Strongly quasiconvex functions: what we know (so far)

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Optimization Theory: Concepts, Methods, and Applications

- $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ with a convex domain[1] is
  - *strongly convex*: $\exists \gamma \in]0, +\infty[$ s.t. $\forall x, y \in \mathrm{dom}\ h\ \forall \lambda \in [0,1]$

$$h(\lambda y + (1-\lambda)x) \le \lambda h(y) + (1-\lambda)h(x) - \lambda(1-\lambda)\frac{\gamma}{2}\|x-y\|^2$$

  - *strongly quasiconvex*[2]: $\exists \gamma \in]0, +\infty[$ s.t. $\forall x, y \in \mathrm{dom}\ h$ $\forall \lambda \in [0,1]$

$$h(\lambda y + (1-\lambda)x) \le \max\{h(y), h(x)\} - \lambda(1-\lambda)\frac{\gamma}{2}\|x-y\|^2$$

- both properties can be considered on a set $U \subseteq \mathrm{dom}\, h$, too
- strongly convex $\Rightarrow$ strongly quasiconvex
- $\|\cdot\|$ is strongly quasiconvex on any bounded convex $U \subseteq \mathbb{R}^n$ but *not* strongly convex
- $\sqrt{\|\cdot\|}$ is strongly quasiconvex on any bounded convex $U \subseteq \mathbb{R}^n$ but *not* convex
- any constant function is convex but *not* strongly quasiconvex

[1] $\mathrm{dom}\, h = \{x \in \mathbb{R}^n : h(x) < +\infty\}$

[2] introduced by Polyak, 1966

- the composition of a strongly quasiconvex function with a linear operator is strongly quasiconvex
- the maximum of finitely many strongly quasiconvex functions is strongly quasiconvex
- a strongly quasiconvex function cannot be bounded from above on an unbounded set[3]
- $K \subseteq \mathbb{R}^n$ solid convex cone, $h : K \to \mathbb{R}$ quadratic: h strongly quasiconvex $\Leftrightarrow$ h strongly convex[4]
- $\emptyset \neq K \subseteq \mathbb{R}^n$ convex, $h : K \to \mathbb{R}$: h strongly quasiconvex $\Leftrightarrow$
$$t \in \mathbb{R} \mapsto h\left(x + \frac{t}{\|x-y\|}(x-y)\right)$$
strongly quasiconvex on $[0, \|x-y\|]$ $\forall x \neq y \in K$[5]

[3] Jovanović, 1996

[4] Jovanović, 1993

[5] Jovanović, 1996

- a strongly quasiconvex function is 2-supercoercive[6]
- a proper strongly quasiconvex function has at most one minimizer on a convex set that touches its domain
- a proper lsc strongly quasiconvex function has *one* minimizer on a closed convex set[7]
- $\emptyset \neq K \subseteq \mathbb{R}^n$ closed convex, $h : K \to \mathbb{R}$ lsc strongly quasiconvex, $\beta > 0 \Rightarrow \mathrm{Prox}_{\beta h + \delta_K}(\cdot) \neq \emptyset$[8] (but not necessarily singleton)
- $K \subseteq \mathbb{R}^n$ closed convex, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper lsc strongly quasiconvex, $K \subseteq \mathrm{dom}\ h$, $\beta > 0 \Rightarrow$
$$\mathrm{Fix}\left(\mathrm{Prox}_{\beta h + \delta_K}\right) = \arg\min_K h$$[9]

[6] $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\liminf_{\|x\| \to +\infty} \frac{h(x)}{\|x\|^2} > 0$

[7] Lara, 2022

[8] Lara, 2022

[9] Korablev, 1980; Kabgani & Lara, 2022

- $\emptyset \neq K \subseteq \mathbb{R}^n$ convex, $h : K \to \mathbb{R}$ strongly quasiconvex, $\arg\min_K h = \{\overline{x}\} \Rightarrow$

$$h(x) - h(\overline{x}) \geq \frac{\gamma}{8}\|x - \overline{x}\|^2 \; \forall x \in K$$

