Discrete and Continuous Logistic Models with Conditional Hyers–Ulam Stability

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Outline

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1. Hyers–Ulam Stability (HUS)
2. $\mathbb{T} = \mathbb{R}$: Logistic Equation
3. HUS: Logistic
4. $\mathbb{T} = h\mathbb{Z}$: h-Logistic Equation
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6. Future Directions

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2. $\mathbb{T} = \mathbb{R}$: Logistic Equation
3. HUS: Logistic
4. $\mathbb{T} = h\mathbb{Z}$: h-Logistic Equation
5. HUS: h-Logistic
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Abstract

This study investigates the conditional Hyers–Ulam stability of first-order nonlinear logistic models, both continuous and discrete. Identifying bounds on both the relative size of the perturbation and the initial population size is an important issue for nonlinear Hyers–Ulam stability analysis. Utilizing a novel approach, for h-difference equations we derive explicit expressions for the optimal lower bound of the initial value region and the upper bound of the perturbation amplitude, surpassing the precision of previous research. Furthermore, we obtain a sharper Hyers–Ulam stability constant, which quantifies the error between true and approximate solutions, thereby demonstrating enhanced stability. The Hyers–Ulam stability constant is proven to be in terms of the step-size h and the growth rate but independent of the carrying capacity. Detailed examples are provided illustrating the applicability and sharpness of our results on conditional stability.

Hyers–Ulam Stability (HUS)

Stanislaw Ulam, in **A Collection of Mathematical Problems**, 1960, posed the following question:

When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?

Hyers–Ulam Stability (HUS)

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When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?

Definition

The equation

$$y^{\Delta}(t) - F(t,y) = 0, \quad t \in \mathbb{T},$$

is Hyers–Ulam stable if there exists a constant $\kappa > 0$ with the following property: For any $\varepsilon > 0$, and for any function $\varphi : \mathbb{T} \to \mathbb{C}$ such that

$$\left| \varphi^{\Delta}(t) - F(t,\varphi) \right| \le \varepsilon, \quad t \in \mathbb{T},$$

there exists a solution $y : \mathbb{T} \to \mathbb{C}$ of $y^{\Delta}(t) - F(t,y) = 0$ such that

$$|\varphi(t) - y(t)| \le \kappa\varepsilon, \quad t \in \mathbb{T}.$$

Such κ is called an Ulam stability constant for $y^{\Delta}(t) - F(t,y) = 0$.

Hyers–Ulam Stability (HUS)

Definition

For any $\varepsilon > 0$, and for any function $\varphi : \mathbb{T} \to \mathbb{C}$ such that

$$\left|\varphi^{\Delta}(t) - F(t, \varphi)\right| \leq \varepsilon, \quad t \in \mathbb{T},$$

there exists a solution $y : \mathbb{T} \to \mathbb{C}$ of $y^{\Delta}(t) - F(t, y) = 0$ such that

$$|\varphi(t) - y(t)| \leq \kappa\varepsilon, \quad t \in \mathbb{T}.$$

Questions:

- Given the approximate solution φ, is the exact solution y unique?
- What is the minimum κ?
- Are there conditions on ε?
- Are there conditions on F?

Logistic Differential Equation

For $\mathbb{T} = \mathbb{R}$, consider the differential equation

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right),$$

where $P(t)$ is the population at time t, r is the growth rate and K is the carrying capacity.

Conditional Hyers–Ulam Stability (CHUS)

The autonomous equation

$$\frac{dy}{dt} - F(y) = 0, \quad t \in [0, T_y),$$

is conditionally Ulam stable in the class

$$\mathcal{C} = \left\{ y \in C^1[0, T_y) : \; y(0) \in D, T_y > 0 \right\}$$

if $\exists$ constant $\kappa > 0$ such that for each $\varepsilon \in (0, \varepsilon_{\max}]$ and every approximate solution $\varphi \in \mathcal{C}$ satisfying

$$\left| \frac{d\varphi}{dt} - F(\varphi) \right| \leq \varepsilon, \quad t \in [0, T_\varphi)$$

$\exists$ a solution $y \in \mathcal{C}$ of $\frac{dy}{dt} - F(y) = 0$ such that

$$|\varphi(t) - y(t)| \leq \kappa\varepsilon, \quad t \in [0, \min\{T_y, T_\varphi\}).$$

First Result (2018): The rescaled logistic differential equation

$$y'(t) = y(t)\,(1 - y(t))$$

is conditionally Hyers–Ulam stable in the class

$$\mathcal{C} = \left\{ y \in C^1[0,\infty) : \; y(0) \geq \frac{1}{2} \right\}$$

for each $\varepsilon \in (0, \frac{1}{4}]$ with HU stability constant $\kappa = 2$.

