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# PRODUCT OF TOEPLITZ MATRIX AND KTH-ORDER SLANT TOEPLITZ MATRIX.

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**Abstract-**In this paper, we will study the product of a Toeplitz matrix with a kth-order slant Toeplitz matrix. We will find the necessary and sufficient conditions for this product to be a kth-order slant Toeplitz matrix.
**Keywords:** Toeplitz matrix, kth-order slant Toeplitz matrix, Hankel matrix, kth-order slant Hankel matrix.

## 1. INTRODUCTION

Some authors have deduced several properties of kth-order slant Toeplitz (or Hankel) matrices based on the kth-order slant Toeplitz operators, such as [3]. In [2], Gu gives necessary and sufficient conditions for a product of two Toeplitz matrices to be a Toeplitz matrix, and in [1], Bensliman and Yagoub give necessary and sufficient conditions for a product of two rectangular Toeplitz matrices to be a rectangular Toeplitz matrix.

The main objective of this paper is to study the product of two matrices $A$ and $B$, where $A$ is a Toeplitz matrix and $B$ is a kth-order slant Toeplitz matrix, and we find the conditions under which the product $AB$ is a kth-order slant Toeplitz matrix. It is easy to see that if $k = 1$, we will find the necessary and sufficient conditions for a product of two Toeplitz matrices to be a Toeplitz matrix. Gu discussed this case in [**?**].

## 2. PRELIMINARIES

Let $M_{n\times n}(\mathbb{C})$ denote the set of $n \times n$ matrices with entries in $\mathbb{C}$. Throughout this paper, the rows and columns of any $n \times n$ matrix will be indexed from 0 to $n-1$. Accordingly, we denote the standard basis of $\mathbb{C}^n$ by $\{e_0, e_1, ..., e_{n-1}\}$. Let $J$ and $S$ be two $n \times n$ matrices defined as follows

$$J = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ & & 1 & 0 \\ & \iddots & & \vdots \\ & & & \\ 1 & & & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & 0 & \cdots & & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & \cdots & & 1 & 0 \end{bmatrix}.$$

It is easy to verify that $J_n^2 = I_n$

Recall that a Toeplitz matrix is a matrix of the form

$$A=\begin{bmatrix} a_0 & \overline{\alpha}_1 & \cdots & \overline{\alpha}_{n-1} \\ a_1 & a_0 & & \vdots \\ \vdots & \ddots & & \\ a_{n-1} & & & \end{bmatrix},$$

and the kth-order slant Toeplitz matrix is a matrix of the form

$$B=\begin{bmatrix} b_0 & b_{-1} & \cdots & b_{-n+1} \\ b_k & b_{k-1} & & b_{k-n+1} \\ b_{2k} & b_{2k-1} & & b_{2k-n+1} \\ \vdots & & & \\ b_{(n-1)k} & & & b_{(n-1)k-n+1} \end{bmatrix},$$

where k is a positive integer such that $1 \leq k \leq n-1$.

In this work, the set of $n \times n$ Toeplitz matrices is denoted by $\mathcal{T}(n)$, and the set of $n \times n$ kth-order slant Toeplitz matrices is denoted by $\mathcal{T}(n,k)$.

For two vectors $x$ and $y$ in $\mathbb{C}^n$ .the notation $x \otimes y$ denotes a tensor product of $x$ and $y$, which is defined by

$$(x \otimes y)z = \langle z, y\rangle x \qquad \text{for all } z \in \mathbb{C}^n,$$

where $x \otimes y = x\overline{y}^T$, and $\langle z, y\rangle = z_1\overline{y}_1 + z_1\overline{y}_2 + ... + z_n\overline{y}_n$.

Let $\Delta A = A - SAS^{*k}$ be the displacement of matrix $A \in M_{n\times n}(\mathbb{C})$.

It is worth noting the following elementary lemma, which characterizes when a matrix in $M_{n\times n}(\mathbb{C})$ is a Toeplitz matrix or a kth-order slant Toeplitz matrix.

**Lemma 1.** *Let $A$ be $n \times n$ matrix. Then*

(1) $A \in \mathcal{T}(n,k)$ *if and only if*

$$\Delta A = \sum_{i=0}^{k-1} u_i \otimes e_i + e_0 \otimes v,$$

*for some $u$ and $v$ be vectors in $\mathbb{C}^n$.*

(2) $A \in \mathcal{T}(n)$ *if and only if*

$$A - SAS^* = u \otimes e_0 + e_0 \otimes v,$$

*for some $u$ and $v$ be vectors in $\mathbb{C}^n$.*

*Proof.* (1) Let $A$ be $n \times n$ Toeplitz matrix, then the matrix $S_nAS_n^{*k}$ is the same as matrix $A$ except for the k-1 first rows and the first column of the matrix $SAS^{*k}$, which are zero. Thus, $A - SAS^{*k}$ is $n \times n$ matrix all of its coefficients are zero except those in the k-1 first rows and the first column.