- $h : \mathbb{R}^n \to \mathbb{R}$ proper with $\operatorname{dom} h$ convex - t.f.a.e.[10]
  (a) h convex
  (b) $h + \frac{1}{2\beta}\|z - \cdot\|^2$ strongly convex $\forall z \in \mathbb{R}^n \; \forall \beta > 0$
  (c) $h + \frac{1}{2\beta}\|z - \cdot\|^2$ quasiconvex $\forall z \in \mathbb{R}^n \; \forall \beta > 0$
  (d) $h + \frac{1}{2\beta}\|z - \cdot\|^2$ strongly quasiconvex $\forall z \in \mathbb{R}^n \; \forall \beta > 0$

- some properties also hold in Hilbert spaces

[10] Lara, 2022

- $\emptyset \neq K \subseteq \mathbb{R}^n$, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper, $\beta > 0$
- the (β, γ, K)-*strong subdifferential* of h at $\overline{x} \in K \cap \operatorname{dom} h$ is[11]

$$\begin{aligned}\partial^K_{\beta,\gamma} h(\overline{x}) &= \Big\{z \in \mathbb{R}^n : \max\{h(y), h(\overline{x})\} \geq h(\overline{x}) + \frac{t}{\beta} z^\top (y - \overline{x}) \\ &+ \frac{t}{2}\left(\gamma - \frac{t}{\beta} - t\gamma\right) \|y - \overline{x}\|^2 \ \forall y \in K \quad \forall t \in [0,1]\Big\}\end{aligned}$$

- when $K = S_{h(\overline{x})}(h)$, $\partial_{\beta,\gamma} h(\overline{x}) := \partial^{S_{h(\overline{x})}(h)}_{\beta,\gamma} h(\overline{x})$ is the (β, γ)-*strong sublevel subdifferential* of h at $\overline{x}$
- K closed convex & h strongly quasiconvex $\Rightarrow \partial^K_{\beta,\gamma} h(x) \neq \emptyset$ closed convex $\forall x \in K$

[11] Kabgani & Lara, 2022

- $\emptyset \neq K \subseteq \mathbb{R}^n$ convex, $h : K \to \mathbb{R}$ C^1: h strongly quasiconvex $\Leftrightarrow$

$$h(x) \leq h(y) \implies \nabla h(y)^\top (y - x) \geq \frac{\gamma}{2}\|x - y\|^2, \ \forall x, y \in K$$ [12]

- $\emptyset \neq K \subseteq \mathbb{R}^n$ convex, $h : K \to \mathbb{R}$ C^1: h strongly quasiconvex $\Leftrightarrow \nabla h(x) \in \partial_{1,\gamma} h(x)$ $\forall x \in K \cap \operatorname{dom} h$[13]

- $\emptyset \neq K \subseteq \mathbb{R}^n$ convex, $h : K \to \mathbb{R}$ C^1 & strongly quasiconvex, ∇h L-Lipschitz-continuous, $\arg\min_K h = \{\overline{x}\} \Rightarrow$

$$\|\nabla h(x)\|^2 \geq \frac{2\gamma^2}{L}(h(x) - h(\overline{x})), \ \forall x \in K$$ [14]

[12] Vladimirov & Nesterov & Chekanov, 1978
[13] Lara & Marcavillaca & Vuong, 2024
[14] *Polyak-Łojasiewicz-Kurdyka property*, Korablev, 1980

- $t \in \mathbb{R} \mapsto |t + a|$ $(a \in \mathbb{R})$ is convex, and strongly quasiconvex with modulus $\gamma > 0$ on $[0, 1/\gamma]$ but not strongly convex
- $t \in \mathbb{R} \mapsto -t^2 - t$ is strongly quasiconvex on $[0, 1]$
- $\forall c \in \mathbb{R}$ & $d > 0$ $\exists \delta > 0$: $t \in \mathcal{B}(0, \delta) \mapsto c - de^{-x^2}$ is strongly quasiconvex on $\mathcal{B}(0, \delta)$
- $t \in \mathbb{R} \mapsto t^2 + 3\sin^2 t$ is strongly quasiconvex
- $f : [0, 1] \to \mathbb{R}$, $f(t) = \begin{cases} 0, & t = 0 \\ -\frac{1}{t}, & t \in]0, 1] \end{cases}$ is strongly quasiconvex on $[0, 1]$
- $t \mapsto \sqrt[4]{t^2 + k^2}$ $(k \in \mathbb{R})$ is strongly quasiconvex on any interval $[-c, c] \subseteq \mathbb{R}$
- $\|\cdot\|^\alpha$ $(0 < \alpha < 1)$ is strongly quasiconvex on nonempty bounded convex sets in $\mathbb{R}^n$

