A Multilayer Perceptron and Cost Optimization of a Single-Server Queue with Bernoulli Feedback and Customer Impatience under a Hybrid Vacation Policy

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In a queuing system of a single server, aiming to handle a hybrid vacation, operating within a finite space, and taking account of Bernoulli feedback and balking, alongside reneging and retention. In case of queue emptiness, coming after a normal busy period, the single server makes a shift to a working vacation. The server proceeds to take a vacation in case no customers are queued upon the server's return from a working vacation. For analysis purposes, we employed a recursive method to derive the system's steady-state probabilities, thereby facilitating the evaluation of key performance metrics. Lastly, the Grey Wolf Optimizer is applied to identify the optimal service rates that minimize costs.

I. Introduction

Queueing systems with server vacations have attracted considerable attention due to their applications in computer, manufacturing, service, and communication networks. Early works mainly studied complete vacation models, while later studies introduced the concept of *working vacations*, where the server continues to serve customers but at a lower rate.

In addition, customer impatience (balking and reneging) has become a central aspect of modern queueing theory, reflecting realistic behaviors in areas such as call centers, healthcare, and online services. Recent research has also investigated feedback mechanisms, where dissatisfied customers may rejoin the system.

However, despite these advances, few studies have simultaneously integrated hybrid vacations, feedback, balking, reneging, and retention in a single unified framework. This motivates the present work.

We propose and analyze an $M/M/1/K$ queue with hybrid vacations (combining working and complete vacations), customer impatience, and Bernoulli feedback. Using recursive techniques, we derive steady-state probabilities and evaluate key performance measures. Furthermore, a cost optimization problem is formulated and solved using the Grey Wolf Optimizer (GWO), providing optimal service rates under varying system conditions.

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This integrated model contributes to the literature by offering a more realistic and comprehensive representation of queueing systems, while also addressing practical concerns of service efficiency and cost minimization.

II. Steady-state Solution

We will now examine the bi-variate process $\{(L(t),\ S(t)),\ t \geq 0\}$, where $L(t)$ represents customers roster in the system at time t, and $S(t)$ is the server's status at time t, which can take one of three values, such as $S(t) = 0:$ when the servers are in normal busy period at time t, $S(t) = 1$ when the server is in a working vacation period at time t, and $S(t) = 2$: when the server is on vacation at time t. The combined probability $P_{k,j} = \lim_{t\to\infty} P\{L(t) = k,\ S(t) = j,\ (k, j) \in \Omega\}$, denotes the steady-state probabilities of the system.
where
$(k, j) \in \{\{(k, 0) :\ k = 1, ..., K\} \cup \{(k, 1) :\ k = 0, ..., K\} \cup \{(k, 2) :\ k = 0, ..., K\}\}$. Figure 1 illustrates the transitions in the model represented by a diagram. Using the principle of balance equations

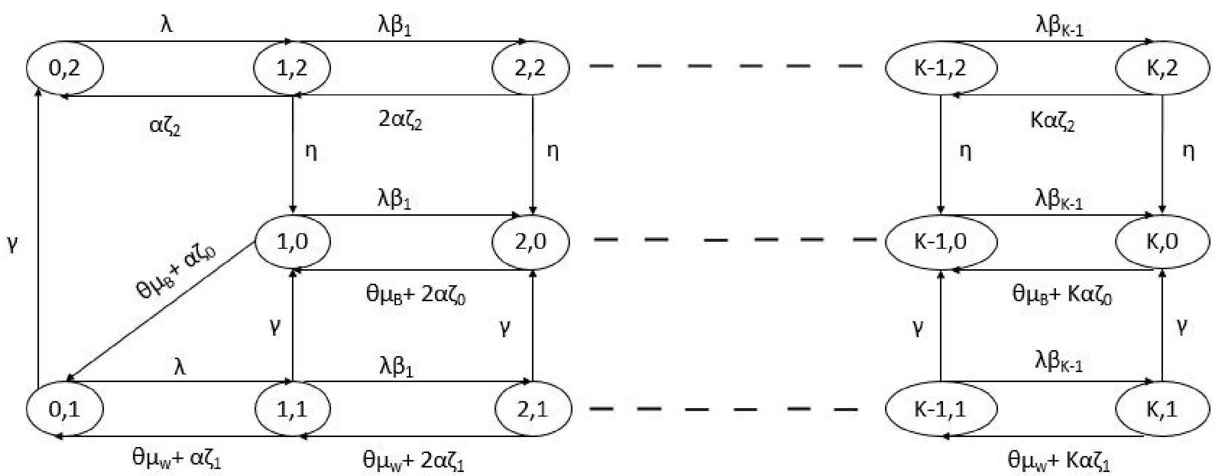

Fig. 1 State transition diagram.

