SOME EXISTENCE RESULTS TO WEAKLY COUPLED SYSTEM OF K SEMI-LINEAR FRACTIONAL EVOLUTION MODELS

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ABSTRACT. In this conversation, examines the long-term existence of solutions for a system of weakly coupled equations involving fractional evolution and various nonlinearities. The main focus is on analyzing the relationship between the regularity of initial data, memory terms, and the allowable range of exponents in a specific equation. Using L^p–L^q estimates for solutions of associated linear fractional -evolution equations with vanishing right-hand sides, along with a fixed-point method, the study establishes the existence of small-data solutions with in certain admissible exponent ranges.

Keywords: the long-term existence ; weakly coupled; small-data ; admissible exponent ranges.

Mathematics Subject Classification: Primary 35R11, , Secondary 35A01.

1. INTRODUCTION

we consider the following the Cauchy problems for weakly coupled systems of semi-linear Fractional σ-evolution equations with nonlinear memory:

$$\begin{cases} \partial_t^{1+\alpha_1} v_1 + (-\Delta)^{\sigma_1} v_1 + S_1^2 v_1 = Z_{\mu_1,p_1}(v_k), \\ \partial_t^{1+\alpha_2} v_2 + (-\Delta)^{\sigma_2} v_2 + S_2^2 v_2 = Z_{\mu_2,p_2}(v_1), \\ \qquad\qquad\vdots \\ \partial_t^{1+\alpha_k} v_k + (-\Delta)^{\sigma_k} v_k + S_k^2 v_k = Z_{\mu_k,p_k}(v_{k-1}), \\ v_\ell(0,x) = v_{0\ell}(x)\ \partial_t v_\ell(0,x) = 0,\ \ell = 1,2,\dots,k \end{cases} \tag{1.1}$$

where for $k \geq 2$, for $\ell = 1,2,\dots,k$,$\alpha_\ell \in (0,1)$, $\sigma_\ell \geq 1$,$S_\ell > 0$, $p_\ell > 1$, $(t,x) \in [0,\infty) \times \mathbb{R}^n$,with

$$Z_{\mu,p}(v)(t,x) = \int_0^t (t-s)^\mu |v(s,x)|^p ds$$

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2. Main Results

Theorem 2.1. *(Loss of decay) Let us assume* $0 < \alpha_\ell < 1$, $\alpha_\ell < \mu_\ell < 1$, $\sigma_\ell \geq 1$, $m_l \geq 1$ *and* $S_\ell > 0$ *for all* $\ell = 1, ..., k$. *Assume that for all* $\delta > 0$

$$\max\left\{1, \frac{\alpha_1 - \mu_1 + 1}{\mu_k - \alpha_k}, \frac{m_k}{m_1} - \delta\right\} < p_1 < \frac{1}{\mu_k - \alpha_k},$$

,

$$\max\left\{1, \frac{\alpha_2 - \mu_2 + 1}{\mu_1 - \alpha_1 - \gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}, \frac{m_1}{m_2} - \delta\right\} < p_2 < \frac{1}{\mu_1 - \alpha_1 - \gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}$$

and for $l = 3, \cdots, k-1$

$$p_\ell < \frac{1}{\mu_{\ell-1} - \alpha_{\ell-1} - \gamma^{(\mu_k, \ldots, \mu_{\ell-2})}_{(\alpha_k, \ldots, \alpha_{\ell-2})}(p_1, ..., p_{\ell-1})},$$

$$p_l > \max\left\{1, \frac{\alpha_\ell - \mu_\ell + 1}{\mu_{\ell-1} - \alpha_{\ell-1} - \gamma^{(\mu_k, \ldots, \mu_{\ell-2})}_{(\alpha_k, \ldots, \alpha_{\ell-2})}(p_1, ..., p_{\ell-1})}, \frac{m_{l-1}}{m_l} - \delta\right\}$$

and

$$p_k > \max\left\{\frac{m_k}{m_{k-1}} - \delta, \frac{1}{\mu_{k-1} - \alpha_{k-1} - \gamma^{(\mu_k, \ldots, \mu_{k-2})}_{(\alpha_k, \ldots, \alpha_{k-2})}(p_1, ..., p_{k-1})}\right\},$$

