T}_{+}$$

and

$$\frac{x}{|x|} x^{\Delta} \leq |x|^{\Delta} \leq -\frac{x}{|x|} x^{\Delta} \text{ for } t \in \mathbb{T}_{-}. \tag{11}$$

In this section we apply the result of Theorem 5 and prove necessary and sufficient conditions for the uniform stability of the zero solution of the integro-dynamic scalar equation

$$x^{\Delta}(t) = a(t)x(t) + \int_{t_0}^{t} b(t,s)x(s)\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{12}$$

Theorem

Suppose that $a(t)$ does not change sign. Then the zero solution of (12) is uniformly stable if and only if there exist a constant K such that

$$a(t) + K \int_{t_0}^{\sigma(t)} |b(t,s)| \Delta s \leq 0, \tag{13}$$

and

$$\min \left\{ a(s)\left(-1 + \frac{1}{K}\right), a(s)\left(1 + \frac{1}{K}\right) \right\} > 0. \tag{14}$$

We make use of the previous Lemma in
$V(s) = |R(t,s)| + \int_s^t \int_{t_0}^s |R(t,\sigma(u))||b(u,v)|\Delta v \Delta u.$

$$
\begin{aligned}
V^{\Delta_s}(s) &= |R(t,s)|^{\Delta_s} - \int_0^{\sigma(s)} |R(t,\sigma(s))||b(s,v)|\Delta v \\
&+ \int_s^t |R(t,\sigma(u))||b(u,s)|\Delta u \\
&\geq \frac{R(t,s)}{|R(t,s)|} R^{\Delta_s}(t,s) - \int_0^{\sigma(s)} |R(t,\sigma(s))||b(s,v)|\Delta v \\
&+ \int_s^t |R(t,\sigma(u))||b(u,s)|\Delta u \\
&\geq -a(s)\frac{R(t,s)}{|R(t,s)|} R(t,\sigma(s)) - \int_{\sigma(s)}^t |R(t,\sigma(u))||b(u,s)|\Delta u \\
&- \int_0^{\sigma(s)} |R(t,\sigma(s))||b(s,v)|\Delta v + \int_s^t |R(t,\sigma(u))||b(u,s)|\Delta u
\end{aligned}
$$

$$V^{\Delta_s}(s) \geq -a(s)\frac{R(t,s)}{|R(t,s)|}R(t,\sigma(s)) - \int_0^{\sigma(s)} |b(s,v)|\Delta v|R(t,\sigma(s))| \tag{15}$$

where we have used

$$\int_s^t |R(t,\sigma(u))||b(u,s)|\Delta u \geq \int_{\sigma(s)}^t |R(t,\sigma(u))||b(u,s)|\Delta u.$$

If $R(t,s)R(t,\sigma(s)) \geq 0$, then
$R(t,s)R(t,\sigma(s)) = |R(t,s)||R(t,\sigma(s))|$. As a consequence, we have from (15) that

$$\begin{aligned} V^{\Delta_s}(s) &\geq \big(-a(s) - \int_{t_0}^{\sigma(s)} |b(u,v)|\Delta v\big)|R(t,\sigma(s))| \\ &\geq a(s)\left(-1+\frac{1}{K}\right)|R(t,\sigma(s))|, \end{aligned} \tag{16}$$

where we have used (13).

Also, if $R(t,s)R(t,\sigma(s)) \leq 0$, then $R(t,s)R(t,\sigma(s)) = -|R(t,s)||R(t,\sigma(s))|$. As a consequence, we have from (15) that

$$\begin{aligned} V^{\Delta_s}(s) &\geq \left(a(s) - \int_0^{\sigma(s)} |b(u,v)|\Delta v\right)|R(t,\sigma(s))| \\ &\geq a(s)\left(1+\frac{1}{K}\right)|R(t,\sigma(s))|, \end{aligned} \tag{17}$$

where we have used (13), again. Thus, (16) and (17) imply that $V^{\Delta_s}(s) \geq \gamma(s)|R(t,\sigma(s))|$ where

$$\gamma(s) := \min\left\{a(s)\left(-1+\frac{1}{K}\right), a(s)\left(1+\frac{1}{K}\right)\right\}.$$

This along with (14) yields that for any τ_0 with $t \geq \tau_0$, $V(\tau_0) \leq V(t) = |R(t,t)| = 1$. Thus, (8) is satisfied since

$$|R(t,\tau_0)| + \int_{\tau_0}^{t}\int_{t_0}^{\tau_0} |R(t,\sigma(u))||b(u,v)|\Delta v\Delta u = V(\tau_0) \leq 1.$$

