Insights into Numerical Range and Numerical Radius Inequalities

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Outline

Definition 1

(**Numerical range**) Let T be a bounded linear operator on the complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. The numerical range of an operator T is the subset of $\mathbb{C}$, denoted by $W(T)$, and is defined as

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

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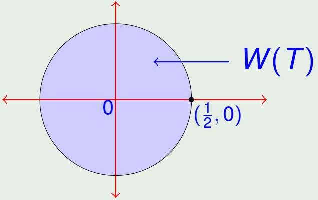

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- The unit sphere of a finite-dimensional space is a compact set and the quadratic map $x \mapsto \langle Tx, x\rangle$ is continuous.

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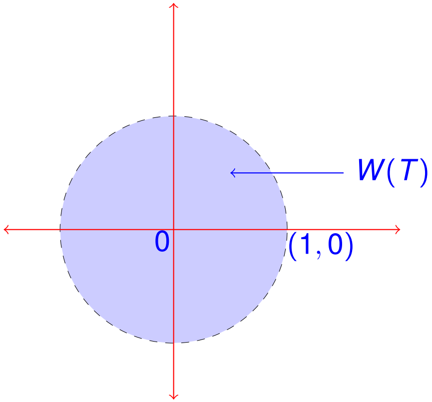
- Here $W(T) = \{z : |z| < 1\}$, which is not compact.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(d) $W(T + S) \subseteq W(T) + W(S)$.

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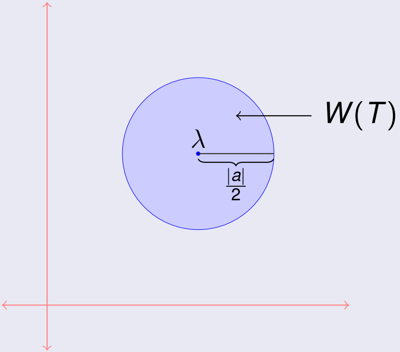

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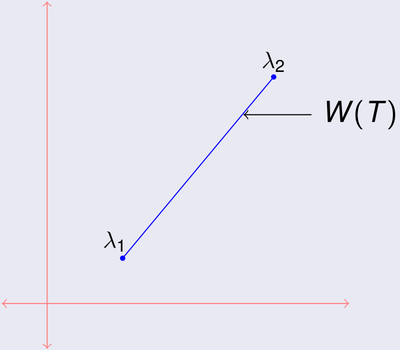

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$$T - \frac{\lambda_1 + \lambda_2}{2} I = \begin{pmatrix} \frac{\lambda_1 - \lambda_2}{2} & a \\ 0 & \frac{\lambda_2 - \lambda_1}{2} \end{pmatrix} = \begin{pmatrix} re^{i\theta} & a \\ 0 & -re^{i\theta} \end{pmatrix},$$

where $\frac{\lambda_1 - \lambda_2}{2} = re^{i\theta}$, $0 \leq \theta \leq 2\pi$.

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- Consider $A = e^{-i\theta}\left[T - \frac{\lambda_1+\lambda_2}{2} I\right]$.
- Let $z = (f, g)$ and $|f|^2 + |g|^2 = 1$, $f = e^{i\alpha}\cos\theta, g = e^{i\beta}\sin\theta$, $\theta \in [0, \frac{\pi}{2}], \alpha, \beta \in [0, 2\pi]$.

$\square$

Proof.

- Then $\langle Az, z\rangle = r(\cos^2\theta - \sin^2\theta) + be^{i(\beta-\alpha)}\sin\theta\cos\theta = x + iy$, where $x = r\cos 2\theta + \frac{|b|}{2}\sin 2\theta\cos(\beta-\alpha+\gamma)$, $y = \frac{|b|}{2}\sin 2\theta\sin(\beta-\alpha+\gamma)$, $\gamma = \arg b$.

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- Differentiating with respect to ϕ, we get $(x - r\cos\phi)r = \frac{|b|^2}{4}\cos\phi$.
- Eliminating ϕ between the last two equations, we get $\frac{x^2}{r^2+(|b|^2/4)} + \frac{y^2}{(|b|^2/4)} = 1$.

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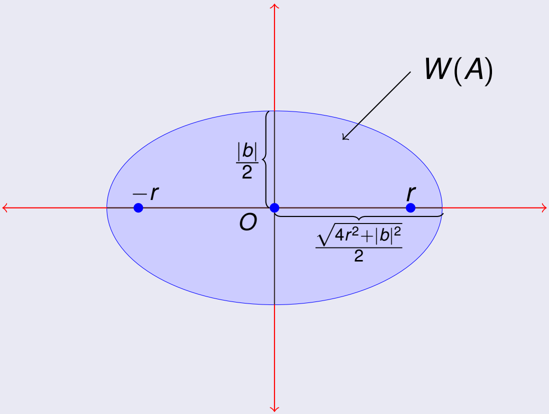

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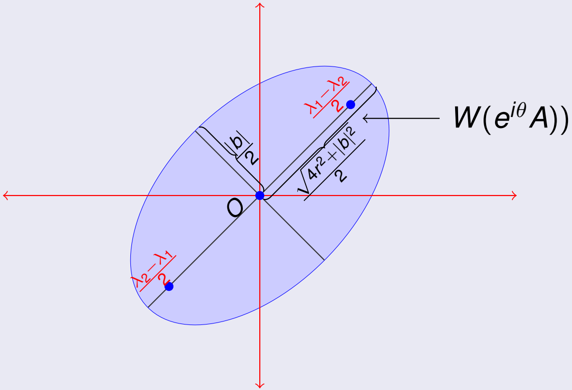

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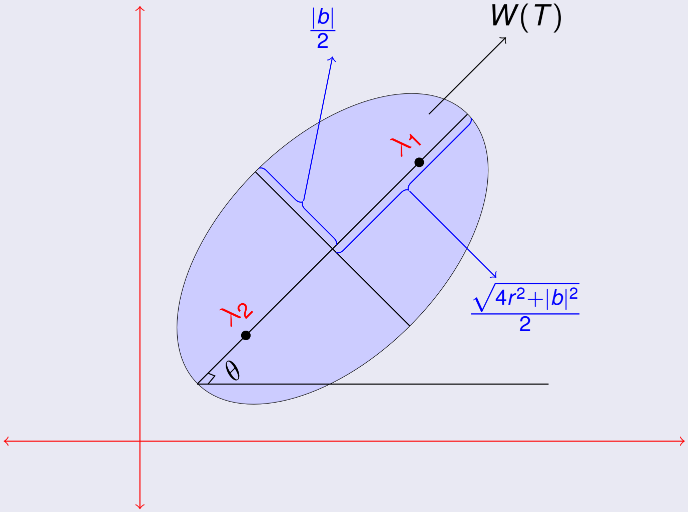