where, for $l = 3, \cdots, k-1$

$$\begin{cases} \gamma^{(\mu_k)}_{(\alpha_k)}(p_1) = 1 - p_1(\mu_k - \alpha_k) \\ \gamma^{(\mu_k, \mu_1)}_{(\alpha_k, \alpha_1)}(p_1, p_2) = 1 - p_2(\mu_1 - \alpha_1) + p_2 \gamma^{(\mu_k)}_{(\alpha_k)}(p_1) \\ \gamma^{(\mu_k, \ldots, \mu_{\ell-1})}_{(\alpha_k, \ldots, \alpha_{\ell-1})}(p_1, ..., p_\ell) = 1 - p_\ell(\mu_\ell - \alpha_\ell) + p_\ell \gamma^{(\mu_k, \ldots, \mu_{\ell-2})}_{(\alpha_k, \ldots, \alpha_{\ell-2})}(p_1, ..., p_{\ell-1}). \end{cases} \tag{2.2}$$

Then there exists a positive constant ε *such that for any data*

$$(v_{01}, .., v_{0k}) \in \mathcal{A}_k := \prod_{\ell=1}^{k} \big(L^{m_l}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big) \quad \text{with } \|(v_{01}, .., v_{0k})\|_{\mathcal{A}_k} \leq \varepsilon$$

, we have a uniquely determined global (in time) Sobolev solution

$$v \in \prod_{\ell=1}^{k} \mathcal{C}\big([0, \infty), L^{m_l}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big)$$

to the Cauchy problem (1.1). *Moreover, the solution satisfies for all* $s \geq 0$ *and* $l = 2, \cdots, k-1$, *the decay estimate*

$$\|v_1(s, \cdot)\|_{L^q} \lesssim (1+s)^{\alpha_1 - \mu_1 + \gamma^{\mu_k}_{\alpha_k}(p_1)} \|v_{01}\|_{L^{m_1} \cap L^\infty} \quad \text{for all} \quad q \in [m_1, \infty],$$

$$\|v_\ell(s, \cdot)\|_{L^q} \lesssim (1+s)^{\alpha_\ell - \mu_\ell + \gamma^{(\mu_k, \ldots, \mu_{\ell-1})}_{(\alpha_k, \ldots, \alpha_{\ell-1})}(p_1, \ldots, p_\ell)} \|v_{0l}\|_{L^{m_l} \cap L^\infty} \quad \text{for all} \quad q \in [m_l, \infty],$$

$$\|v_k(s, \cdot)\|_{L^q} \lesssim (1+s)^{\alpha_k - \mu_k} \|v_{0k}\|_{L^{m_k} \cap L^\infty} \quad \text{for all} \quad q \in [m_k, \infty].$$

Lemma 2.1. *Suppose that $\theta \in [0,1), a \geq 0$ and $b \geq 0$. Then there exists a constant $C = C(a,b,\theta) > 0$ such that for all $t > 0$ the following estimate holds:*

$$\int_0^t (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}\,d\tau \leq \begin{cases} C(1+t)^{-\min\{a+\theta,b\}} & \text{if} \quad \max\{a+\theta,b\} > 1, \\ C(1+t)^{-\min\{a+\theta,b\}}\ln(2+t) & \text{if} \quad \max\{a+\theta,b\} = 1, \\ C(1+t)^{1-a-\theta-b} & \text{if} \quad \max\{a+\theta,b\} < 1. \end{cases} \tag{2.3}$$

Proof. We introduce for all $T > 0$ the space $X^k(T)$ as follows:

$$X^k(T) := \prod_{\ell=1}^{k} C\big([0,T], L^{r_\ell}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|v\|_{X^k(T)} := \sup_{0\leqslant t\leqslant T}\Big\{(1+t)^{-\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}M_1(t,v_1) + \sum_{l=2}^{k-1}(1+t)^{-\gamma^{(\mu_k,\dots,\mu_{\ell-1})}_{(\alpha_k,\dots,\alpha_{\ell-1})}(p_1,\dots,p_\ell)}M_\ell(t,v_l) + M_k(t,v_k)\Big\},$$

we offer the operator N by

$$N \ : \ v \in X^k(T) \to N(v) = N(v)(t,x) := v^{ln}(t,x) + v^{nl}(t,x).$$

We will prove that the operator N satisfies the following two inequalities:

$$\|N(v)\|_{X^k(T)} \lesssim \|(v_{01}, v_{02}, ...v_{0k})\|_{\mathcal{A}_k} + \sum_{\ell=1}^{\ell=k}\|v\|^{p_\ell}_{X^k(T)}, \tag{2.4}$$

$$\|N(v) - N(\overline{v})\|_{X^k(T)} \lesssim \|u-\overline{v}\|_{X^k(T)}\sum_{\ell=1}^{\ell=k}\Big(\|v\|^{p_\ell-1}_{X^k(T)} + \|\overline{v}\|^{p_l-1}_{X^k(T)}\Big) \tag{2.5}$$

