Disjoint $\mathcal{F}$-semi-transitivity in Banach algebras

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Definition

Let $\mathcal{S}$ be a set, $\mathcal{F}$ be a family of subsets of $\mathcal{S}$ and $\{T_{t,1}\}_{t\in\mathcal{S}}, \ldots, \{T_{t,N}\}_{t\in\mathcal{S}}$ be families of bounded linear operators on a Banach space X. We say that $\{T_{t,1}\}_{t\in\mathcal{S}}, \ldots, \{T_{t,N}\}_{t\in\mathcal{S}}$ are disjoint $\mathcal{F}$-semi-transitive, or shortly $d\mathcal{F}$-semi-transitive, if for every collection of non-empty open subsets $\mathcal{O}, V_1, \ldots, V_N$ of X, there exists some $F \in \mathcal{F}$ such that for all $t \in F$ there exists some $\lambda_t \in \mathbb{R}^+$ satisfying that

$$\mathcal{O} \cap \lambda_t T_{t,1}^{-1}(V_1) \cap \cdots \cap \lambda_t T_{t,N}^{-1}(V_N) \neq \varnothing.$$

Let now $\mathcal{A}$ be a non-unital Banach algebra such that $\mathcal{A}$ is a left ideal in a unital normed algebra $\mathcal{A}_1$. We do not assume that the norm on $\mathcal{A}$ extends to $\mathcal{A}_1$.
We will say that $\mathcal{A}$ satisfies *the condition (E)* with respect to $\mathcal{A}_1$ if

$$\|ba\| \leq \|b\|_1 \|a\|$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{A}_1$, where $\| \ \|_1$ denotes the norm on $\mathcal{A}_1$.
Let $\{p_\alpha\}_\alpha$ be a set in $\mathcal{A}$ satisfying that given any open subset O of $\mathcal{A}$ and $x \in O$ there exists some $p_{\alpha_0} \in \{p_\alpha\}_\alpha$ such that $p_{\alpha_0}^3 x \in O$ as well.
This condition will be called *the condition* ($\mathcal{P}$.)

Let

$$\{\Phi_{t,1}\}_{t\in\mathcal{S}},\ \{\Phi_{t,2}\}_{t\in\mathcal{S}},\ \ldots,\ \{\Phi_{t,N}\}_{t\in\mathcal{S}}$$

be families of isometric algebra isomorphisms of $\mathcal{A}_1$ such that

$$\Phi_{t,j}(\mathcal{A}) = \mathcal{A} \qquad (t \in \mathcal{S},\ 1 \le j \le N).$$

and such that $\Phi_{t,j|\mathcal{A}}$ is an isometry for all $j \in \{1, \ldots, N\}$.

We will assume that the system

$$\{\Phi_{t,1}\}_{t\in\mathcal{S}},\ \{\Phi_{t,2}\}_{t\in\mathcal{S}},\ \ldots,\ \{\Phi_{t,N}\}_{t\in\mathcal{S}}$$

is disjoint aperiodic, that is for each fixed p_α and every $H \in \mathcal{F}$ there exist some $F \subseteq H$ with $F \in \mathcal{F}$ such that

$$p_\alpha\,\Phi_{t,\ell}(p_\alpha) = 0, \quad p_\alpha\,\Phi_{t,\ell}^{-1}(p_\alpha) = 0, \quad \text{and} \quad p_\alpha\,\Phi_{t,\ell}\big(\Phi_{t,r}^{-1}(p_\alpha)\big) = 0$$

for all $r, \ell \in \{1, \ldots, N\}$ with $r \neq \ell$ and all $t \in F$.

Further, we will assume that for each α and all $t \in \mathcal{S}$, $r, \ell \in \{1, \ldots, N\}$, and $a \in \mathcal{A}$, it holds that

$$\|a\, p_\alpha\| \leq \|a\|, \quad \|a\, \Phi_{t,\ell}(p_\alpha)\| \leq \|a\|,$$

$$\|a\, \Phi^{-1}_{t,\ell}(p_\alpha)\| \leq \|a\|, \qquad \|a\, \Phi_{t,\ell}\big(\Phi^{-1}_{t,r}(p_\alpha)\big)\| \leq \|a\|.$$

This condition will be called *the condition (R)* throughout the presentation.