- $\emptyset \neq K \subseteq \mathbb{R}^n$, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^n \to \mathbb{R}$: $K \cap \operatorname{dom} h \neq \emptyset$, $g(K) \subseteq]0, M[$, h is strongly convex $\Rightarrow$ h/g is strongly quasiconvex on K when one of the following holds
  - (a) g is affine
  - (b) h is nonnegative on $\operatorname{dom} h$ and g is concave
  - (c) h is nonpositive on $\operatorname{dom} h$ and g is convex

- $A, B \in \mathbb{R}^{n\times n}$, $A \in \mathcal{S}^n_{++}$, $a, b \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, $K = \{x \in \mathbb{R}^n : m \leq (1/2)x^\top Bx + b^\top x + \beta \leq M\}$ $(0 < m < M) \Rightarrow$

$$x \in K \mapsto \frac{\frac{1}{2}x^\top Ax + a^\top x + \alpha}{\frac{1}{2}x^\top Bx + b^\top x + \beta} \in \mathbb{R}$$

  is strongly quasiconvex on K if any of the following holds
  - (a) $B = 0 \in \mathbb{R}^{n\times n}$
  - (b) $(1/2)x^\top Ax + a^\top x + \alpha \geq 0$ for all $x \in K$ and $B \in -\mathcal{S}^n_+$
  - (c) $(1/2)x^\top Ax + a^\top x + \alpha \leq 0$ for all $x \in K$ and $B \in \mathcal{S}^n_+$

- $K \subseteq \mathbb{R}^n$ affine subspace
- $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper lsc, strongly quasiconvex on K
- $K \subseteq \operatorname{dom} h$

$$\min_{x \in K} h(x) \tag{COP}$$

Step 0. let $x^0 = x^{-1} \in K$, $\alpha \in [0, 1[$, $0 < \rho' \leq \rho'' < 2$, $\{c_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_{++}$, $k = 0$

Step 1. choose $\alpha_k \in [0, \alpha]$, set

$$y^k = x^k + \alpha_k(x^k - x^{k-1}) \qquad [\textit{inertial} \text{ step}]$$

and compute[15]

$$z^k \in \mathrm{Prox}_{c_k(h+\delta_K)}(y^k) \qquad [\textit{proximal} \text{ step}]$$

Step 2. if $z^k = y^k$: STOP $\Rightarrow y^k \in \arg\min_K h$

Step 3. choose $\rho_k \in [\rho', \rho'']$ and update

$$x^{k+1} = (1 - \rho_k)y^k + \rho_k z^k \qquad [\textit{relaxation} \text{ step}]$$

Step 4. $k = k + 1$ and go to Step 1

[15] *indicator fct* of K: $\delta_K : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\delta(x) = 0$, $x \in K$; $\delta(x) = +\infty$, $x \notin K$

- if $\alpha = 0$ & $\rho_k = 1$, $k \geq 0$ the algorithm collapses to PPA[16]
- K can be taken closed convex
  - when $0 < \rho' \leq \rho'' < 1$ with the alternate relaxation step

$$x^{k+1} = (1 - \rho_k)x^k + \rho_k z^k$$

  - by adding a projection on K in the relaxation step

$$x^{k+1} = \mathrm{Pr}_K((1 - \rho_k)y^k + \rho_k z^k)$$

  - by removing the inertial steps ($\alpha = 0$) and taking $0 < \rho' \leq \rho'' < 1$
- Bregman PPA[17]

