A Rectifying Reformation and Some Improved Wirtinger-Type Inequalities on Time Scales

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Outline

- **Auxiliary Inequalities**
- **Existing Dynamic Wirtinger Inequalities**
- **A Flaw in the Dynamic Wirtinger Inequality**
- **Reformulation of the Inequality**
- **Some Improved Dynamic Wirtinger-Type Inequalities**
- **Concluding Remark**

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- **Concluding Remark**

Ravi P. Agarwal, Donal O'Regan, and Samir Saker,
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Springer, Dordrecht, 1993.

Auxiliary Inequalities

Theorem 1

[2, Page 500] *For any $\gamma > 0$ and $u, v > 0$, we have the following*

$$(u+v)^\gamma \leqslant \begin{cases} 2^\gamma(u^\gamma + v^\gamma) & \text{for} \quad \gamma > 0, \\ u^\gamma + v^\gamma & \text{for} \quad 0 < \gamma \leqslant 1, \\ 2^{\gamma-1}(u^\gamma + v^\gamma) & \text{for} \quad \gamma > 1. \end{cases}$$

Theorem 2

[2, Page 500] *Let $u, v \in \mathbb{R}$ and $\gamma \in \mathbb{N}$. Then $|u+v|^\gamma \leqslant 2^{\gamma-1}(|u|^\gamma + |v|^\gamma)$.*

Theorem 3

[2, Page 518] *For $m \in \mathbb{R}$ and $\gamma \in \mathbb{N} \cup \{0\}$, we have*

$$(1+m)^{\gamma+1} \geqslant (\gamma+1)m + \frac{(\gamma+1)\gamma}{2} m^2.$$

Theorem 4

[2, Page 518] *Let* $u, v \in \mathbb{R}$ *be such that* $u > v > 0$ *and* $L(\gamma) > 1$, *where* $\gamma \in \mathbb{N} \cup \{0\}$. *Then* $(u+v)^{\gamma+1} \leqslant u^{\gamma+1} + (\gamma+1)u^{\gamma}v + L(\gamma)u^{\gamma}v$.

Theorem 5

[2, Page 518] *Let* $\gamma > 2$ *be an integer and* $u, v \geqslant 0$. *Then there exists a positive constant* $A(\gamma)$ *that does not depend on* u *and* v *such that*

$$(u+v)^{\gamma} \leqslant u^{\gamma} + \gamma u^{\gamma-1}v + A(\gamma)(v^{\gamma} + u^{\gamma-2}v^2).$$

Theorem 6

[2, Page 518] *For any integer* γ *and* $u, v \geqslant 0$, *we have*

$$u^{\gamma} + (\gamma-1)v^{\gamma} - \gamma u v^{\gamma-1} \geqslant 0.$$

Theorem 7 (Hölder's Inequality)

[1, Theorem 2.3.1] *Let p and q be real numbers such that* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then, for any* $f, g \in \mathrm{C_{rd}}([\mathrm{a}, \mathrm{b}]_{\mathbb{T}})$, *we have*

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left(\int_a^b |f(t)|^p \Delta t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \Delta t\right)^{\frac{1}{q}}.$$

Existing Dynamic Wirtinger Inequalities

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Theorem 8 ([1, Theorem 6.2.1])

Suppose $\gamma \geqslant 1$ *is an odd integer. For a positive* $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *satisfying* $M^\Delta > 0$ *or* $M^\Delta < 0$ *on* $\mathbb{I}^\kappa$, *we have*

$$\int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t \geqslant \frac{1}{\psi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b \left|M^\Delta(t)\right| (y(t))^{\gamma+1}\Delta t \tag{1}$$

for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *with* $y(a) = y(b) = 0$, *where* $\psi(\alpha,\beta,\gamma)$ *is the largest root of equation*

$$x^{\gamma+1} - 2^{\gamma-1}(\gamma+1)\alpha x^\gamma - 2^{\gamma-1}\beta = 0. \tag{2}$$

Theorem 9 ([1, Theorem 6.3.1], (Wirtinger inequality))