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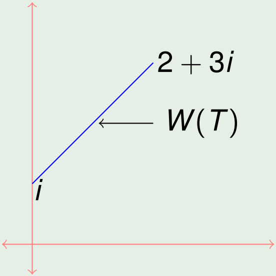

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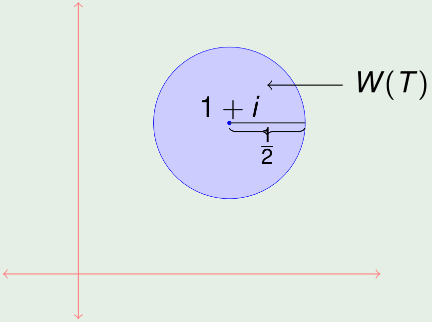

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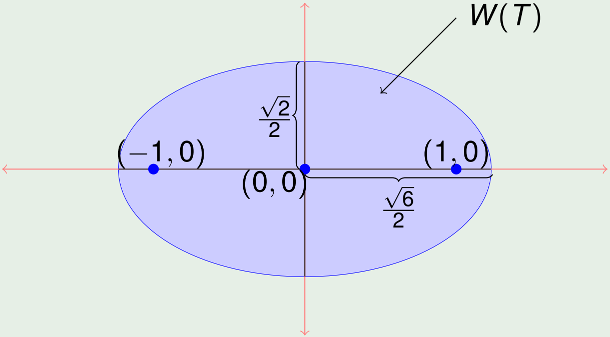

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Then $W(T)$ is the convex hull of the points $2i$, $1+i$ and $-1-i$.

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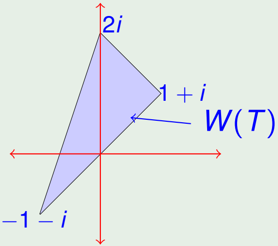
Example 10

Let T be the linear operator on $\mathbb{C}^3$ defined by the matrix

$$T = \begin{pmatrix} 2i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & -1-i \end{pmatrix}.$$

Then $W(T)$ is the convex hull of the points $2i$, $1+i$ and $-1-i$.



Theorem 11

(Toeplitz-Hausdorff Theorem) *The numerical range $W(T)$ of a bounded linear operator T defined on a Hilbert space $\mathcal{H}$ is always convex.*

Proof.

- Let $\xi, \eta \in W(T)$, where $\xi = \langle Tf, f\rangle, \|f\| = 1$, $\eta = \langle Tg, g\rangle, \|g\| = 1$.
- For $t = 0$ or $t = 1$, $(1-t)\xi + t\eta \in W(T)$.
- Consider $t \in (0, 1)$.
- For each $\lambda \in \mathbb{C}$, $\frac{f+\lambda g}{\|f+\lambda g\|} \in S_{\mathcal{H}}$.
- $\{(1-t)\xi + t\eta\} = \langle T\frac{(f+\lambda g)}{\|f+\lambda g\|}, \frac{(f+\lambda g)}{\|f+\lambda g\|}\rangle$.

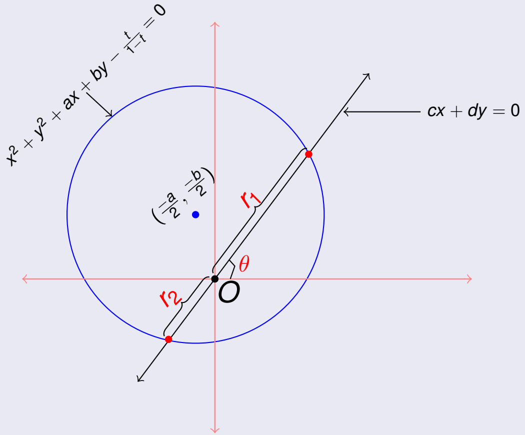
$\Rightarrow$ $|\lambda|^2 + \frac{1}{(1-t)(\xi-\eta)}[2\{(1-t)\xi + t\eta\}Re(\overline{\lambda}\langle f, g\rangle) - 2Re(\overline{\lambda}\langle Tf, g\rangle)] - \frac{t}{1-t} = 0$.

$\square$

Proof.

$\Rightarrow \mid \lambda \mid^2 + C\lambda + D\overline{\lambda} - \frac{t}{1-t} = 0$, where C and D are complex constants independent of λ.

- Considering $\lambda = x + iy$, $x^2 + y^2 + ax + by - \frac{t}{1-t} = 0$ and $cx + dy = 0$.



Proof.

$\Rightarrow$ There exists two $\lambda \in \mathbb{C}$ such that

$$(1-t)\xi + t\eta = \langle Th, h\rangle, \text{ where } h = \frac{f+\lambda g}{\|f+\lambda g\|}.$$

$\Rightarrow$ The line segment joining ξ and η is in $W(T)$.

$\square$

Proof.

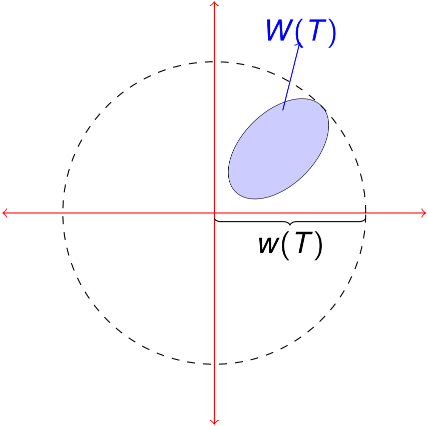
Another proof.

- Let $\xi, \eta \in W(T)$, where $\xi = \langle Tf, f\rangle, \|f\| = 1$, $\eta = \langle Tg, g\rangle, \|g\| = 1$.
- Let V be the subspace spanned by f and g, and P be the orthogonal projection of $\mathcal{H}$ on V so that $Pf = f$ and $Pg = g$.
- Consider $PTP : V \longrightarrow V$ so that $\langle PTPf, f\rangle = \langle Tf, f\rangle$ and $\langle PTPg, g\rangle = \langle Tg, g\rangle$.
- $W(PTP)$ is an ellipse and $W(PTP)$ contains the line segment joining α and β, and $W(PTP)$ is contained in $W(T)$.

We next use numerical radius to study the numerical range of some special class of operators.

Definition 12

The numerical radius $w(T)$ of an operator T on $\mathcal{H}$ is defined as $w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$.

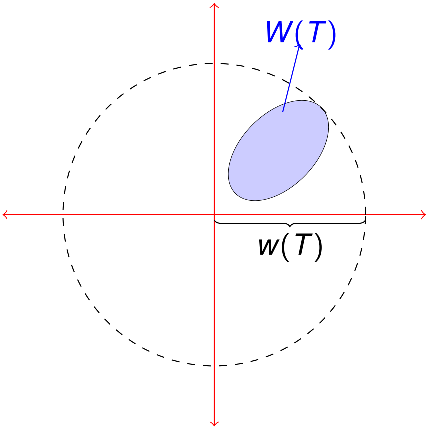


We next use numerical radius to study the numerical range of some special class of operators.