(2) The proof relies on the previous assertion for $k = 1$.

□

Throughout this paper, we assume that $a = (0, a_1, a_2, ..., a_{n-1})^T$, $\alpha = (0, \alpha_1, \alpha_2, ..., \alpha_{n-1})^T$, $\beta = (0, .., 0, b_{-k}, ..., b_{-n+1})^T$, and $b_{(i,k)} = (b_{-i}, b_{k-i}, b_{2k-i}, ..., b_{(n-1)k-i})^T$, for $i = 0, 1, ..., n-1$.

**Lemma 2.** *Let $A \in \mathcal{T}(n)$ and $B \in \mathcal{T}(n,k)$. Then we have*

1) $Ae_0 = a + a_0e_0$.
2) $SAe_{n-1} = \hat{\alpha}$, *where* $\hat{\alpha} = (0, \overline{\alpha}_{n-1}, ..., \overline{\alpha}_1)^T$.
3) $S^kB^*e_{n-1} = b^\sharp$, *where* $b^\sharp = (0, ..., \overline{b}_{(n-1)k}, \overline{b}_{(n-1)k-1}, ..., \overline{b}_{(n-1)k-n+k+1})^T$.

**Lemma 3.** *Let $x_1, x_2, y_1$ and $y_2$ be vectors in $\mathbb{C}^n$ . If the matrices $x_1 \otimes y_1$ and $x_2 \otimes y_2$ are equal, then one of the following cases is satisfied*

1) $x_1 = x_2 = 0$ *or* $y_1 = y_2 = 0$.

2) $x_1 = y_2 = 0$ *or* $y_1 = x_2 = 0$.
3) $x_1 = \lambda x_2$ *and* $\overline{\lambda} y_1 = y_2$ *for some nonzero scalar* $\lambda$.

*By convention* $\frac{1}{\infty} = 0$*, we can propose that* $x_1 \otimes y_1 = x_2 \otimes y_2$*, then one of the following cases is satisfied*

i. *Trivial case:* $x_1 = x_2 = 0$ *or* $y_1 = y_2 = 0$.
ii. $x_1 = \lambda x_2$ *and* $\overline{\lambda} y_1 = y_2$ *for some scalar* $\lambda \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

The following lemma plays a pivotal role in our analysis, as it facilitates the manipulation of the product between a Toeplitz matrix and a kth-order slant Toeplitz matrix.

**Lemma 4.** *Let* $A \in \mathcal{T}(n)$ *and* $B \in \mathcal{T}(n,k)$*. Then we have*

$$\Delta(AB) = \sum_{i=0}^{k-1} Ab_{(i,k)} \otimes e_i + a \otimes \beta + e_0 \otimes (\overline{a_0}\beta + S^k B^* S^* \alpha) - \hat{\alpha} \otimes b^\sharp.$$

## 3.PRODUCT OF TOEPLITZ MATRIX AND kTH-ORDER SLANT TOEPLITZ MATRIX.

**Theorem 1.** *Let* $A \in \mathcal{T}(n)(A \neq 0)$ *and* $B \in \mathcal{T}(n,k)(B \neq 0)$ *with* $1 \leq k \leq n-1$*. Then* $AB \in \mathcal{T}(n,k)$ *if and only if one of the following conditions is met*

1) $A = a_0 I$*, for some* $a_0 \in \mathbb{C}$.
2) $A$ *is an upper triangular Toeplitz matrix and* $B$ *is kth-order slant Toeplitz matrix with* $b^\sharp = 0$.
3) $A$ *is an lower triangular Toeplitz matrix and* $B$ *is kth-order slant Toeplitz matrix with* $\beta = 0$.
4) $B$ *is kth-order slant Toeplitz matrix with* $\beta = b^\sharp = 0$.
5) $A$ *is a Toeplitz matrix with* $a = \lambda\hat{\alpha}$*, and* $B$ *is kth-order slant Toeplitz matrix with* $b^\sharp = 0\overline{\lambda}\beta$*, for some* $\lambda \in \mathbb{C}^*$.

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