FURTHER GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

Vuk Stojiljković

[1]Mathematical Institute of Serbian Academy of Sciences and Arts,
Kneza Mihaila 36,
Belgrade 11000,
Serbia
vuk.stojiljkovic@turing.mi.sanu.ac.rs

November 2, 2025

This is a joint work with prof. Fuad Kittaneh.

Work plan

Abstract

We introduce a new generalized numerical radius, $w_{h,g}^{Re}(A)$, which is defined based on the generalized real and imaginary parts of an operator, as defined by Kittaneh and Stojiljković in a recent paper. We will demonstrate that this new quantity, $w_{h,g}^{Re}(A)$, is a norm on the C^*-algebra of bounded linear operators, $B(H)$, and that it is equivalent to the standard operator norm, $\|\cdot\|$. A key strength of this new concept is its generality; it encompasses and refines existing definitions and inequalities, including those previously established by Sheikhhosseini et al. [23] and Kittaneh. Furthermore, we will explore various new inequalities, including those involving powers of operators and operator matrices, providing extensions and refinements to previous results in the field. The adaptability of $w_{h,g}^{Re}(A)$ through the functions h and g suggests promising applications and significant contributions to the ongoing refinement and extension of operator inequalities in functional analysis. Results in this presentation are based on the recent paper submitted by Kittaneh and Stojiljković [19].

Terminology

Let $\mathcal{H}$ denote a complex Hilbert space with an inner product $\langle . \rangle$. Specifically, the numerical range is given by $W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$, while the numerical radius is defined as:

$$w(A) = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\}.$$

It is well known that $w(.)$ forms a norm on the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators. This numerical radius norm is equivalent to the standard operator norm, $\|\cdot\|$, through a relationship expressed by the sharp two-sided inequality

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$

This inequality holds with equality under specific conditions, such as $A^2 = 0$ for the lower bound and A being a normal operator for the upper bound.

Introduction to the topic

In addition to the investigation of inequalities of the numerical radius, recently Sheikhhousseini et al. [23] introduced the generalized real and imaginary part of an operator, given by

$$Re_v(A) = vA + (1-v)A^*$$

and

$$Im_v(A) = \frac{vT - (1-v)T^*}{i},$$

where $v \in [0,1]$. When $v = \frac{1}{2}$ they reduce to the well-known real and imaginary parts. Utilizing these generalized real and imaginary parts, the authors defined the $v-$weighted numerical radius as:

$$w_v(A) = \sup_{\phi \in \mathbb{R}} \left\| \mathrm{Re}_v(e^{i\phi}) \right\|.$$

In their work, Sheikhhosseini et al. [23] also explored several properties of this weighted numerical radius, deriving various inequalities, including the inequality

$$w_v(A) \leq \sqrt{\|v|A|^2 + (1-v)^2|A^*|^2\| + 2v(1-v)w(A^2)}.$$

In this work we will generalize their concept, prove that our generalization is indeed a norm on C^*-algebra of all bounded linear operators, that it is equivalent to the operator norm. We will also obtain various inequalities that generalize the existing ones.

Auxiliary results

To achieve our results, we will need some lemmas, as follows. The following result was given by Dragomir [8].

Lemma

Let $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$. Then

$$w^r(B^*A) \leq \frac{1}{2} \left\| |A|^{2r} + |B|^{2r} \right\|. \tag{3.1}$$

The next result concerns with non-negative convex functions and can be found in [4].

Lemma

Let f be a non-negative convex function on $[0, +\infty)$ and $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then

$$\left\| f\left(\frac{A+B}{2}\right) \right\| \leq \left\| \frac{f(A)+f(B)}{2} \right\|. \tag{3.2}$$

The direct sum of two copies of $\mathcal{H}$ is denoted by $\mathcal{H} \oplus \mathcal{H}$. If $A, B, C, D \in \mathcal{B}(\mathcal{H})$, then the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator on $\mathcal{H} \oplus \mathcal{H}$, and is defined by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} x = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$ for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$.
The following lemma is well known. See, e.g., [13].

Lemma

Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| = \max\{\|A\|, \|B\|\}. \tag{3.3}$$

We define the following class of functions, which will be used throughout the paper. We say that h, g belong to the class of Q if $h, g : K \subset \mathbb{R} \to \mathbb{R}$.