We now set up new families $\{b_{t,1}\}_{t\in\mathcal{S}}, \cdots, \{b_{t,N}\}_{t\in\mathcal{S}}$ of elements in $\mathcal{A}_1$, and for each $t \in \mathcal{S}$ and $\ell \in \{1, \ldots, N\}$, we define the operator

$$T_{t,\ell} : \mathcal{A} \to \mathcal{A} \quad \text{by} \quad T_{t,\ell}(a) = b_{t,\ell}\, \Phi_{t,\ell}(a).$$

Due to the condition (E), it is not hard to check that $T_{t,\ell}$ is a bounded linear operator on $\mathcal{A}$.

Further, we will assume that for each α there exists so called "α-inverse" for $b_{t,\ell}$ denoted by $b^{-1}_{t,\ell,\alpha}$, that is for each α there exists an element $b^{-1}_{t,\ell,\alpha} \in \mathcal{A}_1$ satisfying that $b_{t,\ell}b^{-1}_{t,\ell,\alpha}ap_\alpha = b^{-1}_{t,\ell,\alpha}b_{t,\ell}ap_\alpha = ap_\alpha$ for all $a \in \mathcal{A}$.
Also, we will assume that for each $t \in \mathcal{S}$ and $\ell \in \{1, \ldots, N\}$ it holds that $b_{t,\ell}a = 0$ if and only if $a = 0$. Such condition will be called *the condition (C)*

Theorem

Under the above notation and assumptions, the following statements are equivalent.
(1) The families $\{T_{t,1}\}_{t\in\mathcal{S}},\cdots,\{T_{t,N}\}_{t\in\mathcal{S}}$ *are* $d\mathcal{F}$*-semi-transitive.*
(2) For each fixed p_α *and* $\varepsilon > 0$ *there exists some* $F \in \mathcal{F}$ *and families* $\{d_t\}_{t\in F}$, $\{g_{t,1}\}_{t\in F}\ldots,\{g_{t,N}\}_{t\in F}$ *in* $\mathcal{A}$ *such that*
$\|d_t - p_\alpha^2\| < \varepsilon, \quad \|g_{t,l} - p_\alpha^2\| < \varepsilon$, *and*

$$\|b_{t,r,\alpha}^{-1}(g_{t,r})\|\|\Phi_{t,l}^{-1}(b_{t,l})d_t\| < \varepsilon^2 \quad \text{for all } t \in F \text{ and } r,l \in \{1,\ldots,N\}.$$

Moreover, for each distinct $r,l \in \{1,\ldots,N\}$ *and* $t \in F$ *it holds that*

$$\|\Phi_{t,r}^{-1}(b_{t,r})\Phi_{t,l}^{-1}(b_{t,l,\alpha}^{-1}g_{t,l})\| < \varepsilon.$$

Corollary

Let now $\mathcal{F}$ be the family of all infinite subsets of $\mathbb{N}$. If there exist dense subsets $\mathcal{A}_0, \mathcal{B}_1, \cdots, \mathcal{B}_N$ of $\mathcal{A}$ and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that for each $s, \ell \in \{1, \ldots, N\}$, every fixed α and all $x \in \mathcal{A}_0, y_s \in \mathcal{B}_s$ we have

$$\lim_{k\to\infty} \|b^{-1}_{n_k,s,\alpha} y_s\| \|\Phi^{-1}_{n_k,l}(b_{n_k,l})x\| = 0,$$

and, in addition, for each distinct $s, \ell \in \{1, \ldots, N\}$ it holds that

$$\lim_{k\to\infty} \|\Phi^{-1}_{n_k,\ell}(b_{n_k,\ell})\Phi^{-1}_{n_k,s}(b^{-1}_{n_k,s,\alpha} y_s)\| = 0,$$

then the families $\{T_{n,1}\}_{n\in\mathbb{N}}, \cdots, \{T_{n,N}\}_{n\in\mathbb{N}}$ are $d\mathcal{F}$-semi-transitive.

