Products and sums of $m-$symmetric and skew $m-$symmetric operators

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Outlines

Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote to the algebra of all bounded linear operators on $\mathcal{H}$. For every $T \in \mathcal{B}(\mathcal{H})$, we denote by T^* the adjoint of T. Throughout this paper $\mathbb{N}$ denotes the set of strictly positive integer numbers i.e., $\mathbb{N} = \{1, 2, 3, \cdots\}$.

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An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be $m-$symmetric for some $m \in \mathbb{N}$ if

$$\sum_{k=0}^{m}(-1)^{m-k}\left(\begin{array}{c} m \\ k \end{array}\right)T^{*(m-k)}T^{k} = 0,$$

where $\left(\begin{array}{c} m \\ k \end{array}\right)$ is the binomial coefficient.

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where $\left(\begin{array}{c} m \\ k \end{array}\right)$ is the binomial coefficient.The authors in [**3**] showed the if $T, S \in \mathcal{B}(\mathcal{H})$ are double commuting operators i.e., $TS = ST$ and $T^*S = ST^*$ such that T is $m-$symmetric and S is $n-$symmetric, then TS and $T + S$ are $(m + n - 1)-$symmetric.

$T \in \mathcal{B}(\mathcal{H})$ is said to be skew $m-$symmetric for some $m \in \mathbb{N}$ if it satisfies

$$\sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} T^{*(m-k)} T^{k} = 0.$$

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It is proved in [**1**] that if $T, S \in \mathcal{B}(\mathcal{H})$ are double commuting operators such that T is skew $m-$symmetric and S is skew $n-$symmetric, then TS is $(m+n-1)-$symmetric and $T+S$ is skew $(m+n-1)-$symmetric.

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It is proved in [**1**] that if $T, S \in \mathcal{B}(\mathcal{H})$ are double commuting operators such that T is skew $m-$symmetric and S is skew $n-$symmetric, then TS is $(m+n-1)-$symmetric and $T+S$ is skew $(m+n-1)-$symmetric. For $T \in \mathcal{B}(\mathcal{H})$, set

$$\alpha_m(T) = \sum_{k=0}^{m} (-1)^{m-k} \left(\begin{array}{c} m \\ k \end{array} \right) T^{*(m-k)} T^{k}.$$

And

$$\delta_m(T) = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) T^{*(m-k)} T^{k}.$$

For $T, S \in \mathcal{B}(\mathcal{H})$, consider the operators $L_T, R_S \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ given by

$$L_T : X \mapsto TX, \quad R_S : X \mapsto XS,$$

i.e., L_T is the left multiplication by T on $\mathcal{B}(\mathcal{H})$ and R_S is the right multiplication by S on $\mathcal{B}(\mathcal{H})$. Note that L_T and R_S commute even though it may be the case that T and S do not commute. Moreover, for any $k \in \mathbb{N}$, we have

$$L_T^k = L_{T^k}, \quad R_S^k = R_{S^k}.$$

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Let $T, X \in \mathcal{B}(\mathcal{H})$. Then for any $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
(L_{T^*} - R_T)^m(X) &= \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} L_{T^*}^{(m-k)} R_T^k(X) \\
&= \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} L_{T^{*(m-k)}} R_{T^k}(X) \\
&= \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} L_{T^{*(m-k)}}(XT^k)
\end{aligned}$$

$$= \sum_{k=0}^{m} (-1)^k \left(\begin{array}{c} m \\ k \end{array} \right) T^{*(m-k)} X T^k.$$

Therefore,

$$T \text{ is an } m\text{−symmetric operator} \iff (L_{T^*} - R_T)^m(I) = 0.$$

Similarly,

$$T \text{ is an skew } m\text{−symmetric operator} \iff (L_{T^*} + R_T)^m(I) = 0.$$

Main results

Lemma

Let $T, S \in \mathcal{B}(\mathcal{H})$ be commuting operators. Then for every $m \in \mathbb{N}$, we have

$$\alpha_m(TS) = \sum_{k=0}^{m}\sum_{i=0}^{k}\binom{m}{k}\binom{k}{i}(-1)^i T^{*k} S^{*(k-i)}\alpha_{m-k}(T) S^{m-k+i}$$

$$= \sum_{k=0}^{m}\sum_{i=0}^{m-k}\binom{m}{k}\binom{m-k}{i}(-1)^i T^{*(m-i)}\alpha_k(S) T^i S^{m-k}.$$