Suppose $\gamma \geqslant 1$ *is an odd integer. For a positive* $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *satisfying* $M^\Delta > 0$ *or* $M^\Delta < 0$ *on* $\mathbb{I}^\kappa$*, we have*

$$\int_a^b \frac{M^\gamma(t) M(\sigma(t))}{|M^\Delta(t)|^\gamma} (y^\Delta(t))^{\gamma+1} \Delta t \geqslant \frac{1}{\phi^{\gamma+1}(\alpha, \beta, \gamma)} \int_a^b |M^\Delta(t)| (y(t))^{\gamma+1} \Delta t \tag{3}$$

for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *with* $y(a) = y(b) = 0$*, where* $\phi(\alpha, \beta, \gamma)$ *is the largest root of the equation*

$$(\gamma+1)x + \frac{(\gamma+1)\gamma}{2} x^2 - (\gamma+1)x^\gamma - 2^{\gamma-1}(\gamma+1)(\alpha+1)x^\gamma - 2^{\gamma-1}\beta = 0. \tag{4}$$

Notations

For an odd integer $\gamma \geqslant 1$ and positive $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$, let

$$\alpha = \sup_{t \in \mathbb{I}^\kappa} \left(\frac{M(\sigma(t))}{M(t)} \right)^{\frac{\gamma}{\gamma+1}} \quad \text{and} \quad \beta = \sup_{t \in \mathbb{I}^\kappa} \left(\frac{\mu(t)|M^\Delta(t)|}{M(t)} \right)^{\gamma}, \tag{5}$$

$$A = \int_a^b |M^\Delta(t)|(y(t))^{\gamma+1}\Delta t \quad \text{and} \quad B = \int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t, \tag{6}$$

where M and y are defined in Theorem 9.
Set

$$C = \frac{A^{\frac{1}{\gamma+1}}}{B^{\frac{1}{\gamma+1}}}.$$

Proof

We claim that

$$B \geqslant \frac{1}{\phi^{\gamma+1}(\alpha, \beta, \gamma)} A. \tag{7}$$

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Notice that

$$\begin{aligned} A &= \int_a^b |M^{\Delta}(t)|(y(t))^{\gamma+1}\Delta t \\ &= \operatorname{sgn}(M^{\Delta}(a)) \int_a^b M^{\Delta}(t)(y(t))^{\gamma+1}\Delta t. \end{aligned}$$

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Performing integration by parts, we get

$$A = \operatorname{sgn}(M^\Delta(a)) \int_a^b [(M(t)(y(t))^{\gamma+1})^\Delta - M(\sigma(t))((y(t))^{\gamma+1})^\Delta]\Delta t. \tag{8}$$

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Employing the Pötzche chain rule, we have

$$(y^{\gamma+1})^{\Delta}(t) = \left\{ \int_0^1 (\gamma+1)(y(t) + h\mu(t)y^{\Delta}(t))^{\gamma} dh \right\} y^{\Delta}(t),$$

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which yields

$$|(y^{\gamma+1})^{\Delta}(t)| \leqslant \left| \int_0^1 (\gamma+1)(y(t) + h\mu(t)y^{\Delta}(t))^{\gamma} dh \right| |y^{\Delta}(t)|. \tag{10}$$

Now, applying Theorem 2 with $u = y(t)$ and $v = h\mu(t)y^{\Delta}(t)$ in (10), we get

$$|(y^{\gamma+1})^{\Delta}(t)| \leqslant 2^{\gamma-1}(\gamma+1)|y^{\Delta}(t)| \left(|y(t)|^{\gamma} + \frac{|\mu(t)y^{\Delta}(t)|^{\gamma}}{\gamma+1} \right), \qquad (11)$$

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and substituting (11) in (9), we get

$$\begin{aligned} A \leqslant\ & 2^{\gamma-1}(\gamma+1) \int_a^b \left(M(\sigma(t)) \frac{M^{\gamma}(t)}{(M^{\Delta})^{\gamma}(t)} \right)^{\frac{1}{\gamma+1}} |y(t)|^{\gamma} \\ & \times \left(M(\sigma(t)) \frac{(M^{\Delta})^{\gamma}(t)}{M^{\gamma}(t)} \right)^{\frac{\gamma}{\gamma+1}} |y^{\Delta}(t)| \Delta t \\ & + 2^{\gamma-1} \int_a^b \left(\frac{\mu(t) M^{\Delta}(t)}{M(t)} \right)^{\gamma} \left(\frac{M(\sigma(t)) M^{\gamma}(t)}{(M^{\Delta})^{\gamma}(t)} \right) |y^{\Delta}(t)|^{\gamma+1} \Delta t. \qquad (12) \end{aligned}$$