Example

Let $\mathbb{T} = \{0, 1, 2...\}$. Consider the scalar equation

$$x(t+1) = \frac{x(t)}{2} - \sum_{s=0}^{t} \frac{x(s)}{(s+1)(s+2)}, \quad t \in \{0, 1, 2...\}, \qquad (18)$$

which can be expressed in the form of Eq. (12) as follows

$$\Delta x(t) = a(t)x(t) + \sum_{s=0}^{t-1} b(t,s)x(s)$$

where $\Delta x(t) := x(t+1) - x(t)$,

$$a(t) = -\frac{1}{2} - \frac{1}{(t+1)(t+2)}, \text{ and } b(t,s) = -\frac{1}{(s+1)(s+2)}.$$

Example-Continued

Example

If we let $K = -1/2$, then we have

$$a(t) + K \int_{t_0}^{\sigma(t)} |b(t,s)| \Delta s = -1 - \frac{1}{(t+1)(t+2)} + \frac{1}{2(t+1)} < 0$$

for $t \in \{0, 1, 2 \ldots\}$ and

$$\min_{t \in \mathbb{T}} \left\{ a(t) \left(-1 + \frac{1}{K} \right), a(t) \left(1 + \frac{1}{K} \right) \right\} = \frac{1}{2} + \frac{1}{(t+1)(t+2)} > 0.$$

This means conditions (13) and (14) hold and the zero solution of (18) is uniformly stable by Theorem 5.

In [Example 3.1] of the paper
Touhid M. Khandaker and Youssef N. Raffoul, Stability properties of linear Volterra discrete systems with nonlinear perturbation. J. Difference Equ. Appl., 8(10):857–874, 2002. In honour of Professor Allan Peterson on the occasion of his 60th birthday the authors ask for the existence of $K \in (0,1)$ such that

$$-\widetilde{a}(t) + K\left[1 - \sum_{s=0}^{t} |b(t,s)|\right] > 0 \tag{19}$$

in order to show that $\Delta V(s) := V(s+1) - V(s) > 0$, which is needed for uniform stability of the zero solution of

$$x(t+1) = \widetilde{a}(t)x(t) + \sum_{s=0}^{t} b(t,s)x(s).$$

In our previous Example we obtained uniform stability of the zero solution of (18) even though there is no such $K \in (0,1)$ such that condition (19) holds. So if we let

$$\widetilde{a}(t) = 1/2 \;\; \text{and} \;\; b(t,s) = -\frac{1}{(s+1)(s+2)},$$

then

$$-\widetilde{a}(t) + K\left[1 - \sum_{s=0}^{t} |b(t,s)|\right] = -\frac{1}{2} + \frac{K}{t+2} < 0$$

whenever $K \in (0,1)$ and $t \in \{0, 1, 2...\}$. Hence condition (19) does not hold.

In [Example 3.1] of the paper
Paul Eloe, Muhammad Islam, and Bo Zhang. Uniform asymptotic stability in linear Volterra integrodifferential equations with application to delay systems. Dynam. Systems Appl., 9(3):331–344, 2000,
the authors considered the scalar equation

$$x'(t) = a(t)x(t) + \int_0^t b(t,s)x(s)ds,$$

and the Lyapunov functional

$$V(s) = |R(t,s)| + \int_s^t \int_0^s |R(t,u)|\,|b(u,v)|\,dvdu.$$

In order to have $\liminf_{h \to 0^+} \frac{1}{h}[V(t+h) - V(t)] \geq 0$ they assumed the existence of a constant $K > 1$ such that for $t \geq 0$

$$a(t) + K\int_0^t |b(t,s)|ds \leq 0, \tag{20}$$

The above condition is needed to show uniform stability. We observe that our conditions (13) and (14) allow K to be negative. For example if we assume $a < 0$ and take $K = -\frac{1}{2}$, then

$$\min\left\{a(s)\left(-1 + \frac{1}{K}\right), a(s)\left(1 + \frac{1}{K}\right)\right\} =$$

$$\min\{-a(s), -3a(s)\} = -a(s) > 0.$$

Therefore, condition (13) is satisfied for any function $b(t,s)$

Necessary and Sufficient Conditions by Lyapunov Functionals

$$x^{\Delta} = A(t)x + \int_0^t C(t,s)x(s)\Delta s, \tag{21}$$

where $A(t)$ is continuous on $t \in [0,\infty)_{\mathbb{T}}$ and $C(t,s)$ is continuous on $t \in [0,\infty)_{\mathbb{T}}$ and rd-continuous with respect the second variable on $s \in [0,\infty)_{\mathbb{T}}$. We establish necessary and sufficient conditions for the stability of the zero solution of scalar (21) Volterra integro-dynamic equation on general time scales. Our approach is based on the construction of suitable Lyapunov functionals.

