On compactness of state spaces in classical Banach spaces

Soumitra Daptari

Institute of Arts and Sciences
Tokyo University of Science

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This is a joint work with Saurabh Dwivedi, Shiv Nadar Institution of Eminence, Delhi NCR.

Outline

Notations

Let X be a Banach space over scalar field $\mathbb{K}$. We will use the following notations throughout the talk.

(1) $X_1 = \{x \in X : \|x\| \leq 1\}$ & $S(X) = \{x \in X : \|x\| = 1\}$.

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(6) Let (Ω, Σ, μ) be a measure space.
$L^1(\mu) = \{f : \Omega \longrightarrow \mathbb{K} : f$ is measurable function & $\int_\Omega |f| d\mu < \infty\}\}$ equipped with the norm $\|f\| = \int_\Omega |f| d\mu$.

Preliminaries

State space in unital C^*-algebra

Let $\mathcal{A}$ be a C^*-algebra with unit I. Then state space of $\mathcal{A}$ is $\mathcal{S}(\mathcal{A}) = \{\rho \in S_{\mathcal{A}^*} : \rho(I) = 1\}$

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Let X be a Banach space $x \in S_X$. The state space of X at the point x is defined by $S_x = \{x^* \in S_{X^*} : x^*(x) = 1\}$

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A few facts on state spaces in Banach spaces

- S_x is non-empty for every $x \in S_X$ by Hahn-Banach extension theorem.
- S_x is a convex subset of B_{X^*}. In fact, it is a face of B_{X^*} i.e if any line segment in B_X^* intersect with S_x, then entire segment should be in S_x.
- S_x is singleton $\iff$ the norm of X is Gautuax differentiable at x.

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Canonical topologies on X^*.

- **Norm-topology** - the topology on X^* induced from the norm in dual space.
- **Weak-topology** - the smallest topology on X^* such that all $x^{**} \in X^{**}$ are continuous.
- **Weak*-topology** - the smallest topology on X^* such that all $J_x \in X^{**}$ are continuous, where $J_x(f) = f(x)\ \forall\ f \in X^*$.

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Compactness of S_x

- S_x is norm, weak, weak* closed.
- S_x is weak* compact.
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Smooth Banach space

A Banach space X is said to be smooth if S_x is singleton for every $x \in S_X$. In this case, S_x is compact with respect to above all three topologies.

Motivations

Motivation 1[2]

Let $\mathcal{A}$ be a C^*-algebra with unit I. If the state space of $\mathcal{A}$ at the unit I is weakly compact, then $\mathcal{A}$ is a finite-dimensional space.

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The norm of X is said to be strongly subdifferentiable (SSD) at $x \in S(X)$ if $\lim_{t\to 0^+} \frac{\|x+ty\|-1}{t}$ exists uniformly for $y \in B_X$.

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Motivation 2[3]

Suppose the norm on X is SSD and all state spaces are norm compact, then strong subdifferentiability passes through the quotients X/Y for a proximinal subspace $Y \subseteq X$.

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Examples in sequence spaces

Example 1

Let $(\alpha_n) \in S(c_0)$. Then,

- (α_n) is smooth in $c_0 \iff (\alpha_n) = ae_N$ for some $N \in \mathbb{N}$ and $|a| = 1$.
- $S_{(\alpha_n)}$ is norm compact for all $(\alpha_n) \in S(c_0)$.
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Example 2

Let $(\alpha_n) \in S(\ell^1)$. Then,

- (α_n) is smooth in $\ell^1 \iff \alpha_n \neq 0$ for all $n \in \mathbb{N}$.
- $S_{(\alpha_n)}$ is norm compact $\iff \Lambda = \{n \in \mathbb{N} : \alpha_n = 0\}$ is finite.
- $S_{(\alpha_n)}$ is weakly compact $\iff \Lambda = \{n \in \mathbb{N} : \alpha_n = 0\}$ is finite.

Examples in function spaces

Example 3

Let μ be a finite measure & $f \in S(L_1(\mu))$. Then,

- f is smooth in $L_1(\mu) \iff f \neq 0$ a. e.
- S_f is norm compact $\iff f$ is a smooth point in $L_1(\mu)$.
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Let $f \in S(C(\Omega))$ & $\Gamma = \{t \in \Omega : |f(t)| = \|f\|\}$. Then,

- f is smooth in $C(\Omega) \iff \Gamma$ is singleton.
- S_f is norm compact $\iff \Gamma$ is finite.
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On spaces of sequences of vectors

Notation:

- $\ell^1(X) := \{(x_n)_1^\infty \subset X : \sum_{n\in\mathbb{N}} \|x_n\| < \infty\}$ equipped with the norm $\|(x_n)_1^\infty\| = \sum_{n\in\mathbb{N}} \|x_n\|$.