We utilize the following generalized definitions of the real and imaginary parts given by Kittaneh and Stojiljković [20].

Definition

Let $A \in B(H)$, and let $h, g \in Q$. We define the generalized h, g real and imaginary parts of an operator,

$$\mathrm{Re}_{h,g}(A) \stackrel{\text{def}}{=} Ah(t) + g(t)A^* \tag{3.4}$$

and

$$\mathrm{Im}_{h,g}(A) \stackrel{\text{def}}{=} \frac{g(t)A - h(t)A^*}{i}.$$

Remark

Setting $g(t) = h(1-t), K = [0,1]$, *and letting* $h(t) = t$, *we recover the generalized real and imaginary parts of an operator defined by Sheikhhosseini et al. [23].*
Setting $f = g = \frac{1}{2}$, *we recover the original real and imaginary parts of an operator, which indeed shows that our definition encaptures the previously defined ones.*

Definition

Let $A \in B(H)$, and let $f, g \in Q$. The generalized numerical radius of A denoted by $w_{h,g}^{\mathrm{Re}}(A)$ is defined as

$$w_{h,g}^{\mathrm{Re}}(A) \stackrel{\mathrm{def}}{=} \sup_{\phi \in \mathbb{R}} \left\| \mathrm{Re}_{h,g}(e^{i\phi} A) \right\| . \tag{4.1}$$

Theorem

Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$ be such that $h, g \neq 0$ and $h \neq -g$. Then the function $w_{h,g}^{Re}(.) : \mathcal{B}(\mathcal{H}) \to [0, +\infty)$ is a norm.

The nonnegativity follows from the fact that $w_{h,g}^{\mathrm{Re}}(A)$ is non-negative by definition. Let us first assume the simple case that $A = 0$. Then clearly we have that $w_{h,g}^{\mathrm{Re}}(A) = 0$ by definition.

Now assume that $w_{h,g}^{\mathrm{Re}}(A) = 0$. We then have $e^{i\phi}(h+g)A + (g-t)A^* e^{-i\phi} = 0$ for all $\phi \in \mathbb{R}$. Setting $\phi = 0$, we obtain $(h+g)A + (g-h)A^* = 0$. Further setting $\phi = \frac{\pi}{2}$, we have $(h+g)A - (g-h)A^* = 0$. Adding these two equations, we obtain $2(g+h)A = 0$. Here we use the fact that $h \neq -g$ and that $h, g \neq 0$ and divide both sides by $2(g+h)$ from which we obtain $A = 0$.

Let $\alpha \in \mathbb{C}$. Then we have

$$\begin{aligned} w_{h,g}^{\mathrm{Re}}(\alpha A) &= \sup_{\phi \in \mathbb{R}} \left\| \mathrm{Re}_{h,g}(\alpha e^{i\phi} A) \right\| \\ &= \sup_{\phi \in \mathbb{R}} \left\| \alpha e^{i\phi}(g+h)A + \overline{\alpha} e^{-i\phi}(g-h)A^* \right\|. \end{aligned}$$

For every non-zero complex number α, there exists $\beta \in \mathbb{R}$ such that $\alpha = |\alpha| e^{i\beta}$.

Utilizing this fact, we have

$$\begin{aligned}
&\sup_{\phi\in\mathbb{R}} \left\| \alpha e^{i\phi}(g+h)A + \overline{\alpha} e^{-i\phi}(g-h)A^* \right\| \\
&= \sup_{\phi\in\mathbb{R}} \left\| |\alpha| e^{i(\phi+\beta)}(g+h)A + |\alpha| e^{-i\phi-i\beta}(g-h)A^* \right\| \\
&= |\alpha| \sup_{\phi\in\mathbb{R}} \left\| e^{i(\phi+\beta)}(g+h)A + e^{-i(\phi+\beta)}(g-h)A^* \right\|.
\end{aligned}$$

Introducing a substitution $\phi+\beta=\beta^{'}$, we have

$$\begin{aligned}
&|\alpha| \sup_{\phi\in\mathbb{R}} \left\| e^{i(\phi+\beta)}(g+h)A + e^{-i(\phi+\beta)}(g-h)A^* \right\| \\
&= |\alpha| \sup_{\beta'-\beta\in\mathbb{R}} \left\| e^{i\beta^{'}}(g+h)A + e^{-i\beta^{'}}(g-h)A^* \right\|.
\end{aligned}$$

Since ϕ ranges over all real numbers, and β is a fixed real number, then $\beta^{'}$ also ranges over all real numbers. Therefore, $\beta'-\beta$ can be replaced with just β' as the set remains the same.