[16] Lara, 2022

[17] Lara & Marcavillaca, 2024

- $0 < \rho' \leq \rho'' < 2$, $\{\rho_k\}_k \subseteq [\rho', \rho'']$, $\alpha \in [0, 1[$, $\{\alpha_k\}_k \subseteq [0, \alpha]$
- $\{\alpha_k\}_k$ is nondecreasing satisfying

$$0 \leq \alpha_k \leq \alpha_{k+1} \leq \alpha < \frac{1}{3} \ \forall \ k \geq 0$$

- $\Omega := \{x \in K : \ h(x) \leq h(z^k) \ \forall k \in \mathbb{N}\}$

$$\Rightarrow$$

- $\forall x^* \in \Omega \ \exists \lim_{k\to\infty} \|x^k - x^*\|$ and

$$\lim_{k\to+\infty} \|x^{k+1} - y^k\| = \lim_{k\to+\infty} \|z^k - y^k\| = 0$$

- if, in addition, $c_k \geq c' > 0 \ \forall k \geq 0$

$$\Rightarrow \begin{cases} x^k \to \overline{x} = \arg\min_K \ h \\ \lim\limits_{k\to+\infty} h(x^k) = \min_K h \end{cases}$$

- $\alpha = 0$, $\rho_k \in (0,1)$, $r_k = \sqrt{1 - \frac{\rho_k c_k \gamma}{1 + c_k \gamma}} \in (0,1)$, $k \geq 0 \Rightarrow$
  - $\|x^{k+1} - \overline{x}\| \leq r_k \|x^k - \overline{x}\| \ \forall k \geq 0$
  - $\|z^k - x^k\|^2 \leq \frac{r_k}{\rho_k(1-\rho_k)} \|x^k - \overline{x}\|^2 \ \forall k \geq 0$
  - $f(z^k) - \min_K f \leq \frac{r_{k-1}}{2c_k} \|x^{k-1} - \overline{x}\|^2 \ \forall k \geq 0$

Step 0. let $\beta, \gamma > 0$, $\{\alpha_j\}_j \subseteq \left]0, \frac{1}{\gamma\beta}\right[$, $x^0 \in K$, $k = 0$

Step 1. if $0_n \in \partial^K_{\beta,\gamma} h(x^k)$: STOP

Step 2. determine

$$\xi^k \in \partial^K_{\beta,\gamma} h(x^k) \qquad [\textit{subgradient}\ \text{step}]$$

Step 3. $x^{k+1} := \mathrm{Pr}_K(x^k - \alpha_k \xi^k)$ [*projection* step]

Step 4. $k = k + 1$ and go to Step 1

- K closed convex, $K \subseteq \operatorname{int}\operatorname{dom} h$
- $\beta > 0$, $M > 0$: $\|\xi\| \leq M \ \forall \xi \in \partial^K_{\beta,\gamma} h(x) \ \forall x \in K$
- $\{\alpha_j\}_j \subseteq]0, 1/(\gamma\beta)[$: $\sum\limits_{k=0}^{+\infty} \alpha_k = +\infty$ & $\sum\limits_{k=0}^{+\infty} \alpha_k^2 < +\infty$

$$\Rightarrow$$

- $x^k \to \overline{x} = \arg\min_K h$
- $\lim\limits_{k \to +\infty} h(x^k) = \min_K h$
- $+\ h$ continuous $\Rightarrow \lim_{k \to +\infty} h(x^k) = h(\overline{x}) = \min_K h$

- $h \in C^1$
- $K = \mathbb{R}^n$

Step 0. let $\alpha > 0$, $\{\alpha_j\}_j \subseteq [\alpha, +\infty[$, $\{\psi^j\}_j \in \ell^1$, $x^0 \in \mathbb{R}^n$ and $k = 0$

Step 1. if $\nabla h(x^k) = 0_n$: STOP

Step 2. determine

$$x^{k+1} = x^k + \alpha_k \nabla h(x^k) + \psi^k \qquad [\textit{gradient} \text{ step}]$$