For a separable Hilbert space H we let $B(H)$ be the space of all bounded linear operators on H, and $\mathcal{C}$ be the C^*-algebra of compact operators on H. For an orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ for H, we let for each $m \in \mathbb{N}$, P_m be the orthogonal projection onto $\mathrm{Span}\{e_{-m}, \dots, e_m\}$. We will denote by $\ell_2(\mathcal{C})$ the standard (right) Hilbert module over $\mathcal{C}$. Notice that $\ell_2(\mathcal{C})$ is a Banach algebra. Indeed, we can define multiplication on $\ell_2(\mathcal{C})$ as pointwise multiplication, i.e., if $\{x_j\}_{j\in\mathbb{Z}}, \{y_j\}_{j\in\mathbb{Z}} \in \ell_2(\mathcal{C})$, then

$$\{x_j\}_{j\in\mathbb{Z}} \cdot \{y_j\}_{j\in\mathbb{Z}} = \{x_j y_j\}_{j\in\mathbb{Z}}.$$

To see that $\{x_j y_j\}_{j\in\mathbb{Z}}$ belongs to $\ell_2(\mathcal{C})$, it suffices to observe that

$$\sum_j y_j^* x_j^* x_j y_j \leq \sum_j y_j^* \|x_j\|^2 y_j \leq \| \{x_j\}_{j\in\mathbb{Z}} \|^2 \sum y_j^* y_j.$$

For each $m, J \in \mathbb{N}$, we let $\tilde{p}_{J,m} \in \ell_2(\mathcal{C})$ be given by

$$(\tilde{p}_{J,m})_i = \begin{cases} P_m, & \text{if } -J \leq i \leq J, \\ 0, & \text{else.} \end{cases}$$

The set $\{\tilde{p}_{J,m}\}_{(J,m)\in\mathbb{N}^2}$ forms a left approximate unit for $\ell_2(\mathcal{C})$.
put $\mathcal{A} = \ell_2(\mathcal{C})$, and

$$\mathcal{A}_1 := \{\{y_j\}_{j\in\mathbb{Z}} \mid y_j \in B(H)\ \forall j,\ \exists M_y > 0 \text{ such that } \|y_j\| \leq M_y\ \forall j\}.$$

The multiplication on $\mathcal{A}_1$ is defined similarly component-wise. We define the norm $\|\cdot\|_1$ on $\mathcal{A}_1$ as $\|\{y_j\}_{j\in\mathbb{Z}}\|_1 = sup_{j\in\mathbb{Z}}\|y_j\|$ for all $\{y_j\}_{j\in\mathbb{Z}} \in \mathcal{A}_1$. It is straightforward to check that the condition *(E)* is satisfied in this case.

For $N \in \mathbb{N}$, let $r_1, \ldots, r_N \in \mathbb{N}$ with $r_1 < r_2 < \cdots < r_N$ and let $\mathcal{U}_1, \ldots, \mathcal{U}_N$ are unitary operators on H. For each $n \in \mathbb{N}$ and $s \in \{1, \ldots, N\}$, we define the map $\Phi_{n,s} : \mathcal{A}_1 \to \mathcal{A}_1$ by

$$\Phi_{n,s}\left(\{y_j\}_{j\in\mathbb{Z}}\right) = \{\mathcal{U}_s^{-r_s n} y_{j-r_s n} \mathcal{U}_s^{r_s n}\}_{j\in\mathbb{Z}},$$

for all $\{y_j\}_{j\in\mathbb{Z}} \in \mathcal{A}_1$.

If $\mathcal{F}$ is a family of subsets of $\mathbb{N}$ which is finitely invariant, that is, for each $n \in \mathbb{N}$ and $F \in \mathcal{F}$ it holds that $F \setminus \{1, \ldots, n\} \in \mathcal{F}$ also, then, by some calculations it can be checked that $\{\Phi_{n,1}\}_{n\in\mathbb{N}}, \cdots, \{\Phi_{n,N}\}_{t\in\mathbb{N}}$ is a disjoint aperiodic system with respect to the left approximate unit $\{\tilde{p}_{J,m}\}_{(J,m)\in\mathbb{N}^2}$ and that the condition *(R)* is satisfied in this case.

For each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, let $\{W_{n,j}^{(\ell)}\}_{j\in\mathbb{Z}}$ be a family of operators in $B(H)$ which is uniformly bounded in norm and such that each $W_{n,j}^{(\ell)}$ has a bounded inverse. Then, for each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, we let $b_{n,\ell} \in \mathcal{A}_1$ be given by

$$(b_{n,\ell})_i = W_{n,i}^{(\ell)} \mathcal{U}_\ell^{r_\ell n},$$

for all $i \in \mathbb{Z}$. Since for each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, the operator $W_{n,i}^{(\ell)} \mathcal{U}_\ell^{r_\ell n}$ is invertible, it follows that the condition (C) is satisfied in this case.