Applying the Hölder inequality, Theorem 7, to (12) with $f = \frac{M^\sigma M^\gamma}{(M^\Delta)^\gamma}|y^\Delta|^{\gamma+1}$, $g = \left(\frac{M^\sigma (M^\Delta)^\gamma}{M^\gamma}\right)|y|^{\gamma+1}$, $p = \gamma + 1$, and $q = \frac{\gamma+1}{\gamma}$, we get

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$$A \leqslant 2^{\gamma-1}(\gamma+1)\alpha \left(\int_a^b \left(\frac{M(\sigma(t))M^\gamma(t)}{(M^\Delta)^\gamma(t)} \right) |y^\Delta(t)|^{\gamma+1} \Delta t \right)^{\frac{1}{\gamma+1}}$$

$$\times \left(\int_a^b |(M^\Delta)^\gamma(t)|\, |y(t)|^{\gamma+1} \Delta t \right)^{\frac{\gamma}{\gamma+1}}$$

$$+ 2^{\gamma-1}\beta \int_a^b \left(\frac{M(\sigma(t))M^\gamma(t)}{(M^\Delta)^\gamma(t)} \right) |y^\Delta(t)|^{\gamma+1} \Delta t.$$

Resubstituting A and B, and rearranging the terms, we get

$$\frac{A^{\frac{1}{\gamma+1}}}{B^{\frac{1}{\gamma+1}}} \leqslant 2^{\gamma-1}\left(\beta\frac{B^{\frac{\gamma}{\gamma+1}}}{A^{\frac{\gamma}{\gamma+1}}} + (\gamma+1)\alpha\right).$$

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That is,

$$C^{\gamma+1} \leqslant 2^{\gamma-1}\left(\beta + (\gamma+1)\alpha C^{\gamma}\right), \tag{13}$$

where $C = \frac{A^{\frac{1}{\gamma+1}}}{B^{\frac{1}{\gamma+1}}} > 0$.

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where $C = \frac{A^{\frac{1}{\gamma+1}}}{B^{\frac{1}{\gamma+1}}} > 0$. Next, applying Theorem 4 with $u = C$, $v = 1$, and $L(\gamma) = 2^{\gamma-1}(\gamma+1)$, we get

$$(C+1)^{\gamma+1} \leqslant C^{\gamma+1} + (\gamma+1)C^{\gamma} + 2^{\gamma-1}(\gamma+1)C^{\gamma}. \tag{14}$$

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Then, from (13) and (14), we can write

$$(C+1)^{\gamma+1} - (\gamma+1)C^{\gamma} - 2^{\gamma-1}(\gamma+1)(\alpha+1)C^{\gamma} - 2^{\gamma-1}\beta \leqslant 0. \tag{15}$$

Using Theorem 3 with $m = C$ in (15), we get

$$(\gamma+1)C + \frac{(\gamma+1)\gamma}{2}C^2 - (\gamma+1)C^\gamma - 2^{\gamma-1}(\gamma+1)(\alpha+1)C^\gamma - 2^{\gamma-1}\beta \leqslant 0. \tag{16}$$

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Thus, we obtain

$$C \leqslant \phi(\alpha,\beta,\gamma), \tag{17}$$

where $\phi(\alpha,\beta,\gamma)$ is the largest root of Equation (4).

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Thus, we obtain

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where $\phi(\alpha, \beta, \gamma)$ is the largest root of Equation (4). On replacing C in terms of A and B in (17), we get

$$A^{\frac{1}{\gamma+1}} \leqslant \phi(\alpha, \beta, \gamma) B^{\frac{1}{\gamma+1}},$$

which is actually (7), and this completes the proof.