Finally, we have the following:

$$|\alpha| \sup_{\beta'-\beta\in\mathbb{R}} \left\| e^{i\beta'}(g+h)A + e^{-i\beta'}(g-h)A^* \right\|$$
$$= |\alpha| \sup_{\beta'\in\mathbb{R}} \left\| e^{i\beta'}(g+h)A + e^{-i\beta'}(g-h)A^* \right\|$$
$$= |\alpha| \sup_{\phi\in\mathbb{R}} \left\| \mathrm{Re}_{h,g}(e^{i\phi}A) \right\| = |\alpha| w^{\mathrm{Re}}_{h,g}(A).$$

The triangle inequality is clear as follows:

$$w^{\mathrm{Re}}_{h,g}(A+B) = \sup_{\phi\in\mathbb{R}} \left\| \mathrm{Re}_{h,g}(e^{i\phi}(A+B)) \right\|$$
$$= \sup_{\phi\in\mathbb{R}} \left\| (g+h)(A+B)e^{i\phi} + (g-h)(A^*+B^*)e^{-i\phi} \right\|$$
$$\leq \sup_{\phi\in\mathbb{R}} \Big[\left\| (g+h)Ae^{i\phi} + (g-h)A^*e^{-i\phi} \right\|$$
$$+ \left\| (g+h)Be^{i\phi} + (g-h)B^*e^{-i\phi} \right\| \Big]$$

$$
\begin{aligned}
&\leq \sup_{\phi\in\mathbb{R}} \left\| (g+h)Ae^{i\phi} + (g-h)A^* e^{-i\phi} \right\| \\
&+ \sup_{\phi\in\mathbb{R}} \left\| (g+h)Be^{i\phi} + (g-h)B^* e^{-i\phi} \right\| \\
&= \sup_{\phi\in\mathbb{R}} \left\| \mathrm{Re}_{h,g}(e^{i\phi}A) \right\| + \sup_{\phi\in\mathbb{R}} \left\| \mathrm{Re}_{h,g}(e^{i\phi}B) \right\| = w_{h,g}^{\mathrm{Re}}(A) + w_{h,g}^{\mathrm{Re}}(B).
\end{aligned}
$$

Therefore, we have proven that $w_{h,g}^{\mathrm{Re}}(A)$ is indeed a norm.

Remark

Setting $h(t) = t, g(t) = 1 - t, K = [0, 1]$ *in* (4.1), *we recover the numerical radius definition given by Sheikhhosseini et al. ([23], Definition 1.5), that is,*

$$
w_{t,1-t}^{Re}(A) = \sup_{\phi\in\mathbb{R}} \left\| Re_t(e^{i\phi}A) \right\|.
$$

Theorem

Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. Then

$$(w_{h,g}^{Re}(A))^2 \leq 2|h(t)g(t)|w(A^2) + \left\| A^*Ah^2(t) + AA^*g^2(t) \right\|. \quad (4.2)$$

Let us consider the following:

$$\left\| Re_{h,g}(e^{i\phi}A) \right\|^2 = \left\| (A^*h(t)e^{-i\phi} + e^{i\phi}Ag(t))(Ae^{i\phi}h(t) + e^{-i\phi}A^*g(t)) \right\|$$

$$= \left\| A^*Ah^2(t) + (A^*)^2h(t)g(t)e^{-2i\phi} + e^{2i\phi}A^2g(t)h(t) + AA^*g^2(t) \right\|$$

$$\leq \left\| e^{2i\phi}A^2h(t)g(t) + e^{-2i\phi}h(t)g(t)(A^*)^2 \right\| + \left\| A^*Ah^2(t) + AA^*g^2(t) \right\|$$

$$= \sup_{\|x\|=1} |\langle (e^{2i\phi}A^2h(t)g(t) + e^{-2i\phi}h(t)g(t)(A^*)^2)x, x\rangle|$$

$$+ \left\| A^*Ah^2(t) + AA^*g^2(t) \right\|$$

$$
\begin{aligned}
&\leq \sup_{\|x\|=1} |h(t)g(t)|\langle A^2x, x\rangle| + |h(t)g(t)| \sup_{\|x\|=1} |\langle (A^*)^2x, x\rangle| \\
&+ \left\| A^*Ah^2(t) + AA^*g^2(t) \right\| \\
&= |h(t)g(t)|w(A^2) + |h(t)g(t)|w((A^*)^2) \\
&+ \left\| A^*Ah^2(t) + AA^*g^2(t) \right\| = 2|h(t)g(t)|w(A^2) \\
&+ \left\| A^*Ah^2(t) + AA^*g^2(t) \right\| .
\end{aligned}
$$

