On the power inequality for the numerical radius

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A general Abstract

In this talk, we give an overview of the power inequality for the numerical radius, which has been known for so long. Then, we present some new progress related to this important inequality.

A Technical Abstract

Using integral representations of the fractional power of matrices, and the geometric intuition of sectorial matrices, we show that for any accretive-dissipative matrix A and any $t \in (0,1)$, the matrix A^t is accretive-dissipative, and that

$$\omega(A^t) \geq \omega^t(A),$$

where $\omega(\cdot)$ is the numerical radius. This inequality complements the well-known power inequality $\omega(A^k) \leq \omega^k(A)$, valid for any square matrix and positive integer power k. As an application, we prove that if A is accretive, then the above fractional inequality holds if $0 < t < \frac{1}{2}$. Other consequences will be given too.

Basic terminologies and notations

- $\mathcal{M}_n$ is the algebra of all $n \times n$ complex matrices.
- The numerical radius and spectral norm are, respectively,

$$\omega(A) = \max_{\|x\|=1} |\langle Ax, x\rangle|, \|A\| = \max_{\|x\|=1} \|Ax\|.$$

Lemma

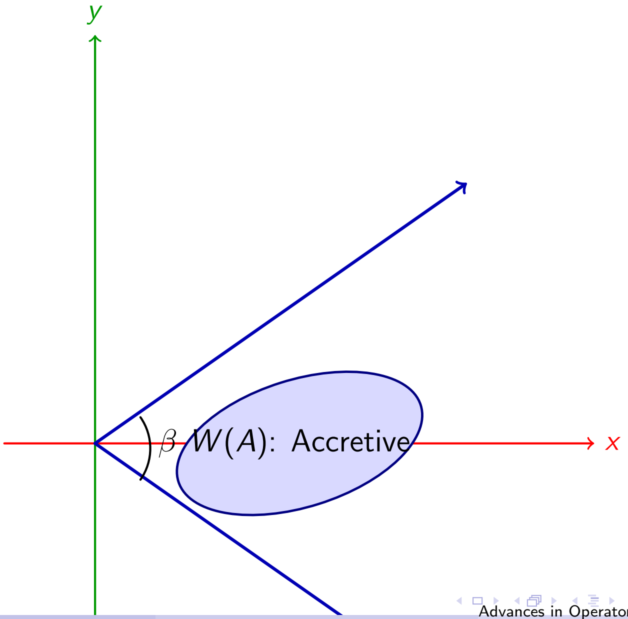
Let $A \in \mathcal{M}_n$. Then

$$\omega(A) = \sup_{\theta\in\mathbb{R}} \left\|\Re\left(e^{i\theta}A\right)\right\|.$$

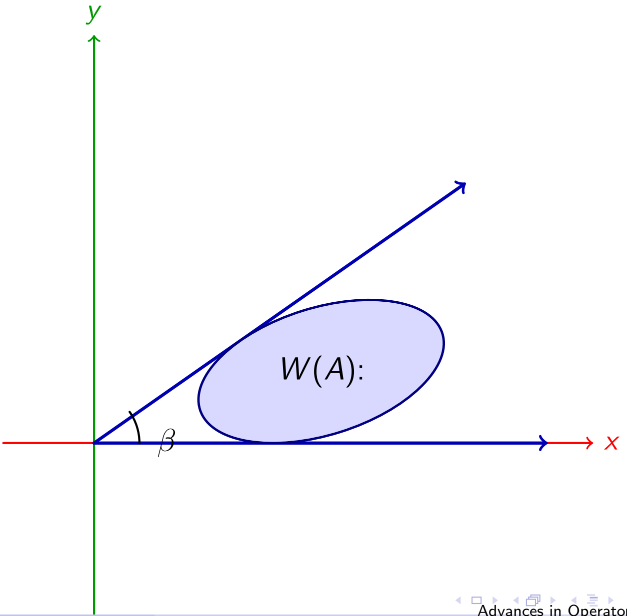
Terminologies

- A is Hermitian if $A^* = A$; A^* being its conjugate transpose.
- A is positive semi-definite if $\langle Ax, x\rangle \geq 0$. We simply write $A \geq O$.
- $|A| = (A^*A)^{\frac{1}{2}}$; the positive semi-definite square root.
- $A > O$ means $A \geq O$ is invertible.
- The real part of A is $\Re(A) = \frac{A+A^*}{2}$. The imaginary part is $\Im(A) = \frac{A-A^*}{2\mathrm{i}}$.
- A is accretive if $\Re A > O$.
- A is accretive-dissipative if $\Re A, \Im A > O$.
- Γ_n is the class of all accretive matrices in $\mathcal{M}_n$.