A Flaw in the Dynamic Wirtinger Inequality

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Example 10

Let $\mathbb{T} = \mathbb{Z}$, $\mathbb{I} = [1, 5]_{\mathbb{T}}$, $\gamma = 3$, and $M(t) = t$. Here $\sigma(t) = t + 1$ and $\mathbb{I}^{\kappa} = \{1, 2, 3, 4\}$. Further, $M(t) > 0$ for $t \in \mathbb{I}$, and $M^{\Delta}(t) = 1 > 0$ for $t \in \mathbb{I}^{\kappa}$. Take $y(t) = (t-1)(t-5)$. From this data, we get

$$\alpha = 2^{\frac{3}{4}}, \quad \beta = 1, \quad A = 418, \quad B = 8934, \quad \text{and} \quad C \approx 0.465.$$

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Now, substituting the values of α and β in (4), we get the cubic equation

$$2x + 3x^2 - (10 + 2^{\frac{15}{4}})x^3 - 2 = 0,$$

which has only one real root given by $\phi(\alpha, \beta, \gamma) \approx -0.46$.

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$$C \leqslant -0.46.$$

This contradicts the fact that $C > 0$ and $C > 1$.

Further,

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Example 11

Let $\mathbb{T} = 2^{\mathbb{N}_0} \cup \{0\}$, $\mathbb{I} = [1, 8]_{\mathbb{T}}$, $\gamma = 3$, and $M \colon \mathbb{I} \to \mathbb{R}$ be defined by $M(t) = t$. Then $\sigma(t) = 2t$ and $\mu(t) = t$ for all $t \in \mathbb{T}$. Clearly, $M > 0$ on $\mathbb{I}$. Also, since $\mathbb{I}$ contains no right-dense points, M is rd-continuous on $\mathbb{I}$ and $M^{\Delta} = 1 > 0$ exists. Hence, $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$. From this data, we get $\alpha = 2^{\frac{3}{4}}$, $\beta = 1$.

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$$(12 + 2^{\frac{19}{4}})x^3 - 6x^2 - 4x + 4 = 0$$

whose largest real root $\phi(\alpha,\beta,\gamma) = -0.488046$.

Let $y(t) = (t-1)(t-8)$. Then we see that $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$, $y(1) = y(8) = 0$, and $y^{\Delta}(t) = t + \sigma(t) - 9 = 3t - 9$.

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$$\frac{A}{\phi^4(\alpha,\beta,\gamma)} = \frac{85536}{(-0.488046)^4}$$

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However, the left side of (3) turns out to be $B = 173664$.

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However, the left side of (3) turns out to be $B = 173664$. Thus,

$$B \ngeq \frac{A}{\phi^4(\alpha,\beta,\gamma)},$$

and hence Inequality (3) does not hold.

Remark

The error in this proof of Theorem 9 is the choice of u and v to obtain (16). In order to use the Inequality of Theorem 4, we must have $C > 1$, since $v = 1$ and $u > v$.

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Proposition

For any $t > 1$ and an odd integer $\gamma > 1$, $p^{\Delta}(t) < 0$, where

$$p(t) = (\gamma+1)t + \frac{(\gamma+1)\gamma}{2}t^2 - (\gamma+1)t^{\gamma} - 2^{\gamma-1}(\gamma+1)(\alpha+1)t^{\gamma} - 2^{\gamma-1}\beta.$$

Observation

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$$t + \frac{\gamma}{2}t^2 < t^\gamma + 2^{\gamma-1}(\alpha+1)t^\gamma + \frac{2^{\gamma-1}\beta}{\gamma+1}.$$

Hence, for $C > 1$ and an odd integer $\gamma > 1$, we get

$$C > \phi(\alpha, \beta, \gamma). \tag{18}$$

Notice that (18) is valid in the context of Example 10. Therefore, the Inequality $C \leqslant \phi(\alpha, \beta, \gamma)$ is an *incorrect statement*.

Next, on expressing C in terms of A and B in (18) and rearranging the terms, we get

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Hence, for $C > 1$ and an odd integer $\gamma > 1$, we obtain

$$\int_a^b \frac{M^\gamma(t) M(\sigma(t))}{|M^\Delta(t)|^\gamma} (y^\Delta(t))^{\gamma+1} \Delta t < \frac{1}{\phi^{\gamma+1}(\alpha, \beta, \gamma)} \int_a^b |M^\Delta(t)| (y(t))^{\gamma+1} \Delta t.$$

This implies that (3) in Theorem 9 is an *incorrect statement*.