$$(\lambda\beta_1 + \mu_b\theta + \alpha\zeta_0)P_{1,0} = (\theta\mu_b + 2\alpha\zeta_0)P_{2,0} + \eta P_{1,2} + \gamma P_{1,1},\ \ k = 1, \tag{1}$$

$$(\lambda\beta_k + \theta\mu_b + k\zeta_0)P_{k,0} = \lambda\beta_{k-1}P_{k-1,0} + (\theta\mu_b + (k+1)\alpha\zeta_0)P_{k+1,0} + \eta P_{k,2} + \gamma P_{k,1},$$
$$2 \leq k \leq K - 1, \tag{2}$$

$$(\theta\mu_b + K\alpha\zeta_0)P_{K,0} = \lambda\beta_{k-1}P_{k-1,0} + \eta P_{K,2} + \gamma P_{K,1},\ \ k = K, \tag{3}$$

$$(\lambda + \gamma)P_{0,1} = (\theta\mu_b + \alpha\zeta)P_{1,1} + (\theta\mu_b + \alpha\zeta_0)P_{1,0},\ \ k = 0, \tag{4}$$

$$(\lambda\beta_1 + \gamma + \theta\mu_w + 2\alpha\zeta_1)P_{1,1} = \lambda P_{0,1} + (\theta\mu_w + 2\alpha\zeta_1)P_{2,1}, \;\; k = 1, \tag{5}$$

$$(\lambda\beta_k + \gamma + \theta\mu_w + k\alpha\zeta_1)P_{k,1} = \lambda\beta_{k-1}P_{k-1,1} + (\theta\mu_w + (n+1)\alpha\zeta_1)P_{k+1,1}, \quad 2 \le k \le K-1, \tag{6}$$

$$(\gamma + \theta\mu_w + K\alpha\zeta_1)P_{K,1} = \lambda\beta_{K-1}P_{K-1,1}, \qquad k = K, \tag{7}$$

$$\lambda P_{0,2} = \gamma P_{0,1} + \alpha\zeta_2 P_{1,2}, \;\; k = 0, \tag{8}$$

$$(\lambda\beta_1 + \alpha\zeta_2 + \eta)P_{1,2} = \lambda P_{0,2} + 2\alpha\zeta_2 P_{2,2}, \qquad k = 1, \tag{9}$$

$$(\lambda\beta_k + k\alpha\zeta_2 + \eta)P_{k,2} = \lambda\beta_{k-1}P_{k-1,2} + (k+1)\alpha\zeta_2 P_{k+1,2}, \;\; 2 \le k \le K-1, \tag{10}$$

$$(\eta + K\alpha\zeta_2)P_{K,2} = \lambda\beta_{K-1}P_{K-1,2}, \qquad k = K. \tag{11}$$

The normalizing condition is

$$\sum_{k=0}^{K}\left(P_{k,0} + P_{k,1} + P_{k,2}\right) = 1. \tag{12}$$

Presented below is the theorem outlining the solution to the above equations. The probabilities describing the system size in different operational periods, namely the vacation period ($P_{2,k}$), working vacation period ($P_{1,k}$), and regular busy period ($P_{0,k}$), in the steady-state are respectively expressed as follows:

$$\begin{aligned} P_{k,2} &= \Gamma_k P_{K,2}, \\ &= \Gamma_k \left(\sum_{k=1}^{K}(\varrho_2\psi_k - \phi_k) + \sum_{k=0}^{K}(\varrho_1\delta_k + \Gamma_k)\right)^{-1}. \end{aligned} \tag{13}$$