Definition 13

The numerical radius $w(T)$ of an operator T on $\mathcal{H}$ is defined as

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$



- Numerical radius is the radius of the smallest circle centred at origin containing the numerical range $W(T)$.
- The spectral radius $r(T)$ of an operator is defined as

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

- $r(T) \leq w(T) \leq \|T\|$.

Theorem 14

(i) T is selfadjoint iff $W(T)$ is real.
(ii) If T is selfadjoint then $\overline{W(T)} = [m, M]$, where
$m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$ *and* $M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$.
(iii) If T is selfadjoint then $\|T\| = w(T) = \max\{|m|, |M|\}$, where
$m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$ *and* $M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$.

Proof.

(i)

- If T is selfadjoint, then $\forall x \in \mathcal{H}$, $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$. Hence $W(T)$ is real.
- if $\langle Tx, x\rangle$ is real for all $x \in \mathcal{H}$, we have $\langle Tx, x\rangle - \langle x, Tx\rangle = 0 = \langle (T - T^*)\, x, x\rangle$. Thus $T - T^* = 0$ or $T = T^*$.

(ii)

- As the numerical range is a bounded subset of the reals so the result follows trivially.

Proof.

(iii)

- $w(T) = \sup\{|m|, |M|\}$.
- For any real $\lambda \neq 0$, $4\|Tx\|^2 = \langle T\left(\lambda x + \lambda^{-1}Tx\right), \lambda x + \lambda^{-1}Tx\rangle - \langle T\left(\lambda x - \lambda^{-1}Tx\right), \lambda x - \lambda^{-1}Tx\rangle$
- $4\|Tx\|^2 \leq w(T)\left[\left\|\lambda x + \lambda^{-1}Tx\right\|^2 + \left\|\lambda x - \lambda^{-1}Tx\right\|^2\right] = 2w(T)\left(\lambda^2\|x\|^2 + \lambda^{-2}\|Tx\|^2\right)$.
- If $x \neq 0$, then taking $\lambda^2 = \frac{\|Tx\|}{\|x\|}$ we get $\|Tx\| \leq w(T)\|x\|$.
- Thus for all $x \in \mathcal{H}$, we get $\|Tx\| \leq w(T)\|x\|$.
- $\|T\| \leq w(T)$
- $\|T\| = w(T) = \max\{|m|, |M|\}$.

□

Theorem 15

If $w(T) = \|T\|$, *then* $r(T) = \|T\|$.

Proof.

- Let $w(T) = \|T\| = 1$.
- Then there is a sequence of unit vectors $\{x_n\}$ such that $\langle Tx_n, x_n\rangle \to \lambda \in W(T), |\lambda| = 1$.
- From$|\langle Tx_n, x_n\rangle| \leq \|Tx_n\| \leq 1$, we have $\|Tx_n\| \to 1$.
- $\|(T - \lambda I)x_n\|^2 = \|Tx_n\|^2 - \langle Tx_n, \lambda x_n\rangle - \langle \lambda x_n, Tx_n\rangle + \|x_n\|^2 \to 0$.
- $\lambda \in \sigma_{\mathrm{app}}(T)$ and $r(T) = 1$.

□

Theorem 16

If $W(T)$ is a line segment then T is normal.

Proof.

- Let T be a bounded linear operator such that $W(T)$ is a line segment.
- Suppose that λ is a point on the line segment with inclination θ.
- Then $W(e^{-i\theta}(T - \lambda I))$ is contained in the real axis.
- Thus $e^{-i\theta}(T - \lambda I)$ is selfadjoint and so $[e^{-i\theta}(T - \lambda I)]^* = e^{-i\theta}(T - \lambda I)$.
- $T^* = \overline{\lambda} I + e^{-2i\theta}(T - \lambda I)$.
- $TT^* = \overline{\lambda} T + e^{-2i\theta} T(T - \lambda I) = T^* T$.
- T is normal.

$\square$

The class of normal operators have some nice properties related to numerical range and numerical radius.

Theorem 17

Let T be a normal operator. Then

(i) $w(T) = \|T\|$.

(ii) $w(T) = r(T)$.

(iii) $\overline{W(T)} = \mathrm{co}(\sigma(T))$.

Theorem 18

In finite dimensional space, the numerical range of a normal operator is a polygon.

Proof.

- Let T be normal operator acts on a finite dimensional space $\mathcal{H}$.
- Thus $W(T)$ is closed.
- It follows from Theorem17 (iii) that $W(T) = co(\sigma(T))$.
- As $\mathcal{H}$ is finite dimensional, $\sigma(T)$ contains atmost finitely many elements.
- So $W(T)$ is a polygon.

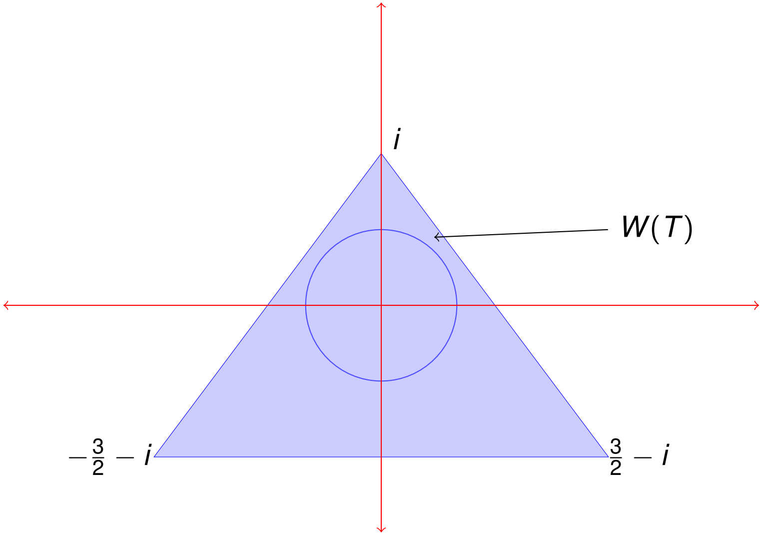
$\square$

Remark 1

(i) If T is unitary then $W(T)$ is a polygon inscribed in the unit circle.
(ii) There exist operator T which is not normal but $W(T)$ is a polygon.

Example 19

- Consider $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and
  $$B = \begin{pmatrix} i & 0 & 0 \\ 0 & \frac{3}{2} - i & 0 \\ 0 & 0 & -\frac{3}{2} - i \end{pmatrix}.$$
- Then $W(T) = co(W(A) \cup W(B)) = co\{i, \frac{3}{2} - i, -\frac{3}{2} - i\}$.
- This is a polygon but T is not normal.


i
$W(T)$
$-\frac{3}{2} - i$
$\frac{3}{2} - i$

Inequality 20

(Classical Numerical radius inequalities)

- *Let $T \in \mathcal{B}(\mathcal{H})$, then*

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1}$$

- *Moreover, if T is normal, then $w(T) = \|T\| = r(T)$. If $T^2 = 0$, then $w(T) = \frac{1}{2}\|T\|$.*

First, we recall the following.