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$$\phi(\alpha, \beta, \gamma)^{-(\gamma+1)} A > B.$$

Hence, for $C > 1$ and an odd integer $\gamma > 1$, we obtain

$$\int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t < \frac{1}{\phi^{\gamma+1}(\alpha, \beta, \gamma)} \int_a^b |M^\Delta(t)|(y(t))^{\gamma+1}\Delta t.$$

This implies that (3) in Theorem 9 is an *incorrect statement*.

Remark

There may be time scales in which Theorem 9 may still hold.

Example 12

Let $\mathbb{T} = \mathbb{N}^2$, $\mathbb{I} = [1, 9]_{\mathbb{T}}$, $\gamma = 3$, and $M(t) = t$. Here $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mathbb{I}^{\kappa} = [1, 4]_{\mathbb{T}}$. Further, $M(t) > 0$ for $t \in \mathbb{I}$, and $M^{\Delta}(t) = 1 > 0$ for $t \in \mathbb{I}^{\kappa}$. Let $y(t) = (t - 1)(t - 9)$.

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$$\alpha = \left(\frac{9}{4}\right)^{\frac{3}{4}}, \beta = 27, \ A = 50625, \ B = 362500, \text{ and } C = \frac{A^{\frac{1}{\gamma+1}}}{B^{\frac{1}{\gamma+1}}} \approx 0.612.$$

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On substituting the values of α and β in (4), we have

$$4x + 6x^2 - \left(20 + 16\left(\frac{9}{4}\right)^{\frac{3}{4}}\right)x^3 - 108 = 0,$$

which has only one real root given by $\phi(\alpha, \beta, \gamma) \approx -1.28$.

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Further,

$$\phi(\alpha,\beta,\gamma)^{-4}A = 18859 < 362000 = B.$$

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Remark

However, there exist time scales with a suitable choice of functions as illustrated above, for which the conditions of Theorem 9 hold true, but $C < 1$. This is explained below.

Example 13

As seen in Example 11, we get $C = \frac{A^{\frac{1}{\gamma+1}}}{B^{\frac{1}{\gamma+1}}} = \frac{(85536)^{\frac{1}{4}}}{(173664)^{\frac{1}{4}}} = 0.837741 < 1$. Hence, Theorem 4 cannot be used to proceed in the proof of Theorem 9.

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Hence, Theorem 4 cannot be used to proceed in the proof of Theorem 9.
Moreover, since α and γ are nonnegative, the leading coefficient of (4) will always be negative. Therefore, keeping in mind the the fact that p decreases, we get

$$\lim_{t \to \infty} p(t) = -\infty. \tag{19}$$

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Now, since $\phi(\alpha, \beta, \gamma)$ is the largest root of Equation (4), (19) would imply that $p(t) < 0$ for all $t > \phi(\alpha, \beta, \gamma)$ (if not, then there would exist a root of $p(t)$ greater than $\phi(\alpha, \beta, \gamma)$, a contradiction).

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$$(\gamma+1)C + \frac{(\gamma+1)\gamma}{2}C^2 - (\gamma+1)C^\gamma - 2^{\gamma-1}(\gamma+1)(\alpha+1)C^\gamma - 2^{\gamma-1}\beta \leqslant 0.$$

Reformulation of the Inequality

Reformulation of the Inequality

Lemma 14

Let $u, v \in \mathbb{R}$, *and* γ *be an odd natural number such that* $2^{\gamma+1}u \geqslant v > 0$. *Then*

$$\left(\frac{v}{2}\right)^{\gamma+1} \geqslant (\gamma+1)v^{\gamma}u - (2^{\gamma+1}u)^{\gamma+1}. \tag{20}$$

Theorem 15 (Reformulation of Theorem 9)