$$P_{k,1} = \varrho_1\delta_k P_{K,2}. \tag{14}$$

$$P_{k,0} = (\varrho_2\psi_k - \phi_k)P_{K,2}, \tag{15}$$

where

$$\Gamma_k = \begin{cases} 1, & k = K, \\ \frac{K\alpha\zeta_2 + \eta}{\lambda\beta_{K-1}}, & k = K-1, \\ \frac{\lambda\beta_{k+1} + (k+1)\alpha\zeta_2 + \eta}{\lambda\beta_k}\Gamma_{k+1} - \frac{(k+2)\alpha\zeta_2}{\lambda\beta_k}\Gamma_{k+2}, & 0 \le k < K-2, \end{cases} \tag{16}$$

$$\delta_k = \begin{cases} 1, & k = K, \\ \frac{\gamma+\theta\mu_w+K\alpha\zeta_1}{\lambda\beta_{K-1}}, & k = k-1, \\ \frac{\lambda\beta_{k+1}+\gamma+\theta\mu_w+(k+1)\alpha\zeta_1}{\lambda\beta_k}\delta_{k+1} - \frac{\theta\mu_w+(k+2)\alpha\zeta_1}{\lambda\beta_k}\delta_{k+2}, & 0 \le k < K-2, \end{cases} \tag{17}$$

$$\varrho_1 = \frac{\lambda\Gamma_0 - \alpha\zeta_2\Gamma_1}{\gamma\delta_0}. \tag{18}$$

$$\psi_k = \begin{cases} 1, & k = K, \\ \frac{\theta\mu_b+K\alpha\zeta_0}{\lambda\beta_{K-1}}, & k = K-1, \\ \frac{\lambda\beta_{k+1}+\theta\mu_b+(k+1)\alpha\zeta_0}{\lambda\beta_k}\psi_{k+1} - \frac{\theta\mu_b+(k+2)\alpha\zeta_0}{\lambda\beta_k}\psi_{k+2}, & 1 \le k < K-2, \end{cases}$$

$$\phi_k = \begin{cases} 0, & k = K, \\ \frac{\eta+\gamma\varrho_1}{\lambda\beta_{K-1}}, & k = K-1, \\ \frac{\eta\Gamma_{k+1}+\gamma\varrho_1\delta_{k+1}}{\lambda\beta_k}, & 0 \le k < K-2, \end{cases}$$

$$\Theta_2 = \frac{\delta_0\varrho_1(\lambda+\gamma) - \delta_1(\theta\mu_w+\alpha\zeta_1)\varrho_1 + \phi_1(\theta\mu_b+\alpha\zeta_0)}{\psi_1(\theta\mu_b+\alpha\zeta_0)}, \tag{19}$$

and

$$P_{K,2} = \left(\sum_{k=1}^{K}(\varrho_2\psi_k - \phi_k) + \sum_{k=0}^{K}(\varrho_1\delta_k + \Gamma_k)\right)^{-1}. \tag{20}$$

by solving equations recursively (9) – (11), we find $P_{k,2} = \Gamma_k P_{K,2}$, such that (16) represent Γ_k. by equations (5) – (7), we get $P_{k,1} = \delta_k P_{K,1}$, such that (17) represent δ_k. We use equation (8) and we obtain (14) – (18). Via equations (2) – (3), we obtain $P_{k,0}$ in terms of $P_{K,0}$ and $P_{K,2}$. Using (4), we can obtain $P_{k,0}$ in terms of $P_{K,2}$ that is given by

(15). Finally, by applying the normalization condition we derive equation (20).

III. Performance measures

▹ The probabilities associated with different server states–normal busy period, working vacation, and vacation–are defined as follows:

$$\begin{aligned} P_b &= P_{K,2}\sum_{k=1}^{K}(\varrho_2\psi_k - \phi_k). \\ P_{wv} &= \varrho_1 P_{K,2}\sum_{k=0}^{K}\delta_k. \\ P_v &= P_{K,2}\sum_{k=0}^{K}\Gamma_k. \end{aligned}$$