Then we let $T_{n,\ell}$ be the operator on $\mathcal{A}$ given by

$$T_{n,\ell}(a) = b_{n,\ell}\, \Phi_{n,\ell}(a), \qquad \text{for all } a \in \mathcal{A}.$$

By some calculations one can check that for all $n, J, m \in \mathbb{N}$, $s \in \{1, \dots, N\}$ it holds that

$$(b^{-1}_{n,s,(J,m)})_i = (b^{-1}_{n,s,J})_i = \mathcal{U}_s^{-r_s n}\, W^{(s)\,-1}_{n,i},$$

for all $i \in \mathbb{Z}$ with $-J \leq i \leq J$, and $(b^{-1}_{n,s,(J,m)})_i = (b^{-1}_{n,s,J})_i = 0$ for all $i \in \mathbb{Z}$ with $i \leq -J-1$ and all $i \in \mathbb{Z}$ with $i \geq J+1$.

Theorem

Under the above notation and assumptions, the following statements are equivalent.
(1) The families of operators $\{T_{n,1}\}_{n\in\mathbb{N}}, \dots, \{T_{n,N}\}_{n\in\mathbb{N}}$ *are* $d\mathcal{F}$*-semi-transitive.*
(2) For every $J, m \in \mathbb{N}$ *and* $\varepsilon > 0$*, there exists some* $F \in \mathcal{F}$ *such that for each* $j \in \mathbb{Z}$ *with* $-J \le j \le J$ *we can find families* $\{D_{n,j}\}_{n\in F}, \quad \{G^{(1)}_{n,j}\}_{n\in F}, \dots, \{G^{(N)}_{n,j}\}_{n\in F}$ *of compact operators on* H *satisfying*

$$\|D_{n,j} - P_m\| < \varepsilon, \quad \|G^{(s)}_{n,j} - P_m\| < \varepsilon,$$

and

$$\|W^{(\ell)}_{n,j+r_\ell n} D_{n,j}\| \, \|W^{(s)\ -1}_{n,j} G^{(s)}_{n,j}\| < \varepsilon^2 \quad \text{for all } n \in F \text{ and } \ell, s \in \{1, \dots, N\},$$

and satisfying in addition that for each distinct $s, \ell \in \{1, \dots, N\}$ *and all* $n \in F$

$$\|W^{(\ell)}_{n,j-r_s n+r_\ell n} \, W^{(s)-1}_{n,j} \, G^{(s)}_{n,j}\| < \varepsilon.$$

From now on we will assume that $\mathcal{F}$ the family of all infinite subsets of $\mathbb{N}$.

Corollary

Under the above notation and assumptions, if for each $j \in \mathbb{Z}$ there exist dense subsets $H_0^{(j)}, H_1^{(j)}, \dots, H_N^{(j)}$ of H and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that for each $s, \ell \in \{1, \dots, N\}$ and every $x^{(j)} \in H_0^{(j)}$, $y_s^{(j)} \in H_s^{(j)}$ we have

$$\lim_{k\to\infty} \|W_{n_k,j}^{(s)\,-1}\, y_s^{(j)}\| \|W_{n_k,j+r_\ell n_k}^{(\ell)}\, x^{(j)}\| = 0,$$

and for each distinct $s, \ell \in \{1, \dots, N\}$ and every $y_s^{(j)} \in H_s^{(j)}$,we have

$$\lim_{k\to\infty} \|W_{n_k,j-r_s n_k+r_\ell n_k}^{(\ell)}\, W_{n_k,j}^{(s)-1} y_s^{(j)}\| = 0,$$

then the sequences of operators $\{T_{n,1}\}_{n\in\mathbb{N}}, \dots, \{T_{n,N}\}_{n\in\mathbb{N}}$ are $d\mathcal{F}$-semi-transitive.