Definition 21

A function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is said to be **convex** if, for all $x_1, x_2 \in I$ and for all $\lambda \in [0, 1]$, the following inequality holds:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

Inequalities using Cartesian decomposition of operator

First, we recall the following.

Definition 21

A function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is said to be **convex** if, for all $x_1, x_2 \in I$ and for all $\lambda \in [0, 1]$, the following inequality holds:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

In [14], Kittaneh gives a refined upper and lower bound of (1) using Cartesian decomposition and the convexity of functions.

Inequality 22

(A refinement of Classical Numerical radius inequalities)
If $T \in \mathcal{B}(\mathcal{H})$, then

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{2}$$

Proof.

- Suppose that $T = B + iC$ is the Cartesian decomposition of the operator A.
- Let $x \in \mathcal{H}$ be arbitrary.
- It follows from the convexity of the function $f(t) = t^2$ that

$$\begin{aligned} |\langle Tx, x\rangle|^2 = \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2 &\geq \frac{1}{2}(|\langle Bx, x\rangle| + |\langle Cx, x\rangle|)^2 \\ &\geq \frac{1}{2}|\langle (B \pm C)x, x\rangle|^2. \end{aligned}$$

$\square$

Proof.

- So

$$
\begin{aligned}
w^2(T) &= \sup\left\{|\langle Tx, x\rangle|^2 : x \in \mathcal{H}, \|x\| = 1\right\} \\
&\geq \frac{1}{2}\sup\left\{|\langle (B \pm C)x, x\rangle|^2 : x \in \mathcal{H}, \|x\| = 1\right\} \\
&= \frac{1}{2}w(B \pm C)^2 = \frac{1}{2}\|B \pm C\|^2 = \frac{1}{2}\left\|(B \pm C)^2\right\|
\end{aligned}
$$

- Hence

$$
\begin{aligned}
2w^2(T) &\geq \frac{1}{2}\left\|(B + C)^2\right\| + \frac{1}{2}\left\|(B - C)^2\right\| \\
&\geq \frac{1}{2}\left\|(B + C)^2 + (B - C)^2\right\| \\
&= \left\|B^2 + C^2\right\| = \frac{1}{2}\left\|T^*T + TT^*\right\|.
\end{aligned}
$$

Proof.

- For every unit vector $x \in \mathcal{H}$, it follows from the Cauchy-Schwarz inequality that

$$
\begin{aligned}
|\langle Tx, x\rangle|^2 = \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2 &\leq \|Bx\|^2 + \|Cx\|^2 \\
&= \left\langle B^2 x, x \right\rangle + \left\langle C^2 x, x \right\rangle \\
&= \left\langle \left(B^2 + C^2\right) x, x \right\rangle .
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
w^2(A) &= \sup \left\{ |\langle Tx, x\rangle|^2 : x \in \mathcal{H}, \|x\| = 1 \right\} \\
&\leq \sup \left\{ \left\langle \left(B^2 + C^2\right) x, x \right\rangle : x \in \mathcal{H}, \|x\| = 1 \right\} \\
&= \left\| B^2 + C^2 \right\| = \frac{1}{2} \|T^*T + TT^*\|
\end{aligned}
$$

Now, we present the following nontrivial improvement of the first inequality in (1), that is, $\frac{1}{2}\|T\| \leq w(T)$ and the second inequality in (2). Note an elementary identity that $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ for $a, b \in \mathbb{R}$.

Inequality 23

Let $T \in \mathcal{B}(\mathcal{H})$, then

$$w(T) \geq \frac{\|T\|}{2} + \frac{\left| \ \|\mathrm{Re}(T)\| - \|\mathrm{Im}(T)\| \ \right|}{2}. \tag{3}$$

Proof.

- Let $x \in \mathcal{H}$ with $\|x\| = 1$.
- Then from the Cartesian decomposition of T we have, $|\langle Tx, x\rangle|^2 = |\langle \mathrm{Re}(T)x, x\rangle|^2 + |\langle \mathrm{Im}(T)x, x\rangle|^2$.
- This implies $w(T) \geq \|\mathrm{Re}(T)\|$ and $w(T) \geq \|\mathrm{Im}(T)\|$.

$\square$

Proof.

- Thus,

$$\begin{aligned} w(T) &\geq \max\{\|\mathrm{Re}(T)\|, \|\mathrm{Im}(T)\|\} \\ &= \frac{\|\mathrm{Re}(T)\| + \|\mathrm{Im}(T)\|}{2} + \frac{\left|\ \|\mathrm{Re}(T)\| - \|\mathrm{Im}(T)\|\ \right|}{2} \\ &\geq \frac{\|\mathrm{Re}(T) + \mathrm{i}\mathrm{Im}(T)\|}{2} + \frac{\left|\ \|\mathrm{Re}(T)\| - \|\mathrm{Im}(T)\|\ \right|}{2} \\ &= \frac{\|T\|}{2} + \frac{\left|\ \|\mathrm{Re}(T)\| - \|\mathrm{Im}(T)\|\ \right|}{2}. \end{aligned}$$

□

Corollary 24

Let $T \in \mathcal{B}(\mathcal{H})$. If $w(T) = \frac{\|T\|}{2}$, then $\|\mathrm{Re}(T)\| = \|\mathrm{Im}(T)\| = \frac{\|T\|}{2}$.

Proof.

- From Inequality (3), we have $w(T) \geq \frac{\|T\|}{2} + \frac{\big|\,\|\mathrm{Re}(T)\| - \|\mathrm{Im}(T)\|\,\big|}{2} \geq \frac{\|T\|}{2}$.
- This implies that if $w(T) = \frac{\|T\|}{2}$, then $\|\mathrm{Re}(T)\| = \|\mathrm{Im}(T)\|$.
- Also $\|\mathrm{Re}(T)\| \leq w(T) = \frac{\|T\|}{2} = \frac{\|\mathrm{Re}(T) + \mathrm{i}\mathrm{Im}(T)\|}{2} \leq \frac{\|\mathrm{Re}(T)\| + \|\mathrm{Im}(T)\|}{2} = \|\mathrm{Re}(T)\|$
- So $\|\mathrm{Re}(T)\| = \|\mathrm{Im}(T)\| = \frac{\|T\|}{2}$.

$\square$

We next derive an improvement of the first inequality in (2).

Theorem 25

Let $T \in \mathcal{B}(\mathcal{H})$, then

$$w^2(T) \geq \frac{1}{4}\|T^*T + TT^*\| + \frac{1}{2}\left| \|\mathrm{Re}(T)\|^2 - \|\mathrm{Im}(T)\|^2 \right|. \qquad (4)$$

Proof.