▹ The expressions for the expected number of customers in the system (L_s) and in the queue (L_q) are defined as follows:

$$\begin{aligned} L_s &= \sum_{k=c}^{k} k(P_{k,0}+P_{k,1}+P_{k,2}) \\ L_s &= P_{K,2}\left[\sum_{k=1}^{K}(\varrho_2 k\psi_k - k\phi_k + \varrho_1 k\delta_k + k\Gamma_k)\right]. \end{aligned} \tag{21}$$

$$\begin{aligned} L_q &= \sum_{k=1}^{K}(k-1)(P_{k,0}+P_{k,1}) + \sum_{k=1}^{K} kP_{k,2}) \\ L_q &= P_{K,2}\left[\sum_{k=1}^{K}(\varrho_2(k-1)\psi_k - (k-1)\phi_k + \varrho_1(k-1)\delta_k + k\Gamma_k)\right]. \end{aligned} \tag{22}$$

▹ The expected balking rate:

$$\begin{aligned} B_r &= \lambda\sum_{k=1}^{K}(1-\beta_k)(P_{k,0}+P_{k,1}+P_{k,2}) \\ B_r &= \lambda P_{K,2}\left[\sum_{k=c}^{K}(\varrho_2\beta_k^{'}\psi_k - \beta_k^{'}\phi_k + \varrho_1\beta_k^{'}\delta_k + \beta_k^{'}\Gamma_k)\right]. \end{aligned} \tag{23}$$

▹ The expressions for the expected waiting time of customers in the system (W_s) and in the queue (W_q) are given by:

$$W_s = \frac{L_s}{\lambda'}, \ \text{ where } \ \lambda^{'} = \lambda - B_r, \ \ W_q = \frac{L_q}{\lambda'}. \tag{24}$$

▹ The expression for E_{cs} (expected number of customers served per time unit) is given by:

$$E_{cs} = \mu_b\theta\sum_{k=1}^{K} kP_{0,n} + \mu_w\theta\sum_{k=1}^{K} kP_{1,n} \tag{25}$$

▹ The expected reneging rate:

$$R_r = \alpha\zeta_0 \sum_{k=1}^{K} kP_{k,0} + \alpha\zeta_1 \sum_{k=1}^{K} kP_{k,1} + \alpha\zeta_2 \sum_{k=1}^{K} kP_{2,k}$$

$$R_r = \alpha P_{K,2}\left[\sum_{k=1}^{K}(\zeta_0\varrho_2 k\psi_k - \zeta_0 k\phi_k + \zeta_1\Theta_1 k\delta_k)\right] \tag{26}$$

$$+\alpha P_{K,2}\left[\zeta_2 \sum_{k=1}^{K} k\Gamma_k\right].$$

▹ The expected retention rate:

$$R_t = \alpha^{'}\zeta_0 \sum_{k=1}^{K} kP_{k,0} + \alpha\zeta_1 \sum_{k=1}^{K} kP_{k,1} + \alpha\zeta_2 \sum_{k=1}^{K} kP_{2,k}$$

$$R_t = \alpha^{'} P_{K,2}\left[\sum_{k=1}^{K}(\zeta_0\varrho_2 k\psi_k - \zeta_0 k\phi_k + \zeta_1\Theta_1 k\delta_k)\right] \tag{27}$$

$$+\alpha^{'} P_{K,2}\left[\zeta_2 \sum_{k=1}^{K} k\Gamma_k\right].$$

A. Numerical cost optimum

This subsection seeks the minimization of the total cost expected to be incurred by the system. Concretely using an evaluation of the cost function Λ based on the parameters μ_b and μ_w.

Due to the complexity and significant non-linearity of optimization problems, analytical solutions are often challenging to obtain. However, by utilizing suitable nonlinear optimization techniques, we can derive optimal solutions in the cost model. In this instance, we define the parameters and apply the grey wolf optimizer algorithm to obtain the optimal values (u_b^*, u_w^*) for the service rates. We write the problem designed to optimize:

$$\min_{\mu_b,\mu_w} \Lambda(\mu_b, \mu_w)$$
$$\text{s.t}\begin{cases}\mu_b - \mu_w > 0,\\ \mu_w > 0,\\ (\mu_b, \mu_w) \in \mathbb{R}_+^2.\end{cases}$$

To proceed with analyzing the cost optimization of the queueing model, we firstly set the parameters regulating the cost: $C_b = 80$, $C_v = 60$, $C_{wv} = 70$, $C_q = 55$, $C_r = 30$, $C_t = 15$, $C_{\mu_b} = 3$, $C_{\mu_w} = 2$, $C_f = 7$, $C_a = 3$.