Next we let Ω be a locally compact Hausdorff space, $\alpha_1, \ldots, \alpha_N$ be homeomorphisms of Ω, $\{w_j^{(1)}\}_{j\in\mathbb{Z}}, \ldots, \{w_j^{(N)}\}_{j\in\mathbb{Z}} \subseteq C_b(\Omega)$, where $C_b(\Omega)$ denotes the space of all bounded continuous functions on Ω. We will assume that $w_j^{(\ell)} > 0$ for all $j \in \mathbb{Z}$ and $\ell \in \{1, \ldots, N\}$ and that $\|w_j^{(\ell)}\|_\infty < M$ for all $j \in \mathbb{Z}$, $\ell \in \{1, \ldots, N\}$ and some $M > 0$. Now we consider $\mathcal{A} = \ell_2(C_0(\Omega))$, and

$$\mathcal{A}_1 := \{\{y_j\}_{j\in\mathbb{Z}} \mid y_j \in C_b(\Omega)\ \forall j,\ \exists M_y > 0 \text{ such that } \|y_j\|_\infty \le M_y\ \forall j\}.$$

The multiplication and the norm on $\mathcal{A}_1$ are defined in the same way as in the previous case.

For each pair of compact subset K_1 and K_2 of Ω with $K_1 \subseteq K_2$, we let $u_{(K_1,K_2)} \in C_0(\Omega)$ such that $u_{(K_1,K_2)|_{K_1}} = 1$, $\operatorname{supp} u_{(K_1,K_2)} = K_2$ and $0 \leq u_{(K_1,K_2)} \leq 1$. For $J \in \mathbb{N}$, we let $\tilde{p}_{(K_1,K_2,J)} \in \mathcal{A}$ be given by

$$(\tilde{p}_{(K_1,K_2,J)})_j = \begin{cases} u_{(K_1,K_2)}, & \text{if } -J \leq j \leq J, \\ 0, & \text{else.} \end{cases}$$

Then it is easily checked that $\{\tilde{p}^3_{(K_1,K_2,J)}\}_{K_1 \subseteq K_2 \subseteq \Omega,\, K_1,K_2 \text{ compact } ,J \in \mathbb{N}}$ satisfies the condition $(\mathcal{P}.)$

For each $\ell \in \{1, \ldots, N\}$ and $k \in \mathbb{N}$, we let $\tilde{\Phi}_{k,\ell} : \mathcal{A} \to \mathcal{A}$ be given by

$$\tilde{\Phi}_{k,\ell}\big(\{f_j\}_{j\in\mathbb{Z}}\big) = \{f_{j-k} \circ \alpha_\ell^k\}_{j\in\mathbb{Z}},$$

for all $\{f_j\}_{j\in\mathbb{Z}} \subset \mathcal{A}$. Let now $r_1, \ldots, r_N \in \mathbb{N}$ with $r_1 < r_2 < \cdots < r_N$. For each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, set

$$\Phi_{n,\ell} = \tilde{\Phi}_{r_\ell n,\ell}.$$

Clearly, the system of isometric isomorphisms $\{\Phi_{k,1}\}_{k\in\mathbb{N}}, \ldots, \{\Phi_{k,N}\}_{k\in\mathbb{N}}$ is disjoint aperiodic with respect to

$$\big\{\tilde{p}_{(K_1,K_2,J)}\big\}_{K_1\subseteq K_2\subseteq\Omega,\ K_1,K_2 \text{ compact },J\in\mathbb{N}}.$$

Also, the condition *(R)* is satisfied in this case.

For each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, let $b_{n,\ell} \in \mathcal{A}_1$ be given by

$$(b_{n,\ell})_{j+r_\ell n} = \prod_{k=1}^{r_\ell n} \left(w_{j+k}^{(\ell)} \circ \alpha_\ell^{n-k} \right) \quad \text{for all } j \in \mathbb{Z}.$$

Since for each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, the function $w_j^{(\ell)} > 0$, it follows that the condition (C) is satisfied in this case. Then, for all $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, we let $T_{n,\ell} : \mathcal{A} \to \mathcal{A}$ be given by

$$T_{n,\ell}(a) = b_{n,\ell}\, \Phi_{n,\ell}(a) \quad \text{for all } a \in \mathcal{A}.$$