Remark

Setting $h(t) = t, g(t) = 1 - t, K = [0, 1]$ in (4.2), we recover the result given by Sheikhhosseini et al. ([23], eq. 3.4)

$$
w^{Re}_{t,1-t}(A) = w_t(A) \leq \sqrt{\| t^2|A|^2 + (1-t)^2|A^*|^2 \| + 2t(1-t)w(A^2)}.
$$

Remark

Setting $h(t) = g(t) = \frac{1}{2}$ *in* (4.2), *we recover the inequality given by Abu-Omar and Kittaneh in [1], that is,*

$$w^2(A) \le \frac{1}{2} w(A^2) + \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\|. \tag{4.3}$$

In the following corollary, we recover the inequality given by El-Haddad and Kittaneh [10], in the case of $l = 1$.

Corollary

Let $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^2(A) \le \frac{1}{2} w(A^2) + \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| \le \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|.$$

Starting from (4.3), we proceed as follows:

$$\begin{aligned} w^2(A) &\le \frac{1}{2} w(A^2) + \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| \\ &\le \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| \quad (\text{by} \quad (3.1)) \\ &= \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|. \end{aligned}$$

Theorem

Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$ be nonnegative and $m(A) = \inf_{\|x\|=1} |\langle Ax, x \rangle|$. Then the following inequality holds:

$$\begin{aligned} &\left\| h^2(t)|A|^2 + g^2(t)|A^*|^2 \right\| + 2h(t)g(t)m(A^2) \le (w_{h,g}^{Re}(A))^2 \\ &\le \left\| h^2(t)|A|^2 + g^2(t)|A^*|^2 \right\| + 2h(t)g(t)w_{h,g}^{Re}(A^2). \end{aligned}$$

Let $x \in \mathcal{H}$ be an arbitrary unit vector and let $\phi \in \mathbb{R}$ be such that $e^{2i\phi}\langle A^2x, x \rangle = |\langle A^2x, x \rangle|$.

Using $\left\|\mathrm{Re}_{h,g}(e^{i\phi}A)\right\| \leq w^{\mathrm{Re}}_{h,g}(A)$, we have the following:

$$
\begin{aligned}
&\left\|\mathrm{Re}_{h,g}(e^{i\phi}A)\right\|^2 \\
&= \left\|(A^* h(t)e^{-i\phi} + e^{i\phi}Ag(t))(Ae^{i\phi}h(t) + e^{-i\phi}A^* g(t))\right\| \\
&\geq |\langle(|A|^2 h^2(t) + |A^*|^2 g^2(t))x, x\rangle + 2h(t)g(t)\langle \mathrm{Re}(e^{2i\phi}A^2)x, x\rangle| \\
&= |\langle(|A|^2 h^2(t) + |A^*|^2 g^2(t))x, x\rangle + 2h(t)g(t)\mathrm{Re}(\langle e^{2i\phi}A^2)x, x\rangle)| \\
&= |\langle(|A|^2 h^2(t) + |A^*|^2 g^2(t))x, x\rangle + 2h(t)g(t)|\langle A^2 x, x\rangle| \\
&\geq \langle(|A|^2 h^2(t) + |A^*|^2 g^2(t))x, x\rangle + 2h(t)g(t)m(A^2).
\end{aligned}
$$

Taking the supremum over all unit vectors, we obtain the desired inequality.