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Proof

Note that

$$\begin{aligned}(y_1y_2 - x_1x_2)^m &= \left((y_1 - x_1)x_2 + y_1(y_2 - x_2) \right)^m\\ &= \sum_{k=0}^{m}\binom{m}{k} y_1^k(y_2 - x_2)^k x_2^{m-k}(y_1 - x_1)^{m-k}.\end{aligned}$$

Since T and S are commuting, we obtain

$$R_{TS} = R_S R_T = R_T R_S$$

$$L_{(TS)^*} = L_{T^*} L_{S^*} = L_{S^*} L_{T^*}.$$

Therefore,

$$R_S(L_{T^*} - R_T) = (L_{T^*} - R_T)R_S$$

$$L_{T^*}(L_{S^*} - R_S) = (L_{S^*} - R_S)L_{T^*}.$$

Hence

$$\begin{aligned}
\alpha_m(TS) =& (L_{(TS)^*} - R_{TS})^m(I) \\
=& (L_{T^*} L_{S^*} - R_T R_S)^m(I) \\
=& \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) L_{T^*}^k (L_{S^*} - R_S)^k R_S^{m-k} (L_{T^*} - R_T)^{m-k}(I) \\
=& \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) L_{T^*}^k (L_{S^*} - R_S)^k (\alpha_{m-k}(T) S^{m-k}).
\end{aligned}$$

Observe that

$$
\begin{aligned}
(L_{S^*} - R_S)^k(\alpha_{m-k}(T)S^{m-k}) &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i S^{*(k-i)}\alpha_{m-k}(T)S^{m-k}S^i \\
&= \sum_{i=0}^{k} \binom{k}{i} (-1)^i S^{*(k-i)}\alpha_{m-k}(T)S^{m-k+i}.
\end{aligned}
$$

Thus

$$
\alpha_m(TS) = \sum_{k=0}^{m}\sum_{i=0}^{k} \binom{m}{k}\binom{k}{i} (-1)^i T^{*k} S^{*(k-i)}\alpha_{m-k}(T)S^{m-k+i}.
$$

On the other hand, we have

$$\sum_{i=0}^{k}\left(\begin{array}{c}k\\ i\end{array}\right)(-1)^i S^{*(k-i)}\alpha_{m-k}(T)S^{m-k+i}$$

$$= \sum_{i=0}^{k}\left(\begin{array}{c}k\\ i\end{array}\right)(-1)^i S^{*(k-i)}\left(\sum_{j=0}^{m-k}\left(\begin{array}{c}m-k\\ j\end{array}\right)(-1)^j T^{*(m-k-j)}T^j\right)S^{m-k+i}$$

$$= \sum_{i=0}^{k}\sum_{j=0}^{m-k}\left(\begin{array}{c}k\\ i\end{array}\right)\left(\begin{array}{c}m-k\\ j\end{array}\right)(-1)^i(-1)^j S^{*(k-i)}T^{*(m-k-j)}T^j S^{m-k+i}$$

$$= \sum_{j=0}^{m-k}\left(\begin{array}{c}m-k\\ j\end{array}\right)(-1)^j T^{*(m-k-j)}\left(\sum_{i=0}^{k}\left(\begin{array}{c}k\\ i\end{array}\right)(-1)^i S^{*(k-i)}S^i\right)T^j S^{m-k}$$

$$= \sum_{j=0}^{m-k}\left(\begin{array}{c}m-k\\ j\end{array}\right)(-1)^j T^{*(m-k-j)}\alpha_k(S)T^j S^{m-k}.$$

Therefore,

$$\sum_{i=0}^{k}\left(\begin{array}{c} k \\ i \end{array}\right)(-1)^i S^{*(k-i)}\alpha_{m-k}(T)S^{m-k+i}$$

$$=\sum_{j=0}^{m-k}\left(\begin{array}{c} m-k \\ j \end{array}\right)(-1)^j T^{*(m-k-j)}\alpha_k(S)T^jS^{m-k}.$$

Hence

$$\alpha_m(TS)=\sum_{k=0}^{m}\sum_{j=0}^{m-k}\left(\begin{array}{c} m \\ k \end{array}\right)\left(\begin{array}{c} m-k \\ j \end{array}\right)(-1)^j T^{*(m-j)}\alpha_k(S)T^jS^{m-k}.$$

As desired.

Theorem

Suppose that $T, S \in \mathcal{B}(\mathcal{H})$ are commuting operators. If T is $m-$symmetric and S is $n-$symmetric, thus TS is an $(m+n-1)-$symmetric operator.