Now, for each $K_2 \subseteq \Omega$ compact, we can find some $v_{K_2} \in C_0(\Omega)$ with $0 \le v_{K_2} \le 1$ such that $v_{K_2|_{K_2}} = 1$. Recalling that the approximate unit for $\mathcal{A}$ in this case is indexed over all triples (K_1, K_2, J) with $K_1 \subseteq K_2 \subseteq \Omega$, K_1, K_2 compact, $J \in \mathbb{N}$, for each $n \in \mathbb{N}$ and $\ell \in \{1, \dots, N\}$, and every fixed triple (K_1, K_2, J) we let $b^{-1}_{n,\ell,(K_1,K_2,J)} \in \mathcal{A}_1$ be given by

$$(b^{-1}_{n,\ell,(K_1,K_2,J)})_{j+r_\ell n} = \frac{v_{K_2}}{\prod_{k=1}^{r_\ell n} \left(w^{(\ell)}_{j+k} \circ \alpha_\ell^{n-k} \right)}$$

for all $j \in \mathbb{Z}$ with $-J - r_\ell n \le j \le J - r_\ell n$ and $(b^{-1}_{n,\ell,(K_1,K_2,J)})_{j+r_\ell n} = 0$ else.

Corollary

Under the above notation and assumptions, the following statements are equivalent:
(1) The sequences of operators

$$\{T_{n,1}\}_{n\in\mathbb{N}}, \ \ldots, \ \{T_{n,N}\}_{n\in\mathbb{N}}$$

are $d\mathcal{F}$-semi-transitive on $\mathcal{A}$.
(2) For every $J \in \mathbb{N}$ and each compact subset K of Ω, there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty}\left(\sup_{t\in K}\frac{1}{\prod_{i=1}^{r_s n_k}\left(w^{(s)}_{j+i-r_s n_k}\circ\alpha_s^{r_s n_k - i}\right)(t)}\right)\left(\sup_{t\in K}\prod_{i=1}^{r_\ell n_k}\left(w^{(\ell)}_{j+i}\circ\alpha_\ell^{-i}\right)(t)\right)=0$$

for all $j\in\mathbb{Z}$ with $-J\leq j\leq J$ and every $l, s\in\{1,\ldots,N\}$. Moreover, for each distinct $s,\ell\in\{1,\ldots,N\}$ and all $j\in\mathbb{Z}$ with $-J\leq j\leq J$, it holds that

$$\lim_{k\to\infty}\left(\sup_{t\in K}\frac{\prod_{i=1}^{r_\ell n_k}\left(w^{(\ell)}_{j-r_\ell n_k-i}\circ\alpha_\ell^{-i}\circ\alpha_s^{r_s n_k}\right)(t)}{\prod_{i=1}^{r_s n_k}\left(w^{(s)}_{j-r_s n_k+i}\circ\alpha_s^{n_k-i}\right)(t)}\right)=0.$$

In sequel, we assume that $\Omega = \mathbb{R}$. and $\mathcal{C} = C_0(\mathbb{R})$. For $\tau \in C_b(\mathbb{R})$, we put

$$\mathcal{C}_\tau := \{f \in \mathcal{C} : \sum_{k=0}^{\infty} \|f\tau^k\|_\infty < \infty\}.$$

For each $f \in \mathcal{C}_\tau$ we define

$$\|f\|_\tau := \sum_{k=0}^{\infty} \|f\tau^k\|_\infty.$$

Then, $\mathcal{C}_\tau$ is a Banach algebra. We will call this algebra *Segal algebra corresponding to τ.*

For a homeomorphism α of $\mathbb{R}$, we consider corresponding Segal algebra $\mathcal{C}_{\tau\circ\alpha^k}$ for all $k \in \mathbb{Z}$. Clearly, $f \cdot (\tau \circ \alpha^k) \in \mathcal{C}$ for all $k \in \mathbb{Z}$ whenever $f \in C_0(\mathbb{R})$ since $\tau \circ \alpha^k \in C_b(\mathbb{R})$ for all $k \in \mathbb{Z}$. By definition, $\|f\|_\infty \leq \|f\|_{\tau\circ\alpha^k}$ whenever $f \in \mathcal{C}_{\tau\circ\alpha^k}$.

Lemma

Let $\tau \in C_b(\mathbb{R})$ and $\epsilon_1, \epsilon_2 \in (0,1)$ with $\epsilon_2 < \epsilon_1$. Then for every compact subset $K_{\epsilon_2}^{(\tau)} \subseteq |\tau|^{-1}([0,\epsilon_2])$, there exists a function $\mu_{K_{\epsilon_2}^{(\tau)},\epsilon_1} \in \mathcal{C}_\tau$ such that the followings hold:

(1) $\mu_{K_{\epsilon_2}^{(\tau)},\epsilon_1} = 1$ *on* $K_{\epsilon_2}^{(\tau)}$

,

(2) $\mu_{K_{\epsilon_2}^{(\tau)},\epsilon_1} \in C_c(\mathbb{R})$,

(3) $\operatorname{supp} \mu_{K_{\epsilon_2}^{(\tau)},\epsilon_1} \subseteq |\tau|^{-1}([0,\epsilon_1]))$.