− From 1, the minimum expected cost is seen to increase when λ increases. Nevertheless, when the vacation rate is on the rise the minimum expected cost is dropping. This confirms that reducing the vacation rate is a costly endeavor.

Table 1 **The optimal** (μ_b^*, μ_w^*) **and** $\Lambda^*(\mu_b^*, \mu_w^*)$ **for various values of** λ **and** η**, when** $\alpha = 6:1:8$**,** $K = 12$**,** $\alpha = 0.7$**,** $\theta = 0.6$**,** $\gamma = 2$**,** $\eta = [2; 2.5; 3]$**,** $\xi_0 = 0.9$**,** $\xi_1 = 1.6$**,** $\xi_2 = 1.9$.

η	λ	μ_b^*	μ_w^*	$\Lambda^*(\mu_b^*, \mu_w^*)$
	6	10.5074	2.1155	274.9516
2	7	11.8579	3.5971	304.1049
	8	13.1917	5.1614	332.3367
	6	10.7098	2.5624	273.0545
2.5	7	12.0748	4.1477	302.0183
	8	13.4340	5.7542	330.1020
	6	10.8870	2.9701	271.6902
3	7	12.2621	4.6054	300.4876
	8	13.6206	6.1863	328.4362

Table 2 **The optimal** (μ_b^*, μ_w^*) **and** $\Lambda^*(\mu_b^*, \mu_w^*)$ **for various value of** θ, **when** $\lambda = 6.5$**,** $K = 12$**,** $\alpha = 0.7$**,** $\theta = [0.4; 0.6; 0.8]$**,** $\gamma = 2$**,** $\eta = 3$**,** $\xi_0 = 0.9$**,** $\xi_1 = 1.6$**,** $\xi_2 = 1.9$.

θ	μ_b^*	μ_w^*	$\Lambda^*(\mu_b^*, \mu_w^*)$
0.4	13.7709	2.1155	344.1649
0.6	11.5769	3.7784	286.2084
0.8	10.7260	7.7133	238.7107

Table 3 **The optimal** (μ_b^*, μ_w^*) **and** $\Lambda^*(\mu_b^*, \mu_w^*)$ **for various value of** γ**, when** $\lambda = 6.5$**,** $K = 12$**,** $\alpha = 0.7$**,** $\theta = 0.6$**,** $\gamma = [1.5; 2; 2.5]$**,** $\eta = 3$**,** $\xi_0 = 0.9$**,** $\xi_1 = 1.6$**,** $\xi_2 = 1.9$.

γ	μ_b^*	μ_w^*	$\Lambda^*(\mu_b^*, \mu_w^*)$
1.5	11.3372	7.0923	292.0922
2	11.5757	3.7884	286.2084
2.5	11.8271	2.1155	278.8619

Table 4 **The optimal** (μ_b^*, μ_w^*) **and** $\Lambda^*(\mu_b^*, \mu_w^*)$ **for various value of** γ**, when** $\lambda = 6.5$**,** $K = 12$**,** $\alpha = [0.7]$**,** $\theta = 0.6$**,** $\gamma = 2$**,** $\eta = 3$**,** $\xi_0 = 0.9$**,** $\xi_1 = 1.6$**,** $\xi_2 = 1.9$.

α	μ_b^*	μ_w^*	$\Lambda^*(\mu_b^*, \mu_w^*)$
0.3	13.1045	5.5258	297.005
0.5	12.3133	4.6534	291.4246
0.7	11.5754	3.7850	286.2048

– From 2-4, we observe that with the leap of θ, there is a diminution in the minimum expected cost, and it can also be seen that a drop of the optimal anticipated cost with the hike of γ and α. This means that reducing the working vacation time, feedback probability, and retention probability results in additional cost savings.

IV. Conclusion

Our modest paper examines a queue of $M/M/1/K$ model including Bernoulli feedback under a hybrid vacation policy scenario with impatient customers. Employing a recursive method, steady-state probabilities were derived, and metrics were formulated to assess the system's performance. In addition, numerical solutions were achieved through the implementation of the Grey Wolf Optimizer to ensure optimizing the rates of the services and minimizing the function that expresses the expected cost. Finally, experimental computation results were used to emphasize the effects of several parameters on (μ_b^*, μ_w^*) and $\Lambda(\mu_b^*, \mu_w^*)$.

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