Regarding the upper bound, consider the same expression but use triangle inequality, that is

$$
\begin{aligned}
(w_{h,g}^{\mathrm{Re}}(A))^2 &= \sup_{\phi\in\mathbb{R}} \left\| \mathrm{Re}_{h,g}(e^{i\phi}A) \right\|^2 \\
&= \left\| (A^* h(t) e^{-i\phi} + e^{i\phi} A g(t))(A e^{i\phi} h(t) + e^{-i\phi} A^* g(t)) \right\| \\
&\leq \left\| |A|^2 h^2(t) + |A^*|^2 g^2(t) \right\| + \sup_{\phi\in\mathbb{R}} \left\| \mathrm{Re}(e^{2i\phi} A^2) \right\| \\
&= \left\| |A|^2 h^2(t) + |A^*|^2 g^2(t) \right\| + 2h(t)g(t) w_{h,g}^{Re}(A^2).
\end{aligned}
$$

Theorem

Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. Then for $\alpha, \beta \in \mathbb{R}$, we have

$$
w_{h,g}^{Re}(A) = \sup_{\alpha^2+\beta^2=1} \left\| \alpha Re_{h,g}(A) + \beta Im_{h,g}(A^*) \right\|. \tag{4.4}
$$

Consider the generalized numerical radius given by (4.1), and consider the following

$$\begin{aligned}\mathrm{Re}_{h,g}(e^{i\phi}A) &= (\cos\phi + i\sin\phi)Ah(t) + A^*g(t)(\cos\phi - i\sin\phi)\\ &= \cos\phi(Ah(t) + A^*g(t)) + \sin\phi\left(\frac{A^*g(t) - Ah(t)}{i}\right)\\ &= \cos\phi\mathrm{Re}_{h,g}(A) + \sin\phi\mathrm{Im}_{h,g}(A^*).\end{aligned}$$

Now letting $\alpha = \cos\phi, \beta = \sin\phi$, we observe that since $\cos\phi, \sin\phi$ satisfy $\cos^2\phi + \sin^2\phi = 1$ and α, β satisfy $\alpha^2 + \beta^2 = 1$, the sets are the same, and so, we obtain the desired expression.

Remark

Setting $(\alpha,\beta) = (1,0), (\alpha,\beta) = (0,1)$, *respectively in* (4.4), *we obtain the following inequalities, respetively:*

$$w_{h,g}^{Re}(A) \geq \|Re_{h,g}(A)\|, w_{h,g}^{Re}(A) \geq \|Im_{h,g}(A)\|.$$

Corollary

Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. Then

$$w_{h,g}^{Re}(A) \leq \sqrt{\|Re_{h,g}(A)\|^2 + \|Im_{h,g}(A)\|^2}.$$

We start by utilizing the representation of $w_{h,g}^{Re}(A)$ given by (4.4). Then we proceed as follows:

$$\begin{aligned} w_{h,g}^{Re}(A) &= \sup_{\alpha^2+\beta^2=1} \|\alpha Re_{h,g}(A) + \beta Im_{h,g}(A^*)\| \\ &\leq \sup_{\alpha^2+\beta^2=1} \left(|\alpha|\, \|Re_{h,g}(A)\| + |\beta|\, \|Im_{h,g}(A^*)\|\right) \\ &\leq \sup_{\alpha^2+\beta^2=1} \sqrt{(\alpha^2+\beta^2)(\|Re_{h,g}(A)\|^2 + \|Im_{h,g}(A^*)\|^2)} \\ &\quad (\textit{by the Cauchy-Schwarz inequality}). \end{aligned}$$

Theorem

Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$ be of the same sign. Then

$$\frac{\|A\|}{2}(|h(t)+g(t)|-|g(t)-h(t)|) \leq w_{h,g}^{Re}(A) \leq (|h(t)|+|g(t)|)\,\|A\|\,.$$

In particular, the norm $w_{h,g}(.)$ is equivalent to the usual operator norm $\|.\|$.

Consider the inequalities from the Remark 15, that is,

$$w_{h,g}^{\mathrm{Re}}(A) \geq \|\mathrm{Re}_{h,g}(A)\|\,, w_{h,g}^{\mathrm{Re}}(A) \geq \|\mathrm{Im}_{h,g}(A)\|\,.$$