Lemma

For $\tau \in C_b(\mathbb{R})$ *and each* $f \in \mathcal{C}_\tau$, *there is a sequence* (g_n) *in the set* $\{\mu_{K^{(\tau)}_{\epsilon_2},\epsilon_1} :\ 0 < \epsilon_2 < \epsilon_1 < 1,\ K^{(\tau)}_{\epsilon_2}$ *is compact*$\}$ *such that* $fg_n \to f$ *in* $\mathcal{C}_\tau$.

Lemma

For each $k \in \mathbb{Z}$, $f \circ \alpha \in \mathcal{C}_{\tau\circ\alpha^{k+1}}$ *if and only if* $f \in \mathcal{C}_{\tau\circ\alpha^k}$. *Moreover, in this case,* $\|f \circ \alpha\|_{\tau\circ\alpha^{k+1}} = \|f\|_{\tau\circ\alpha^k}$.

Let

$$\mathrm{c}_0^{\mathcal{C},\tau} := \left\{ s := (s_k) \in \prod_{k\in\mathbb{Z}} \mathcal{C}_{\tau\circ\alpha^k} : \lim_{k\to\infty} \|s_k\|_{\tau\circ\alpha^k} = \lim_{k\to\infty} \|s_{-k}\|_{\tau\circ\alpha^{-k}} = 0 \right\}$$

and we equip ${\mathrm{c}_0}^{\mathcal{C},\tau}$ with the norm

$$\|s\|_0 := \sup_{k\in\mathbb{Z}} \|s_k\|_{\tau\circ\alpha^k}.$$

Since we have that $\mathcal{C}_{\tau\circ\alpha^k}$ is Banach algebra for all $k \in \mathbb{Z}$, it is not hard to check that $\mathrm{c}_0^{\mathcal{C},\tau}$ is a non-unital Banach algebra. We will from now on denote ${\mathrm{c}_0}^{\mathcal{C},\tau}$ by $\mathcal{A}$. Moreover, we let $\mathcal{A}_1$ be the same as above in the previous case when we considered $\mathcal{A} = \ell_2(C_0(\Omega))$. It is straightforward to check that the condition *(E)* is satisfied also in this case when ${\mathrm{c}_0}^{\mathcal{C},\tau} = \mathcal{A}$.

For $\epsilon_1, \epsilon_2 \in (0,1)$ with $\epsilon_2 < \epsilon_1, J \in \mathbb{N}$, and any collection of compact subsets $K_{\epsilon_2}^{(\tau\circ\alpha^{-J})}, K_{\epsilon_2}^{(\tau\circ\alpha^{-J+1})}, \cdots, K_{\epsilon_2}^{(\tau\circ\alpha^{J})}$ of $\mathbb{R}$, we let $\tilde{p}_{(K_{\epsilon_2}^{(\tau\circ\alpha^{-J})}, K_{\epsilon_2}^{(\tau\circ\alpha^{-J+1})}, \cdots, K_{\epsilon_2}^{(\tau\circ\alpha^{J})}), \epsilon_1} \in \mathcal{A}$ be given by

$$(\tilde{p}_{((K_{\epsilon_2}^{(\tau\circ\alpha^{-J})}, K_{\epsilon_2}^{(\tau\circ\alpha^{-J+1})}, \cdots, K_{\epsilon_2}^{(\tau\circ\alpha^{J})}), \epsilon_1)})_j = \begin{cases} \mu_{K_{\epsilon_2}^{(\tau\circ\alpha^{j})}, \epsilon_1}, & \text{if } -J \leq j \leq J, \\ 0, & \text{else.} \end{cases}$$

The condition $(\mathcal{P})$ is satisfied in this case.