An accretive Matrix



An accretive-Dissipative Matrix



The power Inequality

A basic inequality

$$\omega(A^k) \leq \omega^k(A), A \in \mathcal{M}_n, k \in \mathbb{N}. \tag{3.1}$$

The reversed verssion

$$\omega(A^{-k}) \geq \omega^{-k}(A), A \in \mathcal{M}_n, k \in \mathbb{N};$$

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Why is the power inequality important?

Sub-multiplicativity

- $\omega(AB) \nleq \omega(A)\omega(B)$.
- However, $\|AB\| \leq \|A\|\,\|B\|$.
- $\omega(A^3) \nleq \omega(A)\omega^2(A)$.

Partial Sub-multiplicativity

$\omega(A^k) \leq \omega(A)\omega(A)\ldots\omega(A)$.

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$\omega(A^k) \leq \omega(A)\omega(A)\ldots\omega(A)$.

The main problem

We cannot find in any reference any work that treats fractional powers of the numerical radius. We have the following question.

A question

What is the relation between $\omega(A^t)$ and $\omega^t(A)$, for $0 < t < 1$?

Here we need to be very careful!

Caution

We need A^t to be well defined.

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Definition of A^t

When we deal with fractional powers of matrices, we look at

Simple cases

- $A \geq O$.
- A is accretive

More generally

Let $A \in \mathcal{M}_n$ be such that $W(A) \cap (-\infty, 0] = \varnothing$, and let Ω_A denote a contour in the resolvent of A that winds once about each eigenvalue of A and avoids $(-\infty, 0]$. Then for any $t \in (0,1)$, the prinicpal fractional power A^t can be defined via the Dunford integral as follows:

$$A^t = \frac{1}{2\pi i} \int_{\Omega_A} z^t \Big(zI - A\Big)^{-1} dz, \tag{4.1}$$

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This coincides with the following formula (see [**?**, **?**]):

Another formula

$$A^t = \frac{\sin(t\pi)}{\pi} \int_0^\infty s^{t-1} A(sI + A)^{-1}\, ds. \qquad (4.2)$$

Watch out

It is important to remember that in (4.1), and in what follows, z^t refers to the principal power of z, and so is A^t.

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One can use (4.2) to prove that

Fractional powers of accretive

If A is accretive, then so is A^t, for any $t \in (0, 1)$.

See [**?**, Lemma A6] and [**?**, Theorem 2.3].

Some Lemmas

Lemma

Let $A > O$ and let $t \in [0,1]$. Then

$$\|A\|^t = \|A^t\|.$$

Lemma

([?, Lemma A1],[?, Corollary 2.4]) Let $A \in \Gamma_n$ be such that $W(A) \subseteq S_\alpha$ for some $\alpha \in \left[0, \frac{\pi}{2}\right)$, and let $t \in [0,1]$. Then $W(A^t) \subseteq S_{t\alpha}$.

Lemma

[?, Proposition 7.1] Let $A \in \Gamma_n$ and let $t \in [0,1]$. Then

$$\Re^t(A) \leq \Re(A^t).$$

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Lemma

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$$\Re^t(A) \le \Re(A^t).$$

A simple motivation

We state the following simple fractional consequence of (3.1).

Proposition

Let $A \in \Gamma_n$*, and let* $k \in \mathbb{N}$*. Then*

$$\omega\left(A^{\frac{1}{k}}\right) \geq \omega^{\frac{1}{k}}(A).$$

Proof.