Utilizing these two inequalities, we obtain

$$\begin{aligned} 2w_{h,g}^{\mathrm{Re}}(A) &\geq \|\mathrm{Re}_{h,g}(A)\| + \|\mathrm{Im}_{h,g}(A)\| \geq \|\mathrm{Re}_{h,g}(A) + i\mathrm{Im}_{h,g}(A)\| \\ &= \|A(h(t)+g(t)) - A^*(h-g)\| \\ &\geq \|A(h(t)+g(t))\| - \|A^*(h-g)\| \\ &= \|A\|\,(|h(t)+g(t)| - |h(t)-g(t)|), \end{aligned}$$

which implies that $w_{h,g}^{\mathrm{Re}}(A) \geq \frac{\|A\|}{2}(|h(t)+g(t)| - |h(t)-g(t)|)$. For the upper bound, we use Definition 7. We proceed as follows:

$$
\begin{aligned}
w_{h,g}^{\mathrm{Re}}(A) &\overset{\text{def}}{=} \sup_{\phi\in\mathbb{R}} \left\|\mathrm{Re}_{h,g}(e^{i\phi}A)\right\| \\
&= \sup_{\phi\in\mathbb{R}} \left\|e^{i\phi}Ah(t) + e^{-i\phi}A^*g(t)\right\| \\
&\leq \sup_{\phi\in\mathbb{R}} \left(\left\|e^{i\phi}Ah(t)\right\| + \left\|e^{-i\phi}A^*g(t)\right\|\right) \\
&\leq \sup_{\phi\in\mathbb{R}} \left(|h(t)|\,\|A\| + |g(t)|\,\|A\|\right) \\
&= \|A\|\,(|h(t)| + |g(t)|).
\end{aligned}
$$

Combining the left and right-hand side inequality, we reach the desired conclusion.

Theorem

Suppose that $T = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$, *where* $A, B \in \mathcal{B}(\mathcal{H})$, *and let* $M = \max\{|h(t)|, |g(t)\}$. *Then*

$$w_{h,g}^{Re}(T) \leq \sqrt{(w_{h,g}^{Re}(A))^2 + |h(t)g(t)| \, \|B\|^2 + Q \, \|B\| \, w_{h,g}^{Re}(A)}. \tag{4.5}$$

Let $\phi \in \mathbb{R}$ be arbitrary. Consider the following:

$$\mathrm{Re}_{h,g}(e^{i\phi}T) = \begin{bmatrix} \mathrm{Re}_{h,g}(e^{i\phi}A) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e^{i\phi}h(t)B \\ e^{-i\phi}g(t)B^* & 0 \end{bmatrix},$$

when squared we have the following:

$$(\mathrm{Re}_{h,g}(e^{i\phi}T))^2 = \begin{bmatrix} (\mathrm{Re}_{h,g}(e^{i\phi}A))^2 & 0 \\ 0 & 0 \end{bmatrix} + h(t)g(t) \begin{bmatrix} BB^* & 0 \\ 0 & B^*B \end{bmatrix} + \begin{bmatrix} 0 & e^{i\phi}h(t)\mathrm{Re}_{h,g}(e^{i\phi}A)B \\ e^{-i\phi}g(t)\mathrm{Re}_{h,g}(e^{i\phi}A)B^* & 0 \end{bmatrix}.$$

Taking norms, and utilizing (3.3), we obtain the following:

$$\begin{aligned}\left\|\mathrm{Re}_{h,g}(e^{i\phi}T)\right\|^2 &\leq \left\|\mathrm{Re}_{h,g}(e^{i\phi}A)\right\|^2 + |h(t)g(t)|\,\|B\|^2 \\ &+ \max\{\left\|\mathrm{Re}_{h,g}(Ae^{i\phi})h(t)e^{i\phi}B\right\|, \\ &\left\|e^{-i\phi}g(t)\mathrm{Re}_{h,g}(e^{i\phi}A)B^*\right\|\} \\ &\leq (w_{h,g}^{Re}(A))^2 + |h(t)g(t)|\,\|B\|^2 + M\,\|B\|\,w_{h,g}^{\mathrm{Re}}(A).\end{aligned}$$

Taking the supremum over $\phi \in \mathbb{R}$, we obtain the desired inequality.