Let now $\alpha = \alpha_1 = \cdots = \alpha_N$. For each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, let $\Phi_{n,\ell}$ be the same as above in the previous case when we considered $\mathcal{A} = \ell_2(C_0(\Omega))$. It is not hard to check that the system $\{\Phi_{k,1}\}_{k\in\mathbb{N}}, \ldots, \{\Phi_{k,N}\}_{k\in\mathbb{N}}$ is again a system of isometric isomorphisms also in this case when ${\rm c_0}^{\mathcal{C},\tau} = \mathcal{A}$. Moreover, it is disjoint aperiodic and satisfies the condition *(R)* with respect to

$$\{\tilde{p}_{((K_{\epsilon_2}^{(\tau\circ\alpha^{-J})}, K_{\epsilon_2}^{(\tau\circ\alpha^{-J+1})}, \cdots, K_{\epsilon_2}^{(\tau\circ\alpha^{J})}), \epsilon_1)}\}.$$

For each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, N\}$, let $\{w_j^{(1)}\}_{j\in\mathbb{Z}}, \ldots, \{w_j^{(N)}\}_{j\in\mathbb{Z}} \subseteq C_b(\mathbb{R})$, $b_{n,\ell} \in \mathcal{A}_1$ and $T_{n,\ell} : \mathcal{A} \to \mathcal{A}$ be as above in the previous case (when we considered $\mathcal{A} = \ell_2(C_0(\Omega))$).

Corollary

Under the above notation and assumptions, the following statements are equivalent:
(1) The sequences of operators

$$\{T_{n,1}\}_{n\in\mathbb{N}},\ \ldots,\ \{T_{n,N}\}_{n\in\mathbb{N}}$$

are $d\mathcal{F}$-semi-transitive on $\mathcal{A}$.
(2) For every $J \in \mathbb{N}$ and finite collection

$$\{K_{\epsilon_J}^{(\tau\circ\alpha^{-J})}, K_{\epsilon_{-J+1}}^{(\tau\circ\alpha^{-J+1})}, \ldots, K_{\epsilon_0}^{\tau}, \ldots, K_{\epsilon_J}^{(\tau\circ\alpha^{J})}\}$$

of compact subsets of $\mathbb{R}$, there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty}\left(\sup_{t\in K_{\epsilon_2}^{(\tau\circ\alpha^{j})}} \frac{1}{\prod_{i=1}^{r_s n_k}\left(w_{j+i-r_s n_k}^{(s)}\circ\alpha^{r_s n_k - i}\right)(t)}\right)\left(\sup_{t\in K_{\epsilon_2}^{(\tau\circ\alpha^{j})}} \prod_{i=1}^{r_\ell n_k}\left(w_{j+i}^{(\ell)}\circ\alpha^{-i}\right)(t)\right)$$

$$= 0$$

for all $j \in \mathbb{Z}$ with $-J \leq j \leq J$ and every $l, s \in \{1, \ldots, N\}$.

Corollary

Moreover, for each distinct $s, \ell \in \{1, \ldots, N\}$ *and all* $j \in \mathbb{Z}$ *with* $-J \leq j \leq J$*, it holds that*

$$\lim_{k\to\infty}\left(\sup_{t\in K_{\epsilon_2}^{(\tau\circ\alpha^j)}} \frac{\prod_{i=1}^{r_\ell n_k}\left(w_{j-r_\ell n_k-i}^{(\ell)}\circ\alpha^{-i}\circ\alpha^{r_s n_k}\right)(t)}{\prod_{i=1}^{r_s n_k}\left(w_{j-r_s n_k+i}^{(s)}\circ\alpha^{n_k-i}\right)(t)}\right)=0.$$

Example

Let $N = 1, r_1 = 1$ and choose some $\{w_j\}_{j\in\mathbb{Z}} \subseteq L^\infty(\mathbb{R})$ such that $w_j > 0$, $w_j^{-1} \in L^\infty(\mathbb{R})$ and $\| w_j \| \leq M$ for all $j \in \mathbb{Z}$ and some $M > 1$. Let $\alpha(t) = t - 1$ for all $t \in \mathbb{R}$. If there exists an $\varepsilon > 0$ such that $w_j\chi_{[0,\infty)} \leq 1 - \varepsilon, w_j\chi_{(-\infty,0)} = 1$ for all $j \geq 0$ and $w_j = 1$ for all $j < 0$; or if there exists an $\varepsilon > 0$ such that $w_j\chi_{[0,\infty)} = 1, w_j\chi_{(-\infty,0)} \geq 1 + \varepsilon$ for all $j < 0$ and $w_j = 1$ for all $j \geq 0$, then the conditions of Corollary 6 and Corollary 10 are satisfied in this case.

Thank you for attention !

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