Since A is accretive, $A^{\frac{1}{k}}$ is a well-defined accretive matrix. Applying (3.1) implies

$$\omega(A) = \omega\left(\left(A^{\frac{1}{k}}\right)^k\right) \leq \omega^k\left(A^{\frac{1}{k}}\right),$$

which is equivalent to the desired statement. □

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This triggers something

Conjecture

Let $A \in \Gamma_n$ and $0 < t < 1$. Then

$$\omega(A^t) \geq \omega^t(A).$$

Comments

- We conjecture this for accretive matrices.
- Notice that the inequality in the conjecture is trivial for positive matrices.
- Our numerical calculations support this conjecture.

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First achievement

Theorem

Let $A \in \mathcal{M}_n$ be accretive-dissipative and let $t \in (0,1)$. Then

$$\omega(A^t) \geq \omega^t(A).$$

Comments

- This does not resolve the conjecture!
- This partially answers our conjecture.
- This requires further study of A^t.

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Corollary

Let $A \in \Gamma_n$ *and let* $0 < t < \frac{1}{2}$*. Then*

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Let $A \in \Gamma_n$ *and let* $0 < t < \frac{1}{2}$. *Then*

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The proof of the accretive case: $0 < t < 1/2$

Proof.

- $A \in \Gamma_n \Rightarrow W(A^{\frac{1}{2}}) \subset S_{\pi/4}$.
- $e^{i\frac{\pi}{4}} A^{\frac{1}{2}}$ is accretive-dissipative.
- Let $t = \frac{r}{2}, 0 < r < 1$:

$$\begin{aligned}
\omega(A^t) = \omega\left(\left(A^{\frac{1}{2}}\right)^r\right) &= \omega\left(e^{i(r\,\frac{\pi}{4})}\left(A^{\frac{1}{2}}\right)^r\right) \\
&= \omega\left(\left(e^{i\,\frac{\pi}{4}} A^{\frac{1}{2}}\right)^r\right) \\
&\geq \omega^r\left(e^{i\,\frac{\pi}{4}}\, A^{\frac{1}{2}}\right) \\
&= \omega^r\left(A^{\frac{1}{2}}\right) \\
&\geq \omega^{\frac{r}{2}}(A) = \omega^t(A).
\end{aligned}$$

A byproduct

The following allows arbitrary powers (m, k are not necessarily integers):

Corollary

Let A be accretive, and assume that A^k is accretive for some positive number $k \geq 2$. Then

$$\omega^m(A) \geq \omega(A^m),$$

for $2 \leq m \leq k$.

What is this?

This is an extension of the power inequality to fractional powers larger than 2.

But

A harsh condition that A and A^k are accretive!

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Technical observation

Proposition

Let $A \in \mathcal{M}_n$ and let $x_0 \in \mathcal{S}_1(\mathbb{C}^n)$ be such that $\omega(A) = |\langle Ax_0, x_0\rangle|$. If $\gamma = Arg(\langle Ax_0, x_0\rangle)$, then

$$\omega(A) = \left\| \Re\left(e^{-i\gamma}A\right)\right\|.$$

Proof.

Notice first that, by the compactness of $\mathcal{S}_1(\mathbb{C}^n)$, we have $\omega(A) = |\langle Ax_0, x_0\rangle|$ for some $x_0 \in \mathcal{S}_1(\mathbb{C}^n)$. Let γ be the principal argument of $\langle Ax_0, x_0\rangle$. Then

$$\langle Ax_0, x_0\rangle = |\langle Ax_0, x_0\rangle|e^{i\gamma} = \omega(A)e^{i\gamma}.$$

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Proof continued

Now, noting that the matrix $\Re\left(e^{-i\gamma}A\right)$ is Hermitian, and that for any matrix X, $\omega(\Re(X)) \leq \omega(X)$, we deduce the following simple consequences:

$$\left\|\Re\left(e^{-i\gamma}A\right)\right\| = \omega\left(\Re\left(e^{-i\gamma}A\right)\right) \leq \omega(e^{-i\gamma}A) = \omega(A).$$