- Let $x \in \mathcal{H}$ with $\|x\| = 1$.
- Then from the Cartesian decomposition of A we get, $|\langle Tx, x\rangle|^2 = |\langle \mathrm{Re}(T)x, x\rangle|^2 + |\langle \mathrm{Im}(T)x, x\rangle|^2$.
- This implies $w(T) \geq \|\mathrm{Re}(T)\|$ and $w(T) \geq \|\mathrm{Im}(T)\|$.

□

Proof.

- So

$$
\begin{aligned}
w^2(T) &\geq \max\left\{\|\mathrm{Re}(T)\|^2, \|\mathrm{Im}(T)\|^2\right\} \\
&= \frac{\|\mathrm{Re}(T)\|^2 + \|\mathrm{Im}(T)\|^2}{2} + \frac{\left|\|\mathrm{Re}(T)\|^2 - \|\mathrm{Im}(T)\|^2\right|}{2} \\
&= \frac{\|(\mathrm{Re}(T))^2\| + \|(\mathrm{Im}(T))^2\|}{2} + \frac{\left|\|\mathrm{Re}(T)\|^2 - \|\mathrm{Im}(T)\|^2\right|}{2} \\
&\geq \frac{\|(\mathrm{Re}(T))^2 + (\mathrm{Im}(T))^2\|}{2} + \frac{\left|\|\mathrm{Re}(T)\|^2 - \|\mathrm{Im}(T)\|^2\right|}{2} \\
&= \frac{1}{4}\|T^*T + TT^*\| + \frac{\left|\|\mathrm{Re}(T)\|^2 - \|\mathrm{Im}(T)\|^2\right|}{2}.
\end{aligned}
$$

$\square$

Inequalities using Cauchy-Schwarz inequality and it's generalizations

For any $x, y \in \mathcal{H}$, the Cauchy-Schwarz inequality asserts that

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

This classical result was later refined in an elegant form by Buzano [8], as stated below.

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This classical result was later refined in an elegant form by Buzano [8], as stated below.

Inequality 26

Let $x, y, e \in \mathcal{H}$ *with* $\|e\| = 1$. *Then*

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}\left(\|x\|\,\|y\| + |\langle x, y\rangle|\right). \tag{5}$$

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Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}\left(\|x\| \|y\| + |\langle x, y\rangle|\right). \tag{5}$$

Abu-Omar and Kittaneh in [1] proved the following inequality

$$w^2(T) \leq \frac{1}{2} w(T^2) + \frac{1}{4}\left\| |T|^2 + |T^*|^2 \right\|. \tag{6}$$

By using Buzano's inequality we extend the above inequality. First we need the following inequality.

Inequality 27

(Hölder–McCarthy inequality)
Let $T \in \mathcal{B}(\mathcal{H})$ be positive and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle, \quad r \geq 1. \tag{7}$$

The inequality is reversed when $0 \leq r \leq 1$.

Inequality 28

Let $T \in \mathcal{B}(\mathcal{H})$ and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then

$$|\langle Tx, x\rangle|^{2r} \leq \frac{1}{2}|\langle T^2x, x\rangle|^r + \frac{1}{4}\left\langle (|T|^{2r} + |T^*|^{2r})x, x \right\rangle \tag{8}$$

for all $r \geq 1$.

Proof.

- By considering $a = Tx$, $b = T^*x$ and $e = x$ in (5), we get $|\langle Tx, x\rangle|^2 \leq \frac{1}{2}\left(|\langle T^2x, x\rangle| + \|Tx\| \ \|T^*x\|\right)$.
- From the convexity of the real function $f(t) = t^r$ $(r \geq 1)$, we derive that

$$\begin{aligned} & |\langle Tx, x\rangle|^{2r} \\ \leq \ & \frac{1}{2}\left(|\langle T^2x, x\rangle|^r + \|Tx\|^r \ \|T^*x\|^r\right) \end{aligned}$$

$\square$

Proof.

- Then

$$
\begin{aligned}
& |\langle Tx, x\rangle|^{2r} \\
\leq\ & \frac{1}{2}\left(|\langle T^2x, x\rangle|^r + \frac{1}{2}(\|Tx\|^{2r} + \|T^*x\|^{2r})\right) \\
=\ & \frac{1}{2}\left(|\langle T^2x, x\rangle|^r + \frac{1}{2}(\langle |T|^2x, x\rangle^r + \langle |T^*|^2x, x\rangle^r)\right) \\
\leq\ & \frac{1}{2}\left(|\langle T^2x, x\rangle|^r + \frac{1}{2}(\langle |T|^{2r}x, x\rangle + \langle |T^*|^{2r}x, x\rangle)\right) \quad \text{(by(7))} \\
=\ & \frac{1}{2}|\langle T^2x, x\rangle|^r + \frac{1}{4}\left\langle (|T|^{2r} + |T^*|^{2r})x, x\right\rangle .
\end{aligned}
$$

□

Inequalities using Cauchy-Schwarz inequality and it's generalizations

Inequality 29

Suppose that $T \in \mathcal{B}(\mathcal{H})$. Then

$$w^{2r}(T) \leq \frac{\alpha}{2} w^r(T^2) + \left\| \frac{\alpha}{4} |T|^{2r} + \left(1 - \frac{3}{4}\alpha\right) |T^*|^{2r} \right\| \qquad (9)$$

for all $r \geq 1$ and for all $\alpha \in [0, 1]$.

Proof.

- Let $x \in \mathcal{H}$ be a unit vector.
- It follows from the Cauchy–Schwarz inequality that $|\langle Tx, x\rangle| = \alpha|\langle Tx, x\rangle| + (1-\alpha)|\langle Tx, x\rangle| \leq \alpha|\langle Tx, x\rangle| + (1-\alpha)\|T^*x\|$.

□

Proof.

- From the convexity of $f(t) = t^{2r}$ $(r \geq 1)$, we infer that

$$
\begin{aligned}
& |\langle Tx, x\rangle|^{2r} \\
\leq\ & \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)\|T^*x\|^{2r} \\
\leq\ & \alpha|\langle Tx, x\rangle|^{2r} + (1-\alpha)\langle |T^*|^{2r}x, x\rangle \quad \text{(by (7))} \\
\leq\ & \frac{\alpha}{2}|\langle T^2x, x\rangle|^r + \frac{\alpha}{4}\left\langle (|T|^{2r} + |T^*|^{2r})x, x\right\rangle + (1-\alpha)\langle |T^*|^{2r}x, x\rangle \\
& \qquad \text{(by 8)} \\
=\ & \frac{\alpha}{2}|\langle T^2x, x\rangle|^r + \left\langle \left\{\frac{\alpha}{4}\left(|T|^{2r} + |T^*|^{2r}\right) + (1-\alpha)|T^*|^{2r}\right\} x, x\right\rangle \\
=\ & \frac{\alpha}{2}|\langle T^2x, x\rangle|^r + \left\langle \left\{\frac{\alpha}{4}|T|^{2r} + \left(1 - \frac{3}{4}\alpha\right)|T^*|^{2r}\right\} x, x\right\rangle \\
\leq\ & \frac{\alpha}{2}w^r(T^2) + \left\|\frac{\alpha}{4}|T|^{2r} + \left(1 - \frac{3}{4}\alpha\right)|T^*|^{2r}\right\|.
\end{aligned}
$$

Inequalities using Cauchy-Schwarz inequality and it's generalizations

Proof.