Popa, Rasa, Viorel (2018), continued

Meaning: Given $0 < \varepsilon \leq \frac{1}{4}$, if $\exists$ approximate solution φ such that

$$\left|\varphi'(t) - \varphi(t)\,(1 - \varphi(t))\right| \leq \varepsilon, \quad \forall t \geq 0,$$

with $\varphi(0) \geq \frac{1}{2}$, then $\exists$ exact solution y of

$$y'(t) - y(t)\,(1 - y(t)) = 0, \quad y(0) = \varphi(0),$$

such that

$$|\varphi(t) - y(t)| \leq 2\varepsilon, \quad \forall t \geq 0.$$

Onitsuka (2021)

Fix $a \neq 0$ and $b \neq 0$. The logistic-type differential equation

$$y'(t) = y(t)\,(a + by(t))$$

is conditionally Hyers–Ulam stable in the class

$$\mathcal{C} = \left\{ y \in C^1[0,\infty) : \; by(0) \leq \frac{-a}{2} \right\}$$

for each $\varepsilon \in \left(0, \frac{a^2}{4|b|}\right]$ with HU stability constant $\kappa = \frac{2}{|a|}$.

Onitsuka (2021), continued

In particular, the logistic differential equation

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right)$$

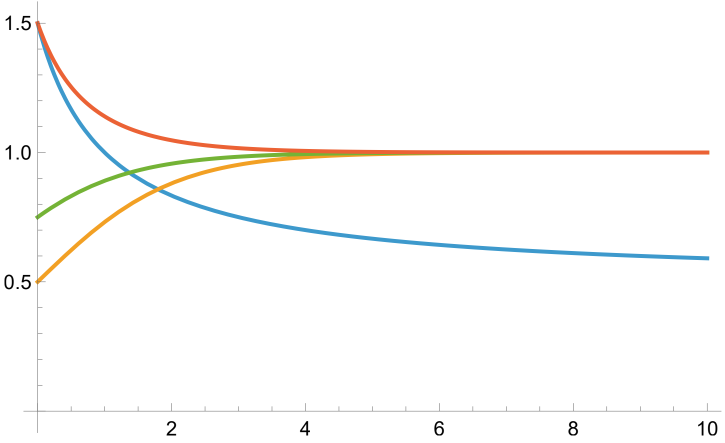
is conditionally Hyers–Ulam stable (CHUS) in the class

$$\mathcal{C} = \left\{ y \in C^1[0,\infty) : \; P(0) \geq \frac{K}{2} \right\}$$

for each $\varepsilon \in \left(0, \frac{rK}{4}\right]$ with HU stability constant $\kappa = \frac{2}{r}$.

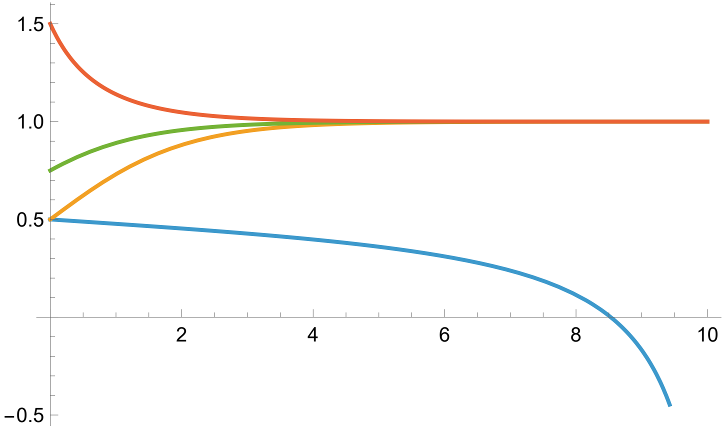
Logistic: Approximate and Exact Solutions

Figure: With small perturbation ε and large initial population, the logistic equation is HU stable.



Logistic: Approximate and Exact Solutions

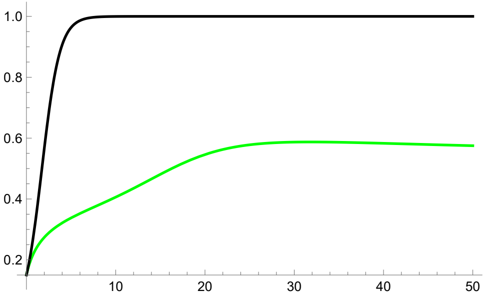
Figure: With large perturbation ε the logistic equation is HU unstable.