On the other hand,

$$\begin{aligned}\left\|\Re\left(e^{-i\gamma}A\right)\right\| &= \omega\left(\Re\left(e^{-i\gamma}A\right)\right)\\ &= \sup_{\|x\|=1}\left|\left\langle\Re\left(e^{-i\gamma}A\right)x, x\right\rangle\right|\\ &\geq \left|\left\langle\Re\left(e^{-i\gamma}A\right)x_0, x_0\right\rangle\right|\\ &= \left|\Re\left(e^{-i\gamma}\langle Ax_0, x_0\rangle\right)\right|\\ &= \left|\Re\left(e^{-i\gamma}e^{i\gamma}\omega(A)\right)\right| = \left|\Re(\omega(A))\right|\\ &= \omega(A).\end{aligned}$$

This completes the proof.

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On the other hand,

$$\begin{aligned}
\left\|\Re\left(e^{-i\gamma}A\right)\right\| &= \omega\left(\Re\left(e^{-i\gamma}A\right)\right) \\
&= \sup_{||x||=1}\left|\langle\Re\left(e^{-i\gamma}A\right)x, x\rangle\right| \\
&\ge \left|\langle\Re\left(e^{-i\gamma}A\right)x_0, x_0\rangle\right| \\
&= \left|\Re\left(e^{-i\gamma}\langle Ax_0, x_0\rangle\right)\right| \\
&= \left|\Re\left(e^{-i\gamma}e^{i\gamma}\omega(A)\right)\right| = \left|\Re(\omega(A))\right| \\
&= \omega(A).
\end{aligned}$$

This completes the proof.

A remark

Remark

If A is accretive-dissipative, then $W(A) \subseteq \{z \in \mathbb{C} : \Re(z), \Im(z) > 0\}$.
This means that $0 < Arg \langle Ax, x \rangle < \frac{\pi}{2}$ for any $x \in \mathbb{C}^n$.
Consequently, Proposition 9 implies that $\omega(A) = \left\| \Re\left(e^{-i\gamma} A \right) \right\|$ for some $\gamma \in (0, \pi/2)$.

A property for accretive-dissipative matrices

If $A \in \mathcal{M}_n$ is accretive, and $0 < t < 1$, then A^t is accretive.
In the following Proposition, we extend this observation to accretive-dissipative matrices.

Proposition

Let $A \in \mathcal{M}_n$ be accretive-dissipative, and let $0 < t < 1$. Then A^t is accretive-dissipative.

Lemma

[?, Lemma 1] Let $A = H + iK$ be accretive-dissipative. Then $A^{-1} = E + iF$, where

$$E = \left(H + KH^{-1}K\right)^{-1} > O,$$

and

$$F = -\left(K + HK^{-1}H\right)^{-1} < O.$$

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Proof of Proposition

Proof.

Let $A \in \mathcal{M}_n$ be accretive-dissipative. Then so is $sI + A$ for $s > 0$. Consequently, by Lemma 12, we have $-\Im\Big((sI + A)^{-1}\Big) > O$.

Now, let $t \in (0, 1)$. Then, by Lemma 3, $\Re(A^t) > O$. So we only need to prove $\Im(A^t) > O$. But, by (4.2), we have

$$A^t = \frac{\sin(t\pi)}{\pi} \int_0^\infty s^{t-1} A(sI + A)^{-1}\, ds.$$

So, it is enough to prove that $\Im\Big(A(sI + A)^{-1}\Big)$ is positive-definite. Since $A(sI + A)^{-1} = I - s(sI + A)^{-1}$, it follows that

$$\Im\Big(A(sI + A)^{-1}\Big) = -\Im\Big((sI + A)^{-1}\Big) > O,$$

which completes the proof. $\square$

Another needed observation

Proposition

Let $A \in \Gamma_n$ and let $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. Then, for any $t \in (0,1)$,

$$\left(e^{i\theta}A\right)^t = e^{it\theta}A^t.$$

Proof.