- Taking the supremum over all unit vectors $x \in \mathcal{H}$ we obtain inequality (9).

$\square$

Remark 2

It is remarkable that with

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we observe that the inequality 9 for $r = 1$ generalize and improve inequality (6).

Inequality 30

(Mixed Cauchy-Schwarz inequality)
Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$|\langle Tx, x\rangle| \leq \langle |T|x, x\rangle^{1/2} \ \langle |T^*|x, x\rangle^{1/2}.$$

for all $x \in \mathcal{H}$,

Kato [13] generalized the mixed Cauchy-Schwarz inequality to a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$. He proved that

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2\alpha} x, x \right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y \right\rangle \tag{10}$$

for all $x, y \in \mathcal{H}$ and all $\alpha \in [0, 1]$.

Next, Kittaneh gives an extension of Kato's generalized mixed Cauchy-Schwarz inequality. It states as follows.

Inequality 31

[16, Th. 1] Let $T \in \mathcal{B}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. If f and g are two non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t, \quad t \geq 0$, then

$$|\langle Tx, y\rangle|^2 \leq \left\langle f^2(|T|)x, x \right\rangle \left\langle g^2(|T^*|)y, y \right\rangle . \tag{11}$$

Inequalities using Cauchy-Schwarz inequality and it's generalizations

By using mixed Cauchy-Schwarz inequality, we obtain the following refined numerical radius inequality of (2).

Inequality 32

For an operator $T \in \mathcal{B}(\mathcal{H})$, it holds that

$$w^2(T) \leq \frac{1}{4}\left\| |T|^2 + |T^*|^2 \right\| + \frac{1}{2} w\left(|T||T^*|\right). \tag{12}$$

Proof.

- Let $x \in \mathcal{H}$ be a unit vector.
- It follows from the inequality (10) that $|\langle Tx, x\rangle|^2 \leq \langle |T|x, x\rangle \langle |T^*|x, x\rangle$.
- An application of the Buzano inequality gives that

$$\langle |T^*|x, x\rangle \langle x, |T|x\rangle \quad \leq \quad \frac{1}{2} \| |T|x\| \, \| |T^*|x\| + \frac{1}{2} |\langle |T^*|x, |T|x\rangle| \, .$$

Proof.

- Hence

$$
\begin{aligned}
& |\langle Tx, x\rangle|^2 \\
\leq\ & \frac{1}{4}\left(\||T|x\|^2 + \||T^*|x\|^2\right) + \frac{1}{2}|\langle |T||T^*|x, x\rangle| \\
=\ & \frac{1}{4}\left(\left\langle |T|^2 x, x\right\rangle + \left\langle |T^*|^2 x, x\right\rangle\right) + \frac{1}{2}|\langle |T||T^*|x, x\rangle| \\
=\ & \frac{1}{4}\left\langle \left(|T|^2 + |T^*|^2\right) x, x\right\rangle + \frac{1}{2}|\langle |T||T^*|x, x\rangle| \\
\leq\ & \frac{1}{4}\left\||T|^2 + |T^*|^2\right\| + \frac{1}{2} w\left(|T||T^*|\right).
\end{aligned}
$$

- Taking the supremum over $\|x\| = 1$ we obtain that

$$
w^2(T) \leq \frac{1}{4}\left\||T|^2 + |T^*|^2\right\| + \frac{1}{2} w\left(|T||T^*|\right).
$$

Inequalities using Cauchy-Schwarz inequality and it's generalizations

Remark 3

The inequality in (12) *refines the second inequality in (2). As* $w\left(|T||T^*|\right) \leq \||T||T^*|\| = \left\|T^2\right\|$, *so*

$$\begin{aligned} & \frac{1}{4}\left\||T|^2 + |T^*|^2\right\| + \frac{1}{2}w\left(|T||T^*|\right) \\ \leq\ & \frac{1}{4}\left\||T|^2 + |T^*|^2\right\| + \frac{1}{2}\left\|T^2\right\| \\ \leq\ & \frac{1}{4}\left\||T|^2 + |T^*|^2\right\| + \frac{1}{4}\left\||T|^2 + |T^*|^2\right\| \\ =\ & \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \end{aligned}$$

Next, we present another refined numerical radius inequality by using generalized Cauchy-Schwarz inequality and convexity of a function $f(t) = t^{2r}$.

Inequality 33

Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$w^{2r}(T) \leq \left\| \alpha |T|^{2r} + (1-\alpha) |T^*|^{2r} \right\|$$

for all $r \geq 1$ and $0 \leq \alpha \leq 1$. In particular,

$$w^2(T) \leq \min_{0 \leq \alpha \leq 1} \left\| \alpha |T|^2 + (1-\alpha) |T^*|^2 \right\|. \tag{13}$$

Inequalities using Cauchy-Schwarz inequality and it's generalizations

Remark 4

- *(13) generalizes the second inequality in (2) to the form*

$$w^2(T) \leq \|\alpha T^* T + (1-\alpha) TT^*\|, \quad \alpha \in [0,1].$$

- *However, the first inequality cannot be generalized to the form*

$$\frac{1}{2}\|\alpha T^* T + (1-\alpha) TT^*\| \leq w^2(T), \quad \alpha \in [0,1]$$

- *Consider* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
- *Then* $\frac{3}{8} = \frac{1}{2}\|\alpha T^* T + (1-\alpha) TT^*\| \not\leq w^2(T) = \frac{1}{4}$ *for* $\alpha = \frac{1}{4}$.

Inequalities using Bernau and Smithies inequality

Inequality 34

(Bernau and Smithies inequality)
Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$\|Tx\|^2 + |\langle T^2x, x\rangle| \leq 2w(T)\|Tx\|\|x\|,$$

for all $x \in \mathcal{H}$.

Proof.

- Let λ and θ be real numbers, $\lambda \neq 0$.
- Then we have

$$\begin{aligned}\|Tx\|^2 + e^{2i\theta}\langle T^2x, x\rangle &= \frac{1}{2}(\lambda e^{2i\theta}T^2x + \lambda^{-1}e^{i\theta}Tx, \lambda e^{i\theta}Tx + \lambda^{-1}x) \\ &\quad - \frac{1}{2}(\lambda e^{2i\theta}T^2x - \lambda^{-1}e^{i\theta}Tx, \lambda e^{i\theta}Tx - \lambda^{-1}x).\end{aligned}$$

□

Proof.