Suppose $\gamma \geqslant 1$ *is an odd integer. For* $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *satisfying* $M^\Delta > 0$ *or* $M^\Delta < 0$ *on* $\mathbb{I}^\kappa$*, we have*

$$\int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t \geqslant \frac{1}{\phi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b |M^\Delta(t)|(y(t))^{\gamma+1}\Delta t \tag{21}$$

for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *with* $y(a) = y(b) = 0$*, where* $\phi(\alpha,\beta,\gamma)$ *is the largest root of the equation*

$$\frac{15}{16}(\gamma+1)\alpha x^\gamma - (2^{\gamma+1}\alpha)^{\gamma+1} - \frac{\beta}{16} = 0. \tag{22}$$

Example 16

The new equation in Example 10 is given by

$$\frac{15}{16}4(2^{\frac{3}{4}})x^3 - 2^{19} - \frac{1}{16} = 0. \tag{23}$$

Example 16

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$$\frac{15}{16}4(2^{\frac{3}{4}})x^3 - 2^{19} - \frac{1}{16} = 0. \tag{23}$$

Also, $C = 0.465 < 2^4(2^{\frac{3}{4}}) = 2^{\gamma+1}\alpha$ and the root of (23) is given by $\phi = 43.64$. On calculating, we have

$$\frac{A}{\phi^4} = \frac{418}{(43.64)^4} < 8934 = B.$$

Hence, (21) holds.

Example 17

In Example 11, we get the same Equation (23) as in Example 16.

Example 17

In Example 11, we get the same Equation (23) as in Example 16. Also, $C = 0.84 < 2^4(2^{\frac{3}{4}}) = 2^{\gamma+1}\alpha$. On calculating, we have

$$\frac{A}{\phi^4} = \frac{85536}{(43.64)^4} < 173664 = B.$$

Hence, (21) holds.

Example 18

In Example 12, we get the equation

$$\frac{15}{4}\left(\frac{9}{4}\right)^{\frac{3}{4}} x^3 - 2^{10}3^6 - \frac{27}{16} = 0. \tag{24}$$

Example 18

In Example 12, we get the equation

$$\frac{15}{4}\left(\frac{9}{4}\right)^{\frac{3}{4}} x^3 - 2^{10}3^6 - \frac{27}{16} = 0. \tag{24}$$

Here $C = 0.612 < 2^4 \left(\frac{9}{4}\right)^{\frac{3}{4}} = 2^{\gamma+1}\alpha$ and the root of Equation (24) is given by $\phi = 47.67$. On calculating we have,

$$\frac{A}{\phi^4} = \frac{50625}{(47.67)^4} < 362500 = B.$$

Hence, (21) holds.

Some Improved Dynamic Wirtinger-Type Inequalities

Theorem 19

Let $A(\gamma)$ *be the smallest positive constant for which* (4) *holds and* $\gamma \in \mathbb{N}$, $\gamma \geqslant 2$ *be such that*

$$\gamma(\gamma+1) > 6A(\gamma).$$

Suppose that $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *is a positive function for which either* $M^{\Delta} > 0$ *or* $M^{\Delta} < 0$, *and*

$$\alpha < \frac{A(\gamma)}{2^{\gamma}(\gamma+1)^2} \quad \text{and} \quad \beta < \frac{A(\gamma)}{2^{\gamma}}.$$

Moreover, let $\phi(\alpha,\beta,\gamma)$ *be the largest positive root of the equation*

$$-(\gamma+1)\left(1+\alpha 2^{\gamma-1}(\gamma+1)\right)x^{\gamma} - A(\gamma)x^{\gamma-1} + \frac{\gamma(\gamma+1)}{2}x^2 + (\gamma+1)x - A(\gamma) - 2^{\gamma-1}\beta = 0.$$

Then, for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ with $y(t_0) = y(a) = 0$, we have the inequality

$$\int_{t_0}^{a} \frac{(M(t))^{\gamma} M(\sigma(t))}{(M^{\Delta}(t))^{\gamma}} (y^{\Delta}(t))^{\gamma+1} \Delta t \geqslant \frac{1}{\phi^{\gamma+1}(\alpha, \beta, \gamma)} \int_{t_0}^{a} |M^{\Delta}(t)| (y(t))^{\gamma+1} \Delta t.$$