Using the definition of the norm in $X^k(T)$, we may conclude:

$$\|v^{ln}\|_{X^k(T)} \lesssim \|(v_{01}, v_{02}, ..., v_{0k})\|_{\mathcal{A}_k}.$$

Hence, in order to complete the proof of (2.4) it is reasonable to we shall show the following inequality

$$\|v^{nl}\|_{X^k(T)} \lesssim \sum_{\ell=1}^{\ell=k} \|v\|^{p_\ell}_{X^k(T)}.$$

If $v \in X^k(T)$, firstly we apply interpolation we derive on all items plus Ona $q \in [r_1, \infty]$, we have

$$\|v_1^{nl}(t,\cdot)\|_{L^q} \lesssim \|v\|^{p_1}_{X(T)} I_1(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_1 q \in [r_k, \infty],$$

We are interested to estimate $I_1(t)$ the right-hand . For this we need the Lemma 2.1. if we assume that $p_1 < \frac{1}{\mu_k - \alpha_k}$.
to obtain

$$I_1(t) \lesssim (1+t)^{\alpha_1 - \mu_1 + \gamma^{\mu_k}_{\alpha_k}(p_1)}.$$

On other hands, the conditions $q \in [r_1, \infty]$ and $p_1 q \in [r_k, \infty]$ implies that $p_1 \geq \frac{r_k}{r_1}$.

For $l = 2, \cdots, k-1$ and $q \in [r_l, \infty]$, we have

$$\begin{aligned}\|v_\ell^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau - s)^{\alpha_\ell - 1} \||v_{\ell-1}(s,\cdot)|^{p_\ell}\|_{L^q}\, ds\, d\tau \\ &\lesssim \|v\|^{p_\ell}_{X(T)} I_\ell(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_\ell q \in [r_{\ell-1}, \infty],\end{aligned}$$

we estimat $I_\ell(t)$, we use the Lemma2.1 if we assume that

$$p_\ell < \frac{1}{(\mu_{\ell-1} - \alpha_{\ell-1}) - \gamma^{(\mu_k,\ldots,\mu_{\ell-2})}_{(\alpha_k,\ldots,\alpha_{\ell-2})}(p_1, \ldots, p_{\ell-1})}$$

and

$$p_{l-1} > \frac{1 + \alpha_{l-1} - \mu_{l-1}}{\mu_{\ell-2} - \alpha_{\ell-2} - \gamma^{(\mu_k,\ldots,\mu_{\ell-3})}_{(\alpha_k,\ldots,\alpha_{\ell-3})}(p_1, \ldots, p_{\ell-2})}$$

On other hands, the conditions $q \in [r_\ell, \infty]$ and $p_\ell q \in [r_{\ell-1}, \infty]$ implies that $p_\ell \geq \frac{r_\ell - 1}{r_\ell}$.
Once more we apply Lemma 2.1 to obtain

$$I_\ell(t) \lesssim (1+t)^{\alpha_\ell - \mu_\ell + \gamma^{(\mu_k,\cdots,\mu_{\ell-1})}_{(\alpha_k,\cdots,\alpha_{\ell-1})}(p_1,\ldots,p_\ell)}$$

Finally, for $q \in [r_k, \infty]$, we have

$$\|v_k^{nl}(t,\cdot)\|_{L^q} \lesssim \|v\|^{p_k}_{X(T)} I_k(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_k q \in [r_{k-1}, \infty],$$

We are interested to estimate $I_k(t)$ the right-hand . For this we need the Lemma 2.1. we obtain $I_k(t) \lesssim (1+t)^{\alpha_k-\mu_k}$,if we assume that

$$p_{k-1} > \frac{\alpha_{k-1} - \mu_{k-1} + 1}{(\mu_{k-2} - \alpha_{k-2}) - \gamma^{(\mu_k,\ldots,\mu_{k-3})}_{(\alpha_k,\ldots,\alpha_{k-3})}(p_1, \ldots, p_{k-2})}.$$

it remains for us to prove(2.5)Since the number of pages is limited , if I am accepted into this global event , I will end this proof in a private meeting. This completes the proof.

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