- Since $|\langle Ty, y\rangle| \leq w(T)\|y\|^2$ for all y, it follows that

$$\begin{aligned}
&|\|Tx\|^2 + e^{2i\theta}\langle T^2x, x\rangle| \\
&\leq \frac{1}{2}w(T)(\|\lambda e^{i\theta}Tx + \lambda^{-1}x\|^2 + \|\lambda e^{i\theta}Tx - \lambda^{-1}x\|^2) \\
&= w(T)(\lambda^2\|Tx\|^2 + \lambda^{-2}\|x\|^2).
\end{aligned}$$

- If $Tx \neq 0$, we choose θ so that $e^{2i\theta}(T^2x, x) = |(T^2x, x)|$, and λ so that $\lambda^2\|Tx\| = \|x\|$.
- Therefore, $\|Tx\|^2 + |\langle T^2x, x\rangle| \leq 2w(T)\|Tx\|\|x\|$.

$\square$

Inequalities using Bernau and Smithies inequality

Inequality 35

(A generalization Bernau and Smithies inequality)
Let $A, T, B \in \mathcal{B}(\mathcal{H})$. Then

$$|\langle A^* TBx, x\rangle| + |\langle B^* TAx, x\rangle| \leq 2w(T)\|Ax\|\|Bx\|,$$

for all $x \in \mathcal{H}$.

Proof.

- Suppose that $x \in \mathcal{H}$ and θ, ϕ are real numbers such that

$$e^{i\phi}\langle B^* TAx, x\rangle = |\langle B^* TAx, x\rangle|,$$

$$e^{2i\theta}\langle e^{-i\phi}A^* TBx, x\rangle = |\langle e^{-i\phi}A^* TBx, x\rangle| = |\langle A^* TBx, x\rangle|.$$

- Consider a nonzero real number λ.

□

Inequalities using Bernau and Smithies inequality

Proof.

- Then

$$
\begin{aligned}
&2e^{2i\theta}\left\langle TBx, e^{i\phi}Ax\right\rangle + 2e^{i\phi}\langle TAx, Bx\rangle \\
&\quad = \left\langle \left\langle e^{i\theta}T\ \left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle\right. \\
&\qquad - \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle.
\end{aligned}
$$

- Hence,

$$
\begin{aligned}
&2e^{2i\theta}\left\langle e^{-i\phi}A^*TBx, x\right\rangle + 2e^{i\phi}\langle B^*TAx, x\rangle \\
&\quad = \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
&\qquad - \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle.
\end{aligned}
$$

Proof.

- So,

$$
\begin{aligned}
&2\left|\langle A^*TBx,x\rangle\right| + 2\left|\langle B^*TAx,x\rangle\right| \\
&\quad = \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
&\qquad - \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
&2\left|\langle A^*TBx,x\rangle\right| + 2\left|\langle B^*TAx,x\rangle\right| \\
&\leq \left|\left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle\right| \\
&+ \left|\left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle\right|.
\end{aligned}
$$

Proof.

- Hence,

$$2\left|\langle A^* TBx, x\rangle\right| + 2\left|\langle B^* TAx, x\rangle\right| \\ \leq w(T)\left(\left\|\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\|^2 + \left\|\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\|^2\right).$$

- Thus,

$$|\langle A^* TBx, x\rangle| + |\langle B^* TAx, x\rangle| \leq w(T)\left(\lambda^2\|Bx\|^2 + \frac{1}{\lambda^2}\|Ax\|^2\right).$$

- If $\|Bx\| \neq 0$, then we can choose $\lambda^2 = \frac{\|Ax\|}{\|Bx\|}$ to get

$$|\langle A^* TBx, x\rangle| + |\langle B^* TAx, x\rangle| \leq 2w(T)\|Ax\|\|Bx\|.$$

- Clearly, this inequality is valid when $\|Bx\| = 0$.

Remark 5

Considering $A = T$ and $B = I$ in the above inequality, we get the Bernau and Smithies inequality

$$\|Tx\|^2 + \left|\left\langle T^2x, x\right\rangle\right| \leq 2w(T)\|Tx\|\|x\|, \quad x \in \mathcal{H}.$$

By using this generalization we get the following numerical radius inequalities for product of three operators.

Theorem 36

Let $A, T, B \in \mathcal{B}(\mathcal{H})$. Then

$$(i)\;\; c(A^*TB) + w(B^*TA) \leq 2w(T)\|A\|\|B\|.$$
$$(ii)\;\; w(A^*TB) + c(B^*TA) \leq 2w(T)\|A\|\|B\|.$$

- In this section, we focus on the numerical range of operators acting on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ equipped with an additional semi-inner product determined by a positive operator A.
- Specifically, this semi-inner product is defined by $\langle x, y\rangle = \langle Ax, y\rangle$ for all x, y in the concerned space.
- $\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \exists\ c > 0$ such that $\|Tx\|_A \leq c\|x\|_A \ \ \forall x \in \mathcal{H}\}$.

Definition 37

(A-Numerical Range) Let $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. The A-numerical range of T, denoted by $W_A(T)$, is defined as the collection of complex scalars $\langle Tx, x\rangle_A$ for $x \in \mathcal{H}$ with $\|x\|_A = 1$, i.e.,
$W_A(T) = \{\langle Tx, x\rangle_A : x \in \mathcal{H}, \|x\|_A = 1\}$.

- Let $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then $W_A(T)$ is a convex subset of $\mathbb{C}$.

Definition 38

(A-eigenvalue) Let $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then $\lambda \in \mathbb{C}$ is said to be an A-eigenvalue of T if there exists $\lambda \in \mathbb{C}$ such that $ATx = \lambda Ax\}$.

Definition 39

(A-reducing eigenvalue) A complex number $\lambda \in \mathbb{C}$ is said to be an A-reducing eigenvalue of $T \in \mathcal{B}_A(\mathcal{H})$ if $ATx = \lambda Ax$ and $AT^{\sharp_A}x = \bar{\lambda}Ax$, for some nonzero $x \in \overline{R(A)}$.

Then we have the following results.

- Let λ be in the boundary of $W_A(T)$. If λ is an A-eigenvalue of T then it is an A-reducing eigenvalue of T.
- Let $T \in \mathcal{B}_A(\mathcal{H})$. If λ, μ are distinct A-eigenvalues of T with λ in the boundary of $W_A(T)$ then A-eigenvectors associated with λ and μ are A-orthogonal to each other.
- Let $T \in \mathcal{B}_A(\mathcal{H})$ be an A-selfadjoint operator. If $\lambda \in W_A(T)$ and $T \leq_A \lambda I$ then λ is an A-reducing eigenvalue of T.
- Every corner point of $W_A(T)$ is an A-reducing eigenvalue of T.