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Theorem (D., Dwivedi)

Let $(x_n)_1^\infty \in S(\ell^1(X))$, $x_n \neq 0$ for each $n \in \mathbb{N}$. Then

- (x_n) is a smooth point in $\ell^1(X) \iff \frac{x_n}{\|x_n\|}$ is a smooth point in X for each $n \in \mathbb{N}$.
- $S_{(x_n)}$ is norm compact in $\ell^\infty(X^*) \iff S_{\frac{x_n}{\|x_n\|}}$ is norm compact in X^* for each $n \in \mathbb{N}$ & $\operatorname{diam}(S_{\frac{x_n}{\|x_n\|}}) \to 0$ as $n \to \infty$.
- $S_{(x_n)}$ is weakly compact in $\ell^\infty(X^*) \iff S_{\frac{x_n}{\|x_n\|}}$ is weakly compact in X^* for each $n \in \mathbb{N}$ & $\operatorname{diam}(S_{\frac{x_n}{\|x_n\|}}) \to 0$ as $n \to \infty$.

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Corollary (D., Dwivedi)

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Let X be a reflexive space & let $(x_n) \in S(\ell^1(X))$. Then, $S_{(x_n)}$ is weakly compact in $\ell^\infty(X^*) \iff$ $\Lambda = \{n \in \mathbb{N} : x_n = 0\}$ is finite & $\operatorname{diam}\left(S_{\frac{x_n}{\|x_n\|}}\right) \to 0$ as $n \to \infty$.

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Notation:

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Let $(x_n) \in S(c_0(X))$ & $\Lambda = \{n \in \mathbb{N} : \|x_n\| = 1\}$. Then,

- (x_n) is a smooth point in $c_0(X) \iff (x_n) = e_N x_N$ for some $N \in \Lambda$ & x_N is a smooth point in X.
- $S_{(x_n)}$ is norm compact in $\ell^1(X^*) \iff S_{x_n}$ is norm compact in X^* for each $n \in \Lambda$.
- $S_{(x_n)}$ is weakly compact in $\ell^1(X^*) \iff S_{x_n}$ is weakly compact in X^* for each $n \in \Lambda$.

On spaces of vector-valued functions

Notation:

- Let $L^1([0,1],X) := \{f : [0,1] \longrightarrow X,\ \mu\text{-measurable function: } \int_{[0,1]} \|f(t)\| d\mu(t) < \infty\}$ equipped with the norm $\|f\| = \int_{[0,1]} \|f(t)\| d\mu(t)$. In this case, we consider μ to be the Lebesgue measure.

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Let $f \in S(L^1([0,1],X))$, $f \neq 0$ a.e., and let X^* be separable. Then,

- f is a smooth point $\Longleftrightarrow$ $\frac{f(t)}{\|f(t)\|}$ is smooth in X a.e. [4]
- S_f is norm compact $\Longleftrightarrow$ f is a smooth point.
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Remark

The result for smooth point first shown by Deeb and Khalil [5]. They mainly used Kuratowski-Ryll-Nardzewski selection theorem. In this work, we have given an alternative proof using Von Neumann's selection theorem.

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Let $f \in L^1(\mu, X)$. We define $|f| : \Omega \to \mathbb{R}$ by $|f|(\omega) = \|f(\omega)\|$. It is easy to see that $|f| \in L^1(\mu)$. One can also check that $f \in S(L^1(\mu, X)) \iff |f| \in S(L^1(\mu))$.

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Let X be a Banach space and let μ be the Lebesgue measure on $[0,1]$. Suppose $f \in S(L^1([0,1], X))$, then

- f is a smooth point $\implies |f|$ is smooth in $L^1(\mu)$.
- S_f is norm compact $\implies |f|$ is smooth in $L^1(\mu)$.
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Lemma (D., Dwivedi)

Let $A \subseteq C(\Omega)$ be a closed subspace such that $1 \in A$. Then every extreme point of A_1^* is the restriction to A of an extreme point of $C(\Omega)_1^*$. Moreover, we have $\overline{\text{ext}}^{w^*}(A_1^*) \subseteq S(A^*)$.

Problems

Notations:

- For $1 \leq p < \infty$, $\ell^p(X) := \{(x_n)_1^\infty \subset X : \sum_{n\in\mathbb{N}} \|x_n\|^p < \infty\}$ equipped with the norm $\|(x_n)_1^\infty\| = (\sum_{n\in\mathbb{N}} \|x_n\|^p)^{\frac{1}{p}}$.
- Let (Ω, Σ, μ) be a measure space and $1 \leq p < \infty$. We denote $L^p([0,1], X) := \{f : [0,1] \longrightarrow X,\ \mu\text{-measurable function: } \int_{[0,1]} \|f(t)\|^p d\mu(t) < \infty\}$ equipped with the norm $\|f\| = (\int_{[0,1]} \|f(t)\|^p d\mu(t))^{\frac{1}{p}}$.

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Can we characterize the state spaces of $\ell^p(X)$ that are weak (norm) compact, for $1 < p < \infty$?

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Problem 2

- Let μ be a finite (non-Lebesgue) measure. We ask: what can be said about the weak or norm compactness of the state space of $L^1(\Omega, X)$?
- Under what conditions is the state space of $L^p(\Omega, X)$ norm (weak) compact for $1 < p < \infty$?

References

S. Daptari, S. Dwivedi, *A study on state spaces in classical Banach spaces*, Colloq. Math. 179 (1), 87–105 (2025).

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Thank You!