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Proof

Set $r = m+n-1$. It follows from the last Lemma that

$$\alpha_r(TS) = \sum_{k=0}^{n-1}\sum_{i=0}^{k}\begin{pmatrix} r \\ k \end{pmatrix}\begin{pmatrix} k \\ i \end{pmatrix}(-1)^i T^{*k} S^{*(k-i)} \alpha_{r-k}(T) S^{r-k+i}$$
$$+ \sum_{k=n}^{r}\sum_{i=0}^{r-k}\begin{pmatrix} r \\ k \end{pmatrix}\begin{pmatrix} r-k \\ i \end{pmatrix}(-1)^i T^{*(r-i)} \alpha_k(S) T^i S^{r-k}.$$

- Since T is $m-$symmetric, then for every $0 \leq k \leq n-1$, we have

$$\alpha_{m+n-1-k}(T) = 0.$$

Hence

$$\sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{r}{k}\binom{k}{i}(-1)^i T^{*k}S^{*(k-i)}\alpha_{r-k}(T)S^{r-k+i}=0.$$

• Since S is $n-$symmetric, thus for any $n \leq k \leq m+n-1$, we have

$$\alpha_k(S)=0,$$

then

$$\sum_{k=n}^{r}\sum_{i=0}^{r-k}\binom{r}{k}\binom{r-k}{i}(-1)^i T^{*(r-i)}\alpha_k(S)T^iS^{r-k}=0.$$

Therefore,

$$\alpha_{m+n-1}(TS)=0.$$

As required.

Lemma

Let $T, S \in \mathcal{B}(\mathcal{H})$ be commuting operators. Then for every $m \in \mathbb{N}$, we have

$$\begin{aligned}\alpha_m(T+S) &= \sum_{k=0}^{m}\sum_{i=0}^{k}\binom{m}{k}\binom{k}{i}(-1)^i S^{*(k-i)}\alpha_{m-k}(T)S^i \\ &= \sum_{k=0}^{m}\sum_{i=0}^{m-k}\binom{m}{k}\binom{m-k}{i}(-1)^i T^{*(m-k-i)}\alpha_k(S)T^i.\end{aligned}$$

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Proof

Since $TS = ST$, we get

$$(L_{T^*} - R_T)(L_{S^*} - R_S) = (L_{S^*} - R_S)(L_{T^*} - R_T).$$

Therefore,

$$\begin{aligned}
\alpha_m(T+S) =& (L_{T^*+S^*} - R_{T+S})^m(I) \\
=& (L_{T^*} - R_T + L_{S^*} - R_S)^m(I) \\
=& \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) (L_{S^*} - R_S)^k (L_{T^*} - R_T)^{m-k}(I) \\
=& \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) (L_{S^*} - R_S)^k (\alpha_{m-k}(T)) \\
=& \sum_{k=0}^{m} \sum_{i=0}^{k} \left(\begin{array}{c} m \\ k \end{array} \right) \left(\begin{array}{c} k \\ i \end{array} \right) (-1)^i S^{*(k-i)} \alpha_{m-k}(T) S^i.
\end{aligned}$$

On the other hand

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^i S^{*(k-i)}\alpha_{m-k}(T)S^i$$

$$= \sum_{i=0}^{k}\binom{k}{i}(-1)^i S^{*(k-i)}\left(\sum_{j=0}^{m-k}\binom{m-k}{j}(-1)^j T^{*(m-k-j)}T^j\right)S^i$$

$$= \sum_{i=0}^{k}\sum_{j=0}^{m-k}\binom{k}{i}\binom{m-k}{j}(-1)^i(-1)^j S^{*(k-i)}T^{*(m-k-j)}T^jS^i$$

$$= \sum_{j=0}^{m-k}\binom{m-k}{j}(-1)^j T^{*(m-k-j)}\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^i S^{*(k-i)}S^i\right)T^j$$

$$= \sum_{j=0}^{m-k}\binom{m-k}{j}(-1)^j T^{*(m-k-j)}\alpha_k(S)T^j.$$

Hence

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^i S^{*(k-i)} \alpha_{m-k}(T) S^i$$

$$= \sum_{j=0}^{m-k} \binom{m-k}{j} (-1)^j T^{*(m-k-j)} \alpha_k(S) T^j.$$

Thus

$$\alpha_m(T+S) = \sum_{k=0}^{m} \sum_{i=0}^{m-k} \binom{m}{k} \binom{m-k}{i} (-1)^i T^{*(m-k-i)} \alpha_k(S) T^i.$$

As desired.

Theorem

Suppose that $T, S \in \mathcal{B}(\mathcal{H})$ are commuting operators. If T is $m-$symmetric and S is $n-$symmetric, thus $T + S$ is an $(m+n-1)-$symmetric operator.