Remark

Setting $h(t) = g(t) = \frac{1}{2}$ in (4.5), we recover the inequality given by Bhunia et al. ([7], Theorem 9), that is,

$$w(T) \leq \sqrt{w^2(A) + \frac{1}{4}\|B\|^2 + \frac{1}{2}\|B\| w(A)}.$$

Corollary

Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. Then

$$w_{h,g}^{Re}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \sqrt{(w_{h,g}^{Re}(A))^2 + |h(t)g(t)|\,\|B\|^2 + M\|B\|\, w_{h,g}^{Re}(A)} + \sqrt{(w_{h,g}^{Re}(D))^2 + |h(t)g(t)|\,\|C\|^2 + M\|C\|\, w_{h,g}^{Re}(D)}. \tag{4.6}$$

We observe that $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is unitary. Utilizing the subaddtivity of the numerical radius together with weak unitary invariance property, that is, $w(U^* A U) = w(A)$ for the unitary U, we obtain

$$\begin{aligned} w_{h,g}^{\mathrm{Re}}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leq w_{h,g}^{\mathrm{Re}}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) + w_{h,g}^{\mathrm{Re}}\left(\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \right) \\ &= w_{h,g}^{\mathrm{Re}}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) + w_{h,g}^{\mathrm{Re}}\left(U^* \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} U \right) \\ &= w_{h,g}^{\mathrm{Re}}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) + w_{h,g}^{\mathrm{Re}}\left(\begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix} \right). \end{aligned}$$

The desired inequality follows by using (4.5) on both operator matrices on the right.

Theorem

Let $A \in \mathcal{B}(\mathcal{H})$, and let h, g be linear functions belonging to Q. Then $f(t) = w^{Re}_{h(t),g(t)}(A)$ is a convex function on $[0,1]$, and we also have the following inequality:

$$\begin{aligned}
&\sup_{\phi \in \mathbb{R}} \left\| Re_{h(\frac{1}{2}),g(\frac{1}{2})}(e^{i\phi}A) \right\| \\
&\leq \int_0^1 \sup_{\phi \in \mathbb{R}} \left\| Re_{h(t),g(t)}(e^{i\phi}A) \right\| dt \\
&\leq \frac{\sup_{\phi \in \mathbb{R}} \left\| Re_{h(1),g(1)}(e^{i\phi}A) \right\| + \sup_{\phi \in \mathbb{R}} \left\| Re_{h(0),g(0)}(e^{i\phi}A) \right\|}{2}.
\end{aligned}$$

To see that $f(t)$ is convex, one can check easily by definition.

For the proof of the inequality, we will utilize the famous Hermite-Hadamard inequality [12], that is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2},$$

where $f : [a, b] \to \mathbb{R}$ is convex.
To obtain the desired inequality, we set $a = 0, b = 1$ and apply the said inequality to the convex function $f(t) = w^{\mathrm{Re}}_{h(t),g(t)}(A)$.

Theorem

Let $A \in B(H)$, and let $h, g \in Q$. Then

$$w_{h,g}^{Re}(A) \geq \frac{1}{2} \max\{t_1, t_2\},$$

where

$$t_1 = \|A(h(t) + g(t)) + A^*(g(t) - h(t))\| + k,$$

and

$$t_2 = \|h(t)(A + A^*) + g(t)(A^* - A)\| + k,$$

where $k = \Big| \|Re_{h,g}(A)\| - \|Im_{h,g}(A)\| \Big|.$

Corollary

Let $A \in B(H)$*, and let* $h, g \in Q$*. If* $w_{h,g}^{Re}(A) = \frac{\|h(t)(A+A^*)+g(t)(A^*-A)\|}{2}$ *or* $w_{h,g}^{Re}(A) = \frac{\|(h(t)+g(t))A+A^*(g(t)-h(t))\|}{2}$*, then*

$$Re_{h,g}(A) = Im_{h,g}(A) = \frac{\|h(t)(A+A^*)+g(t)(A^*-A)\|}{2},$$

or

$$Re_{h,g}(A) = Im_{h,g}(A) = \frac{\|(h(t)+g(t))A+A^*(g(t)-h(t))\|}{2}.$$

Conclusion

This study successfully developes a novel and adaptable framework for numerical radius inequalities by introducing the generalized numerical radius $w_{h,g}^{Re}(A)$, derived from the generalized real and imaginary parts of an operator. We have established that $w_{h,g}^{Re}(A)$ functions as a norm on $B(H)$ and demonstrated its equivalence to the standard operator norm. A key strength of this framework lies in its generality, as it not only recovers numerous existing numerical radius inequalities as special cases, such as those presented by Sheikhhosseini et al. [23] and Kittaneh, but also provides new identities and sharper bounds. The inherent flexibility offered by the functions h and g suggests promising paths for future research, particularly in the analysis of dynamical systems in which operator properties may evolve. This work significantly contributes to the ongoing refinement and extension of operator inequalities in functional analysis.

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