Theorem

Theorem

Suppose $C(t,s)$ is rd-continuous with respect to the second variable. Let $\int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u$ be continuous for $t \in [0,\infty)_{\mathbb{T}}$. Suppose A is a continuous function on $[0,\infty)_{\mathbb{T}}$ and that there are constants $\nu > 1$ and $\alpha, \beta > 0$ such that

$$\begin{aligned}(1 &+ \mu(t)|A(t)|) \int_0^t |C(t,s)|\Delta s + \mu(t)A^2(t) \\ &+ \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u - 2|A(t)| \le -\alpha \qquad (22)\end{aligned}$$

and

$$\mu(t)\big(|A(t)| + \int_0^t |C(t,s)|\Delta s\big) - (\nu - 1) \le -\beta. \qquad (23)$$

Then the zero solution of (21) is stable if and only if $A(t) < 0$ for all $t \in [0,\infty)_{\mathbb{T}}$.

Proof.

Let

$$V(t,x) = x^2(t) + \nu \int_0^t \int_t^\infty |C(u,s)|\Delta u x^2(s)\Delta s. \qquad (24)$$

Assume $A(t) < 0$ for all $t \in [0,\infty)_{\mathbb{T}}$. We have along the solutions of (21) that

$$\begin{aligned} \dot{V}(t,x) &= 2x(t)\Big(A(t)x(t) + \int_0^t C(t,s)x(s)\Delta s\Big) \\ &+ \mu(t)\Big(A(t)x(t) + \int_0^t C(t,s)x(s)\Delta s\Big)^2 \\ &- \nu \int_0^t |C(t,s)|x^2(s)\Delta s + \nu \int_{\sigma(t)}^\infty |C(u,t)|x^2(t)\Delta u \end{aligned}$$

□

Proof

Proof.

$$
\begin{aligned}
\dot{V}(t,x) &= -2|A(t)|x^2(t) + 2x(t)\int_0^t C(t,s)x(s)\Delta s \\
+ & \quad \mu(t)\Big(A^2(t)x^2(t) + 2A(t)x(t)\int_0^t C(t,s)x(s)\Delta s \\
+ & \quad \big(\int_0^t C(t,s)x(s)\Delta s\big)^2\Big) \\
- & \quad \nu\int_0^t |C(t,s)|x^2(s)\Delta s + \nu\int_{\sigma(t)}^{\infty} |C(u,t)|x^2(t)\Delta u. \qquad (25)
\end{aligned}
$$

□

Proof.

Using the fact that $ab \leq a^2/2 + b^2/2$ for any real numbers a and b, we have

$$2\int_0^t |C(t,s)|\,|x(t)||x(s)|\Delta s \leq \int_0^t |C(t,s)|(x^2(t)+x^2(s))\Delta s.$$

Also, using Theorem (Yellow Book Page 16) one obtains

$$\begin{aligned}\Big(\int_0^t |C(t,s)|x(s)\Delta s\Big)^2 &= \Big(\int_0^t |C(t,s)|^{1/2}|C(t,s)|^{1/2}x(s)\Delta s\Big)^2\\ &\leq \int_0^t |C(t,s)|\Delta s \int_0^t |C(t,s)|x^2(s)\Delta s.\end{aligned}$$

$\square$

A substitution of the above two inequalities into (25) yields

$$
\begin{aligned}
\dot{V}(t,x) \leq & \Big[\big(1+\mu(t)|A(t)|\big)\int_0^t |C(t,s)|\Delta s + \mu(t)A^2(t) \\
+ & \nu\int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u - 2|A(t)|\Big]x^2(t) \\
+ & \Big[\mu(t)\big(|A(t)| + \int_0^t |C(t,s)|\Delta s\big) \\
- & (\nu-1)\Big]\int_0^t |C(t,s)|x^2(s)\Delta s \\
\leq & -\alpha x^2(t). \qquad (26)
\end{aligned}
$$