Let $A \in \Gamma_n$ and let $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. Since $W(A) \subseteq S_\alpha$ for some $0 < \alpha < \frac{\pi}{2}$, it follows that $W(e^{i\theta}A)$ avoids $(-\infty, 0]$. Hence, using (4.1), we can write

$$\left(e^{i\theta}A\right)^t = \frac{1}{2\pi i}\int_{\Omega_{e^{i\theta}A}} z^t\left(zI - e^{i\theta}A\right)^{-1} dz,\ 0 < t < 1. \tag{5.1}$$

□

Proof continued

In this identity, $\Omega_{e^{i\theta}A}$ refers to a rotation by θ of a contour Ω_A that lies completely in the right-half plane, and winds exactly once about each eigenvalue of A. Such assumption is justified by the fact that A is accretive. Due to this, $\Omega_{e^{i\theta}A}$ avoids $(-\infty, 0]$.

Proof continued

Now,

$$\begin{aligned}\left(e^{i\theta}A\right)^t &= \frac{1}{2\pi i}\int_{\Omega_{e^{i\theta}A}} z^t\left(zI - e^{i\theta}A\right)^{-1} dz\\ &= \frac{1}{2\pi i}\int_{\Omega_{e^{i\theta}A}} e^{-i\theta}z^t\left(e^{-i\theta}zI - A\right)^{-1} dz.\end{aligned}$$

Proof continued

Setting $u = e^{-i\theta} z$ rotates the contour $\Omega_{e^{i\theta}A}$ by an angle $-\theta$ to obtain Ω_A back. Now, since u is a complex number in the right half plane, then $Arg(u) \in (\frac{-\pi}{2}, \frac{\pi}{2})$ and consequently, $-\pi < Arg(z) = Arg(u) + \theta < \pi$. This ensures the validity of the following factorization

$$z^t = (e^{i\theta} u)^t = e^{it\theta}\, u^t.$$

Hence,

Proof complete

$$
\begin{aligned}
\left(e^{-i\gamma}A\right)^t &= \frac{1}{2\pi i}\int_{\Omega_{e^{i\theta}A}} z^t\left(zI - e^{i\theta}A\right)^{-1}dz\\
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&= \frac{1}{2\pi i}\int_{\Omega_A} (e^{i\theta}u)^t\left(uI - A\right)^{-1}du\\
&= \frac{1}{2\pi i}\int_{\Omega_u} e^{it\theta}u^t\left(uI - A\right)^{-1}du\\
&= e^{it\theta}\left(\frac{1}{2\pi\, i}\int_{\Omega_A} u^t\left(uI - A\right)^{-1}du\right)\\
&= e^{it\theta}A^t.
\end{aligned}
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This completes the proof.

The main result

Theorem

Let $A \in \mathcal{M}_n$ be accretive-dissipative and let $t \in (0,1)$. Then

$$\omega(A^t) \geq \omega^t(A).$$

Proof.

Let $x_0 \in \mathcal{S}_1(\mathbb{C}^n)$ be such that $\omega(A) = |\langle Ax_0, x_0\rangle|$, and let $\gamma = Arg\langle Ax_0, x_0\rangle$. Since $\Re\left(e^{-i\gamma}A\right) = \cos\gamma\Re(A) + \sin\gamma\Im(A)$, and $0 < \gamma < \frac{\pi}{2}$, it follows that $\Re\left(e^{-i\gamma}A\right) > O$, and hence $e^{-i\gamma}A$ is accretive. $\square$

Proof continued

Now,

$$\begin{aligned}
\omega^t(A) &= \left\|\Re\left(e^{-i\gamma}A\right)\right\|^t \quad \text{(by Proposition 9)}\\
&= \left\|\Re^t\left(e^{-i\gamma}A\right)\right\| \quad \text{(by Lemma 2)}\\
&\le \left\|\Re\left((e^{-i\gamma}A)^t\right)\right\| \quad \text{(by Lemma 4)}\\
&= \left\|\Re\left(e^{-it\gamma}A^t\right)\right\| \quad \text{(by Proposition 13)}\\
&\le \sup_{\theta\in\mathbb{R}} \left\|\Re\left(e^{i\theta}A^t\right)\right\|\\
&= \omega(A^t) \quad \text{(by Lemma 1)}.
\end{aligned}$$

This completes the proof.