Theorem 20

Let $A(\gamma)$ be the smallest positive constant for which (4) holds and $\gamma \in \mathbb{N}$, $\gamma \geqslant 3$ be such that

$$(\gamma-2)(\gamma+1) > 6A(\gamma).$$

Suppose that $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ is a positive function for which either $M^\Delta > 0$ or $M^\Delta < 0$, and (19) holds. Moreover, let $\phi(\alpha,\beta,\gamma)$ be the largest positive root of the equation

$$-(\gamma+1)\left(1+\alpha 2^{\gamma-1}(\gamma+1)\right)x^\gamma - A(\gamma)x^{\gamma-1} + \frac{\gamma(\gamma+1)}{2}x^2 - A(\gamma) - 2^{\gamma-1}\beta = 0.$$

Then, for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ with $y(t_0) = y(a) = 0$, we have the inequality

$$\int_{t_0}^{a} \frac{(M(t))^\gamma M(\sigma(t))}{(M^\Delta(t))^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t \geqslant \frac{1}{\phi^{\gamma+1}(\alpha,\beta,\gamma)} \int_{t_0}^{a} |M^\Delta(t)|(y(t))^{\gamma+1}\Delta t.$$

Theorem 21

Let $A(\gamma)$ *be the smallest positive constant for which* (4) *holds and* $\gamma \in \mathbb{N}$, $\gamma \geqslant 2$ *be such that*

$$\gamma + 1 > 20A(\gamma).$$

Suppose that $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ *is a positive function for which either* $M^\Delta > 0$ *or* $M^\Delta < 0$, *and*

$$\alpha < \frac{A(\gamma)}{2^{\gamma-1}(\gamma+1)^2} \quad \text{and} \quad \beta < \frac{A(\gamma)}{2^{\gamma-1}}.$$

Moreover, let $\phi(\alpha, \beta, \gamma)$ *be the largest positive root of the equation*

$$-(\gamma+1)\left(1 + \alpha 2^{\gamma-1}(\gamma+1)\right) x^\gamma - A(\gamma)x^{\gamma-1} + (\gamma+1)x - A(\gamma) - 2^{\gamma-1}\beta = 0.$$

Then, for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I})$ with $y(t_0) = y(a) = 0$, we have the inequality

$$\int_{t_0}^{a} \frac{(M(t))^{\gamma} M(\sigma(t))}{(M^{\Delta}(t))^{\gamma}} (y^{\Delta}(t))^{\gamma+1} \Delta t \geqslant \frac{1}{\phi^{\gamma+1}(\alpha, \beta, \gamma)} \int_{t_0}^{a} |M^{\Delta}(t)| (y(t))^{\gamma+1} \Delta t.$$

Theorem 22

Let $\gamma \in \mathbb{N}$, $\gamma \geqslant 2$. *Suppose that* $M \in \mathcal{C}^1_{rd}(\mathbb{I})$ *is a positive function for which either* $M^\Delta > 0$ *or* $M^\Delta < 0$, *and*

$$\alpha < \frac{1}{2^\gamma(\gamma+1)^2} \quad \text{and} \quad \beta < \frac{1}{2^\gamma}.$$

Moreover, let $\phi(\alpha, \beta, \gamma)$ *be the largest positive root of the equation*

$$-2^{\gamma-1}(\gamma+1)\alpha x^\gamma + (\gamma+1)x - 2^{\gamma-1}\beta - \gamma = 0.$$

Then, for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{J})$ *with* $y(t_0) = y(a) = 0$, *we have the inequality*

$$\int_{t_0}^{a} \frac{(M(t))^\gamma M(\sigma(t))}{(M^\Delta(t))^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t \geqslant \frac{1}{\phi^{\gamma+1}(\alpha,\beta,\gamma)} \int_{t_0}^{a} |M^\Delta(t)|(y(t))^{\gamma+1}\Delta t.$$

Concluding Remark

We identify suitable time scales on which the Wirtinger inequality, [1, Theorem 6.3.1], does not hold. We give proper reasons for the same and reformulate this version by appropriately redefining the auxiliary equation (4). We have also identified certain time scales on which this erroneous result, [1, Theorem 6.3.1], holds. Further, using existing inequalities, we establish some new improved versions of Wirtinger-like inequality on time scales.

Thanks for your attention!