Logistic: Bifurcation for Specific Pertubations

$$\varphi' = \varphi(1-\varphi) - \frac{t}{4(t+1)}, \quad \varphi_0 = \frac{1}{2} - \frac{\text{BesselK}[0,1]}{2\,\text{BesselK}[1,1]} + 0.001$$

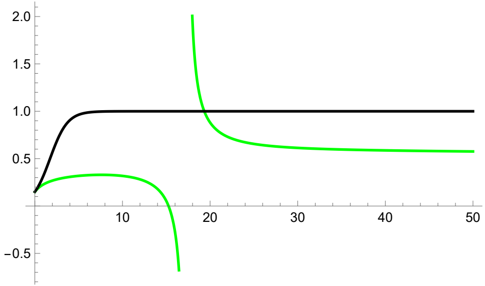
Figure: With small perturbation ε and small initial population, HU stable.



Logistic: Bifurcation for Specific Pertubations

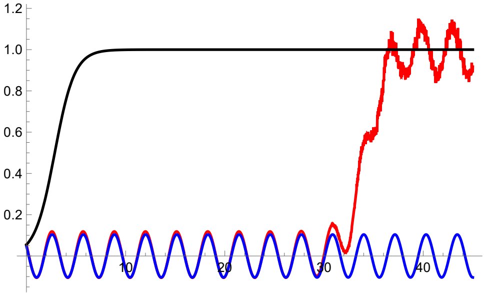
$$\varphi' = \varphi(1-\varphi) - \frac{t}{4(t+1)}, \quad \varphi_0 = \frac{1}{2} - \frac{\text{BesselK}[0,1]}{2\,\text{BesselK}[1,1]} - 0.001$$

Figure: With small perturbation ε and slightly smaller initial population, HU unstable.



$$\varphi' = \varphi(1-\varphi) - \frac{1}{4}\cos(2t), \quad \varphi_0 = 0.05449603614163$$

Figure: With small perturbation ε and smaller initial population but still HUS.



Logistic h-Difference Equation

Given $h > 0$, set $\mathbb{T} := \{0, h, 2h, 3h, \dots\}$, and define

$$\Delta_h P(t) := \frac{P(t+h) - P(t)}{h}.$$

The logistic growth h-difference equation we consider is

$$\Delta_h P(t) = \frac{rP(t)\,(K - P(t))}{K + hrP(t)}, \tag{1}$$

where P is the population size at time t of some species, $r > 0$ is a growth-rate coefficient, $h > 0$ is the step size, and $K > 0$ is the carrying capacity. When $h = 1$, this equation is called the Beverton-Holt equation.

Logistic h-Difference Equation: Perturbations

Let $\varepsilon > 0$ be arbitrarily given. Then the following equations

$$\Delta_h \beta(t) = \frac{r\beta(t)\left(K - \beta(t)\right)}{K + hr\beta(t)} + q(t), \quad |q(t)| \le \varepsilon, \tag{2}$$

$$\Delta_h \ell(t) = \frac{r\ell(t)\left(K - \ell(t)\right)}{K + hr\ell(t)} - \varepsilon, \tag{3}$$

and

$$\Delta_h u(t) = \frac{ru(t)\left(K - u(t)\right)}{K + hru(t)} + \varepsilon \tag{4}$$

for $t \ge 0$, where $q : \mathbb{T} \to \mathbb{R}$, are perturbations of (1) that will play a key role in the analysis that follows below. Throughout this talk, we assume the initial conditions

$$P(0) = \beta(0) = \ell(0) = u(0) = P_0. \tag{5}$$

Definition (Conditional Hyers–Ulam Stability)

Let $[0, T_P)_h := [0, T_P) \cap \mathbb{T}$ be the maximal interval of existence for a function P. Let $D \subseteq \mathbb{R}$ be nonempty. Define the class

$$\mathcal{C}_D := \{P : [0, T_P)_h \to \mathbb{R} : P(0) \in D, T_P > 0\}.$$

Let $\varepsilon \in \mathcal{S} \subseteq (0, \infty)$. The nonlinear h-difference equation

$$\Delta_h P(t) = F(P(t)) \tag{6}$$

is conditionally HU stable in $\mathcal{C}_D$ on $[0, \min\{T_P, T_\phi\})_h$ with $\mathcal{S}$ if $\exists\ \kappa > 0$ s.t. for every $\varepsilon \in \mathcal{S}$ and every approximate solution $\phi \in \mathcal{C}_D$ that satisfies

$$|\Delta_h \phi(t) - F(\phi(t))| \leq \varepsilon \quad \text{for} \quad 0 \leq t < T_\phi, \tag{7}$$