Let $\varepsilon > 0$ be given. We will find a $\delta > 0$ so that for any bounded initial function $\phi : E_{t_0} = [0, t_0]_{\mathbb{T}} \to R$ with $||\phi|| < \delta$, we have $|x(t, t_0, \phi)| < \varepsilon$. Due to (26), V is decreasing and hence for $t \in [0, \infty)_{\mathbb{T}}$ we have that

$$\begin{aligned} x^2 \leq V(t,x) &\leq V(t_0, \phi) \\ &\leq ||\phi||^2 + \nu \int_0^{t_0} \int_{t_0}^{\infty} |C(u,s)| \Delta u \Delta s ||\phi||^2 \\ &= \Big(1 + \nu \int_0^{t_0} \int_{t_0}^{\infty} |C(u,s)| \Delta u \Delta s\Big) ||\phi||^2. \quad (27) \end{aligned}$$

Or,

$$|x(t, t_0, \phi)| \leq \varepsilon, \text{ for } \delta = \left\{ \frac{\varepsilon}{1 + \nu \int_0^{t_0} \int_{t_0}^{\infty} |C(u,s)| \Delta u \Delta s} \right\}^{1/2}.$$

This proves Uniform Stability

Proof-Continued

To prove the other part, we assume $A(t) > 0$ for some $t \in [0, \infty)_{\mathbb{T}}$. Define the functional

$$W(t,x) = x^2(t) - \nu \int_0^t \int_t^\infty |C(u,s)|\Delta u x^2(s)\Delta s.$$

Then along the solutions of (21), we have by a similar argument as for V that

$$\begin{aligned}
\dot{W}(t,x) &\geq \Big[2A(t) - \big(1+\mu(t)|A(t)|\big)\int_0^t |C(t,s)|\Delta s - \mu(t)A^2(t) \\
&- \nu \int_{\sigma(t)}^\infty |C(u,t)|\Delta u\Big] x^2(t) \\
&+ \Big[(\nu - 1) - \mu(t)\big(|A(t)| + \int_0^t |C(t,s)|\Delta s\big)\Big]\int_0^t |C(t,s)| x^2(s)\Delta s \\
&\geq \alpha x^2(t).
\end{aligned}$$

Given any $t_0 \geq 0$ and any $\delta > 0$, we can find a continuous function $\phi : E_{t_0} \to R$ with $||\phi|| < \delta$ with $||\phi|| < \delta$, and $W(t_0, \phi) > 0$ so that if we have $x(t) = x(t, t_0, \phi)$ is a solution of (21), then we have

$$\begin{aligned} x^2(t) \geq W(t, x) \quad &\geq \quad W(t_0, \phi) + \alpha \int_{t_0}^{t} W(t_0, \phi) \Delta s \\ &= \quad W(t_0, \phi) + \alpha W(t_0, \phi)(t - t_0). \end{aligned}$$

As $t \to \infty$, $|x(t)| \to \infty$, which is a contradiction.

Remark

Remark

When $\mathbb{T} = \mathbb{R}$, *condition* (23) *becomes unnecessary if we take* $\nu = 1$ *and condition* (22) *reduces to*

$$\int_0^t |C(t,s)|ds + \int_t^\infty |C(u,t)|du - 2|A(t)| \le -\alpha.$$

Corollary

Assume (22) and (23) hold with $A(t) < 0$ and $A(t)$ is bounded for all $t \in [0, \infty)_{\mathbb{T}}$. Then,
a) $x^2(t)$ is bounded,
b) $x^2(t) \in L^2\big([0, \infty)_{\mathbb{T}}\big)$,
c) $x^{\Delta}(t)$ is bounded,
and
d) the zero solution of (21) is (AS).

Theorem

Consider (21) *with* $A(t) \leq 0$ *for all* $t \in [0, \infty)_{\mathbb{T}}$. *Assume*

$$|A(s)| - \int_{\sigma(s)}^{t} |C(u,s)|\Delta u \geq 0, \ \text{for } s \in [0,\infty)_{\mathbb{T}}. \tag{28}$$

Then the zero solution of (21) *is stable. Moreover, if there exists a* $t_2 \geq 0$ *and an* $\alpha > 0$ *with*

$$|A(s)| - \int_{\sigma(s)}^{t} |C(u,s)|\Delta u \geq \alpha, \ \text{for } s \in [t_2,\infty)_{\mathbb{T}},$$

and if both $\int_0^t C(t,s)\Delta s$ *and* $A(t)$ *are bounded, then* $x = 0$ *is (AS).*