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Proof

Set $r = m+n-1$. By the last Lemma, we have

$$\begin{aligned}\alpha_r(T+S) = &\sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{r}{k}\binom{k}{i}(-1)^i S^{*(k-i)}\alpha_{r-k}(T)S^i \\ &+\sum_{k=n}^{r}\sum_{i=0}^{r-k}\binom{r}{k}\binom{r-k}{i}(-1)^i T^{*(r-k-i)}\alpha_k(S)T^i.\end{aligned}$$

• Since T is $m-$symmetric, then for every $0 \le k \le n-1$, we have

$$\alpha_{m+n-1-k}(T) = 0.$$

Hence

$$\sum_{k=0}^{n-1}\sum_{i=0}^{k}\begin{pmatrix} m+n-1 \\ k \end{pmatrix}\begin{pmatrix} k \\ i \end{pmatrix}(-1)^i S^{*(k-i)}\alpha_{r-k}(T)S^i = 0.$$

- Since S is $n-$symmetric, thus for any $n \leq k \leq m+n-1$, we have

$$\alpha_k(S) = 0,$$

then

$$\sum_{k=n}^{r}\sum_{i=0}^{r-k}\begin{pmatrix} r \\ k \end{pmatrix}\begin{pmatrix} r-k \\ i \end{pmatrix}(-1)^i T^{*(r-k-i)}\alpha_k(S)T^i = 0.$$

Therefore,

$$\alpha_{m+n-1}(T+S) = 0.$$

This completes the proof.

Lemma

Let $T, S \in \mathcal{B}(\mathcal{H})$ be commuting operators. Then for every $m \in \mathbb{N}$, we have

$$\delta_m(TS) = \sum_{k=0}^{m}\sum_{i=0}^{k} \left(\begin{array}{c} m \\ k \end{array}\right)\left(\begin{array}{c} k \\ i \end{array}\right)(-1)^{m-k} T^{*k} S^{*(k-i)} \alpha_{m-k}(T) S^{m-k+i}$$

$$= \sum_{k=0}^{m}\sum_{j=0}^{m-k} \left(\begin{array}{c} m \\ k \end{array}\right)\left(\begin{array}{c} m-k \\ j \end{array}\right)(-1)^{m-k+j} T^{*(m-j)} \delta_k(S) T^j S^{m-k}.$$

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Proof

The following identities follows from the binomial theorem

$$\begin{aligned}(y_1y_2 + x_1x_2)^m &= \Big(\ y_1(y_2+x_2) - (y_1-x_1)x_2\ \Big)^m \\ &= \sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right)(-1)^{m-k}y_1^k(y_2+x_2)^kx_2^{m-k}(y_1-x_1)^{m-k}.\end{aligned}$$

Proceeding as the previous lemmas, we get the desired result.

Theorem

Suppose that $T, S \in \mathcal{B}(\mathcal{H})$ are commuting operators. If T is $m-$symmetric and S is skew $n-$symmetric, thus TS is a skew $(m+n-1)-$symmetric operator.

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$$\begin{aligned}(y_1y_2 - x_1x_2)^m &= \big(\, y_1(y_2+x_2) - (y_1+x_1)x_2 \,\big)^m \\ &= \sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}(-1)^{m-k}y_1^k(y_2+x_2)^k x_2^{m-k}(y_1+x_1)^{m-k}.\end{aligned}$$

Proceeding as the previous lemmas, we get the desired result.

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$$\begin{aligned}\delta_m(T+S) &= \sum_{k=0}^{m} \sum_{i=0}^{k} \binom{m}{k} \binom{k}{i} S^{*(k-i)} \delta_{m-k}(T) S^i \\ &= \sum_{k=0}^{m} \sum_{i=0}^{m-k} \binom{m}{k} \binom{m-k}{i} T^{*(m-k-i)} \delta_k(S) T^i.\end{aligned}$$

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Proof

Since $TS = ST$, we get

$$(L_{T^*}+R_T)(L_{S^*}+R_S) = (L_{S^*}+R_S)(L_{T^*}+R_T).$$

The following identities follows from the binomial theorem

$$(y_1+y_2+x_1+x_2)^m = \sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}(y_2+x_2)^k(y_1+x_1)^{m-k}.$$

Proceeding as the previous lemmas, we get the desired result.

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Suppose that $T, S \in \mathcal{B}(\mathcal{H})$ are commuting operators. If T is skew $m-$symmetric and S is skew $n-$symmetric, thus $T + S$ is a skew $(m + n - 1)-$symmetric operator.

References

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Thanks for your attention