$\exists$ solution $P \in \mathcal{C}_D$ of (6) such that

$$|\phi(t) - P(t)| \leq \kappa\varepsilon \quad \text{for} \quad 0 \leq t < \min\{T_P, T_\phi\}.$$

Proposition

Proposition

Let $P : [0, T_P)_h \to \mathbb{R}$, $\beta : [0, T_\beta)_h \to \mathbb{R}$, $\ell : [0, T_\ell)_h \to \mathbb{R}$, *and* $u : [0, T_u)_h \to \mathbb{R}$ *be the solutions of* (1), (2), (3), *and* (4) *with initial condition* (5), *respectively. If*

$$0 < \varepsilon \le \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} \quad \text{and} \quad P_0 \ge \frac{K\left(\sqrt{1+hr}-1\right)}{hr},$$

then $T_P = T_\beta = T_\ell = T_u = \infty$, *and*

$$\frac{K\left(\sqrt{1+hr}-1\right)}{hr} \le \ell(t) \le \beta(t) \le u(t) \quad \text{and} \quad \ell(t) < P(t) < u(t)$$

hold for all $t \in (0, \infty)_h$.

Example of HUS

Consider (1), (2), (3), and (4) with $h = r = K = 1$. According to Proposition 1, if

$$0 < \varepsilon \leq \left(\sqrt{2}-1\right)^2 \quad \text{and} \quad P_0 \geq \sqrt{2}-1$$

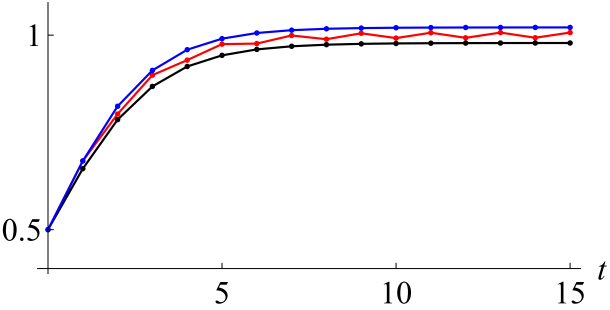
hold, then the solution P and approximate solutions β, ℓ, and u of (1), (2), (3), and (4), respectively, with initial condition (5) satisfy $T_P = T_\beta = T_\ell = T_u = \infty$ and

$$\sqrt{2}-1 \leq \ell(t) \leq \beta(t) \leq u(t) \quad \text{and} \quad \ell(t) < P(t) < u(t)$$

for all $t \in [0,\infty)_1$.

Example of HUS (continued)

Solution orbits of β (red) with $h = r = K = 1$ and $q(t) = 0.01(-1)^t$, ℓ (black), and u (blue), given the initial condition $\beta(0) = \ell(0) = u(0) = 0.5$ and $\varepsilon = 0.01$. Notice that the solution orbit of β (red) is bounded between the others.



Suppose that

$$0 < \varepsilon \le \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} \quad \text{and} \quad P_0 \ge \frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

Let P, ℓ, and u be the solutions of (1), (3), and (4) with initial condition (5), respectively. Then, $T_P = T_\ell = T_u = \infty$,

$$\frac{r(K^2 - hr\ell(t)P(t) - K(\ell(t)+P(t)))}{(K+hr\ell(t))(K+hrP(t))}$$
$$< -\frac{\sqrt{1+hr}-1}{h\sqrt{1+hr}} \frac{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}} =: \mathscr{F}(t)$$

holds for all $t \in (0,\infty)_h$, as it does with ℓ replaced by u.

Technical Lemma #2

Let $\varepsilon > 0$, and from previous slide let

$$\mathscr{F}(t) := -\frac{\sqrt{1+hr}-1}{h\sqrt{1+hr}}\,\frac{(1+hr)^{\frac{t}{2h}+\frac{1}{2}}-(1+hr)^{-\frac{t}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}}+(1+hr)^{-\frac{t}{2h}}}. \tag{8}$$

Then the function

$$\Omega(t) := \varepsilon h\sqrt{1+hr} \cdot \left(e_{\mathscr{F}}(t,0)+\frac{1}{\sqrt{1+hr}-1}\,\frac{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}}-(1+hr)^{-\frac{t-h}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}}+(1+hr)^{-\frac{t-h}{2h}}}\right)$$

solves the linear h-difference equation

$$\Delta_h\Omega(t) = \mathscr{F}(t)\Omega(t)+\varepsilon\sqrt{1+hr}\,\frac{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}}+(1+hr)^{-\frac{t-h}{2h}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}}+(1+hr)^{-\frac{t}{2h}}}$$

with the initial condition $\Omega(0) = 0$.