Let $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N} \text{ and } q > 1\}$. On this time scale (21) takes the form

$$D_q x(t) = A(t)x(t) + \sum_{s \in [1,t)_{q^{\mathbb{N}}}} \mu_q(s) C(t,s) x(s)), \ t \geq 1 \qquad (29)$$

where $[1,t)_{q^{\mathbb{N}}} = [1,t) \cap q^{\mathbb{N}}$;

$$\mu_q(s) := (q-1)s;$$

$$D_q \varphi(t) = \frac{\varphi(qt) - \varphi(t)}{\mu_q(t)}, \ t \in q^{\mathbb{N}};$$

and $C : [1,\infty)_{q^{\mathbb{N}}} \times [1,\infty)_{q^{\mathbb{N}}} \to \mathbb{R}$ is continuous function for $1 \leq s \leq t < \infty$, and $A : [1,\infty)_{q^{\mathbb{N}}} \to \mathbb{R}$ is also continuous. Let $A(t) = -\dfrac{q}{t^3} - \dfrac{t+1}{t}$, $C(t,s) = \dfrac{1}{t^2 s^2}$, for $(t,s) \in [1,\infty)_{q^{\mathbb{N}}} \times [1,\infty)_{q^{\mathbb{N}}}$. Let $t = q^n$. Since,

$$-\int_{\sigma(s)}^{t} |C(u,s)|\Delta u \geq -\int_{1}^{t} |C(u,s)|\Delta u$$

we have that

$$\int_1^t |C(u,s)|\Delta u = \frac{1}{s^2}\{\int\limits_1^{q^n} \frac{1}{u^2} d_q u\}$$

$$= \{\int\limits_{q^0}^{q} \frac{1}{u^2} d_q u + \int\limits_{q}^{q2} \frac{1}{u^2} d_q u + \int\limits_{q^2}^{q^3} \frac{1}{u^2} d_q u + \ldots + \int\limits_{q^{n-1}}^{q^n} \frac{1}{u^2} d_q u\}$$

$$= \frac{1}{s^2} \sum_{k=0}^{n-1} \int\limits_{q^k}^{\sigma(q^k)} \frac{1}{u^2} d_q u$$

$$= \frac{1}{s^2} \sum_{k=0}^{n-1} \mu(q^k) \frac{1}{q^{2k}} = \frac{1}{s^2} \sum_{k=0}^{n-1} (q-1)\, q^k \frac{1}{q^{2k}}$$

$$\leq \frac{1}{s^2}(q-1) \sum_{k=0}^{\infty} \frac{1}{q^k} = \frac{1}{s^2} q.$$

Thus,

$$|A(s)| - \int_{\sigma(s)}^{t} |C(u,s)|\Delta u \geq \frac{1}{s^2}q + \frac{s+1}{s} - \frac{1}{s^2}q \geq 1.$$

All the conditions of the previous Theorem are satisfied with $t_2 = 1$,(based on the definition of stability that we considered) which implies that the zero solution of (29) is (AS).

Theorem (Adivar-Raffoul(mentioned previously) Theorem 5.2)

Suppose that $A(t)$ does not change sign. Then the zero solution of (21) is uniformly stable if and only if there exists a constant K such that

$$A(t) + K \int_{t_0}^{\sigma(t)} |C(t,s)|\Delta s \leq 0, \tag{30}$$

and

$$\min\left\{A(s)\left(-1+\frac{1}{K}\right), A(s)\left(1+\frac{1}{K}\right)\right\} > 0. \tag{31}$$

Suppose $A(t) < 0$ for all $t \in [0, \infty)_{\mathbb{T}}$. Then from our condition (28) we have that $K = 1$. As a consequence, condition (31) can not hold since

$$\min \left\{ A(s)\left(-1 + \frac{1}{K}\right), A(s)\left(1 + \frac{1}{K}\right) \right\} = 0.$$

Comparison

In Example 2.3 of the previously mentioned paper by (Eloe, Islam and Zhang,) when $\mathbb{T} = \mathbb{R}$, the authors considered (21) and assumed the existence of a constant $K > 1$ such that for $t \geq 0$

$$A(t) + K\int_0^t |C(t,s)|ds \leq 0, \tag{32}$$

The above condition is needed to show uniform stability. Condition (32) imposes size limitation on $|C(t,s)|$. Precisely, we must have

$$\int_0^t |C(t,s)|ds \leq -\frac{1}{K}A(t) = \frac{1}{K}|A(t)|, \text{ for } K > 1.$$

On the other hand, our condition (28) is less restrictive since it asks for

$$\int_0^t |C(t,s)|ds \leq |A(t)|.$$

In the spirit of this talk, what can be said about the asymptotic stability of the zero solution of the scalar Volterra integro-dynamic equation

$$x^{\Delta} = A(t)x + \int_0^t C(t,s)x(s)\Delta s.$$

References

- **Adivar-Raffoul(book), Stability, Periodicity and Boundedness in Functional Dynamical Systems on Time Scales, Springer, 2020.**

THANK YOU