Definition 40

(Joint numerical range) For an n-tuple $T = (T_1, \cdots, T_n) \in \mathbb{B}(\mathcal{H})^n$, the joint numerical range of T is defined as

$$W(T) = W(T_1, \cdots, T_n) = \{(\langle T_1 x, x\rangle, \cdots, \langle T_n x, x\rangle) : x \in \mathcal{H}, \|x\| = 1\}.$$

- The joint numerical range does not always satisfy the convexity property.
- Consider $T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- $W(T_1, T_2, T_3)$ is not convex.

Next we state some basic convexity properties of the joint numerical range.

Proposition 41

Let $T = (T_1, \cdots, T_n)$ be an n-tuple of operators on $\mathcal{H}$.

- **(i)** *$W(T)$ is convex if and only if $W(T, I)$ is convex.*
- **(ii)** *If $W(T)$ is convex and $S_1, S_2, \cdots, S_m$ are in span$\{T_1, \cdots, T_n\}$, then $W(S_1, S_2, \cdots, S_m)$ is convex.*
- **(iii)** *Let $\{S_1, S_2, \cdots, S_m\}$ be a basis of span$\{T_1, \cdots, T_n\}$. Then $W(T)$ is convex if and only if $W(S_1, S_2, \cdots, S_m)$ is convex.*

Definition 42

(C-numerical range) For $n \times n$ matrices T and C, the C-numerical range is defined as

$$W_C(T) = \{tr(CU^*TU) : U \text{ is an } n \times n \text{ unitary matrix}\}.$$

Two subclasses of the C-numerical ranges are c-numerical range and k-numerical range.

- For any vector $c = (c_1, c_2, \cdots, c_n) \in \mathbb{C}^n$, the c-numerical range of T is defined as $W_c(T) = \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : \{x_1, x_2, \cdots, x_n\} \text{ orthonormal basis of } \mathbb{C}^n \right\}.$
- For any $k, 1 \leq k \leq n$, the k-numerical range is defined as $W_k(T) = \left\{ \frac{1}{k} \sum_{i=1}^{n} \langle Tx_i, x_i \rangle : \{x_1, x_2, \cdots, x_n\} \text{ orthonormal subset of } \mathbb{C}^n \right\}.$

Remark 6

- $W_c(T)$ is $W_C(T)$ for $C = diag(c_1, c_2, \cdots, c_n)$, $W_k(T)$ is $W_c(T)$ with $c = (1/k, \cdots, 1/k, 0, \cdots, 0)$ in $\mathbb{C}^n$, and $W_k(T)$ with $k = 1$ is the classical numerical range $W(T)$ of T.
- The C-numerical range of an operator is not always convex.
- For example, consider $T = C = diag(0, 1, i)$, then $W_C(T)$ is not convex.

Although the previous example showed non-convexity, the C-numerical range can still be convex for some following particular cases.

(i) If T and C are 2×2 matrices then $W_C(T)$ is a elliptic disc and hence is convex.

(ii) If T and C are $n \times n$ matrices and C is Hermitian then $W_C(T)$ is convex.

Definition 43

(q-numerical range) For a bounded linear operator T on $\mathcal{H}$ and a complex number q with $|q| \leq 1$, the q-numerical range of T, denoted by $W_q(T)$, is defined as

$$W_q(T) = \{\langle Tx, y\rangle : x, y \in \mathcal{H}, \|x\| = 1, \|y\| = 1, \langle x, y\rangle = q\}.$$

- If $q = 1$ then $W_q(T)$ reduces to the classical numerical range $W(T)$.
- For $|q| \leq 1$, $W_q(T)$ is convex.

Problem 1

- Let T be a bounded linear operator on $\mathcal{H}$.
- It is well-known that $w(T) \geq \frac{1}{2}\|T^*T + TT^*\|^{\frac{1}{2}} \geq \frac{\|T\|}{2}$.
- A well-known characterization of the numerical range states that $w(T) = \frac{\|T\|}{2}$ if and only if $\overline{W(T)}$ is a circular disk with center at the origin and radius $\frac{\|T\|}{2}$.
- In [2], we proved that $\overline{W(T)}$ is contained in a closed circular disk and contains a semi-circular disk with center at the origin and radius $\frac{1}{2}\|T^*T + TT^*\|^{\frac{1}{2}}$ if and only if the numerical radius attains its lower bound $\frac{1}{2}\|T^*T + TT^*\|^{\frac{1}{2}}$.

Problem 1

Open Question 1

(1) *Does the equality* $w(T) = \frac{1}{2}\|T^*T + TT^*\|^{\frac{1}{2}}$ *imply that* $\overline{W(T)}$ *is a closed circular disk? If not, then provide an example.*

(2) *Does there exist a lower bound, say,* $l(T)$, *for the numerical radius* $w(T)$ *which satisfies* $w(T) \geq l(T) \geq \frac{1}{2}\|T^*T + TT^*\|^{\frac{1}{2}}$ *and the closure of the numerical range is a circular disk with center at origin when* $w(T) = l(T) \geq \frac{1}{2}\|T^*T + TT^*\|^{\frac{1}{2}}$?

Problem 2

- Let $T \in \mathcal{M}_n(\mathbb{C})$. Then the numerical radius $w(T)$ satisfies $w(T) \leq \|T\|$.
- In [3], we established a generalization and improvement of the above inequality, namely,

$$w(T) \leq \|\alpha T^*T + (1-\alpha)TT^*\|^{\frac{1}{2}},$$

  for all $\alpha \in [0, 1]$.
- It is easy to observe that if T is a normal matrix then $w(T) = \|\alpha T^*T + (1-\alpha)TT^*\|^{\frac{1}{2}}$, for all $\alpha \in [0, 1]$.

Open Question 2

- *It leads us to the question whether the above equality holds for non-normal matrices, for all* $\alpha \in [0, 1]$.

Problem 2

- For a normal matrix T, whose $w(T) = \|\alpha T^*T + (1-\alpha)TT^*\|^{\frac{1}{2}}$ for all $\alpha \in [0,1]$, the boundary of the numerical range $\partial W(T)$ is not smooth.
- Also, there exists a non-normal matrix T for which $W(T) = \|\alpha T^*T + (1-\alpha)TT^*\|^{\frac{1}{2}}$ for all $\alpha \in [0,1]$, and the boundary of $W(T)$ is not smooth.
- For example, consider $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, then $W(T)$ is a convex hull of $\{z \in \mathbb{C} : |z| \leq 1\} \cup \{2\}$, which is not smooth.

In this context, the following two questions arise:

Open Question 3

(1) *If* $w(T) = \|\alpha T^*T + (1-\alpha)TT^*\|^{\frac{1}{2}}$, *for all* $\alpha \in [0,1]$, *then prove or disprove that the boundary of the numerical range* $\partial W(T)$ *is not smooth.*

(2) *If* $w(T) = \|\alpha T^*T + (1-\alpha)TT^*\|^{\frac{1}{2}}$, *for some* $\alpha \in [0,1]$, *then prove or disprove that the boundary of the numerical range* $\partial W(T)$ *is not smooth.*

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Thanks for your kind attention