Step 3. $k = k + 1$ and go to Step 1

- ∇h Lipschitz-continuous & $\{x^k\}_k$ bounded $\Rightarrow$ $x^k \to \overline{x} = \arg\min_K h$[18]
- $\psi^k = 0\ \forall k \geq 0$, ∇h locally Lipschitz-continuous, $\exists \alpha, \bar{\alpha} > 0$: $\alpha \leq \alpha_k \leq \bar{\alpha} < \min\{\gamma/\bar{L}^2, 2/\bar{L}\} \Rightarrow x^k \to \overline{x} = \arg\min_K h$[19]

[18] Rouhani & Rahimi Piranfar, 2021

[19] Lara & Marcavillaca & Vuong, 2024

- $h \in C^1$
- $K = \mathbb{R}^n$

Step 0. let $\alpha, \eta > 0$, $\theta = 1 - \alpha\eta$, $x^0, x^1 \in \mathbb{R}^n$, $k = 1$

Step 1. if $\nabla h(x^k) = 0_n$: STOP

Step 2. determine

$$x^{k+1} = x^k + \theta(x^k - x^{k-1}) - \eta^2 \nabla h(x^k) \qquad [\textit{inertial}\text{ step}]$$

Step 3. $k = k + 1$ and go to Step 1

- ∇h Lipschitz-continuous on bounded sets & $\{x^k\}_k$ bounded

$$\Rightarrow \begin{cases} x^k \to \overline{x} = \arg\min_K h \\ \lim\limits_{k \to +\infty} h(x^k) = \min_K h \end{cases}$$

- $h \in C^1$
- $K = \mathbb{R}^n$

Step 0. let $\eta > 0$, $\{\alpha_j\}_j \subseteq [\eta, +\infty[$, $x^0, x^1 \in \mathbb{R}^n$, $k = 1$

Step 1. if $\nabla h(x^k) = 0_n$: STOP

Step 2. determine

$$x^{k+1} = 2x^k - x^{k-1} + \alpha_k \nabla h(x^k) \qquad \text{[\textit{inertial} step]}$$

Step 3. $k = k + 1$ and go to Step 1

- ∇h Lipschitz-continuous & $\theta \in]0, 1[$

$$\Rightarrow \begin{cases} x^k \to \overline{x} = \arg\min_K h \\ \lim\limits_{k \to +\infty} h(x^k) = \min_K h \end{cases}$$

- first-order [Rouhani & Rahimi Piranfar, 2021], [Lara & Marcavillaca & Vuong, 2024]
- second-order [Rahimi Piranfar & Khatibzadeh, 2021], [Lara & Marcavillaca & Vuong, 2024]

$$\text{find } \overline{x} \in K : \quad f(\overline{x}, y) \geq 0, \ \forall\, y \in K$$

- solution set: $S(K, f)$
- $K \subseteq \mathbb{R}^n$ linear subspace
- $f : K \times K \to \mathbb{R}$
- $f(x, \cdot)$ strongly quasiconvex $\forall x \in K$
- $f(\cdot, y)$ usc $\forall y \in K$
- f (jointly) lsc and pseudomonotone on K, i.e.

$$f(x, y) \geq 0 \implies f(y, x) \leq 0 \ \forall x, y \in K$$

- f satisfies the Lipschitz type condition $\exists \eta > 0$ s.t.

$$f(x, z) - f(x, y) - f(y, z) \leq \eta \left(\|x - y\|^2 + \|y - z\|^2 \right) \ \forall\, x, y, z \in K$$

- $(f(x, x) = 0 \ \forall x \in K)$

- the hypotheses guarantee that $S(K, f)$ is a singleton[20]

[20] [Iusem & Lara, JOTA, 2022]

- proximal point algorithm [Iusem & Lara, 2022]
- regularized algorithm [Iusem & Lara, 2022]
- extragradient algorithm [Muu & Yen, 2023]
- relaxed-inertial proximal point algorithm [**G** & Lara & Marcavillaca, 2024]
- proximal point algorithm with extrapolation terms [Izuchukwu & Ogwo & Shehu, 2024]
- two-step proximal point algorithm [Iusem & Lara & Marcavillaca & Yen, 2024]
- extragradient projection algorithm [Lara & Marcavillaca & Yen, 2024]

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