Technical Lemma #3

Lemma

Let $\varepsilon > 0$, *and let* $\omega(t)$ *satisfy* $\omega(0) = 0$ *and the linear h-difference inequality*

$$\Delta_h \omega(t) \leq \mathscr{F}(t)\omega(t) + \varepsilon$$

for $t \in [0, \infty)_h$, *where* $\mathscr{F}(t)$ *is given previously. Let* $\Omega(t)$ *be given on the previous slide. Then* $\Omega(t) \geq \omega(t)$ *for all* $t \in [0, \infty)_h$.

Main Theorem

Theorem (Conditional HUS)

Suppose that

$$0 < \varepsilon \leq \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} \quad \text{and} \quad P_0 \geq \frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

Let P (exact) and β (approx) be the solutions of (1) and (2) with $P(0) = \beta(0)$, respectively. Then, $T_P = T_\beta = \infty$, and

$$|\beta(t) - P(t)| \leq \frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon$$

holds for all $t \in [0, \infty)_h$. That is, equation (1) is conditionally Hyers–Ulam stable with Hyers–Ulam stability constant $\kappa = \frac{h(1+hr)}{\sqrt{1+hr}-1}$.

Remark on Implications of Main Theorem

The main theorem implies the following fact: (1) is conditionally HUS in class $\mathcal{C}_D$ on $[0,\infty)_h$, with $\varepsilon \in \mathcal{S} = \left(0, \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r}\right]$ and HUS constant $\kappa = \frac{h(1+hr)}{\sqrt{1+hr}-1}$, where $P_0 \in D = \left[\frac{K\left(\sqrt{1+hr}-1\right)}{hr}, \infty\right)$. For the three key constants given here, we note that as the step-size $h > 0$ tends to zero, we have

$$\lim_{h\to 0^+} \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} = \frac{rK}{4}, \quad \lim_{h\to 0^+} \frac{K\left(\sqrt{1+hr}-1\right)}{hr} = \frac{K}{2},$$

and

$$\lim_{h\to 0^+} \frac{h(1+hr)}{\sqrt{1+hr}-1} = \frac{2}{r}.$$

Remark: Y.W. Nam

If $h = 1$, then (1) can be rewritten as the iteration equation

$$P(t+1) = \frac{\sqrt{1+r}P(t)}{\frac{r}{K\sqrt{1+r}}P(t) + \frac{1}{\sqrt{1+r}}}. \tag{9}$$

Letting

$$a = \sqrt{1+r}, \quad b = 0, \quad c = \frac{r}{K\sqrt{1+r}}, \quad d = \frac{1}{\sqrt{1+r}},$$

we see that

$$P(t+1) = \frac{aP(t)+b}{cP(t)+d} \quad \text{with} \quad ad - bc = 1 \quad \text{and} \quad a + d > 2.$$

This is an example of a loxodromic Möbius difference equation. For more on HUS of loxodromic Möbius difference equations, see Nam.

Remark: Jung & Nam Comparison

In 2017, Jung and Nam gave an example of the conditional HUS for the iteration equation

$$P(t+1) = \frac{AP(t)}{CP(t)+1},$$

which is equivalent to (9), where

$$A = 1+r \quad \text{and} \quad C = \frac{r}{K}.$$

Their result, expressed in the terms of our talk, is as follows: equation (9) (resp., (1)) is conditionally HUS in $\mathcal{C}_{D^*}$ on $\mathbb{N}_0$, with $\mathcal{S}^* = \left(0, \frac{A\sqrt{A}-2A+\sqrt{A}}{(A-\sqrt{A}+1)C}\right)$ and HUS constant

$$\kappa^* = \frac{\left(\sqrt{A}+\frac{1}{\sqrt{A}}-1\right)^2}{\left(\sqrt{A}+\frac{1}{\sqrt{A}}-1\right)^2-1}, \; D^* = \left(-\infty, -\frac{A-\sqrt{A}+2}{C}\right)\cup\left(\frac{A-\sqrt{A}}{C}, \infty\right).$$

Remark: Jung & Nam Comparison

We note here that the term "conditional Hyers–Ulam stability" is not used by them, and their original result shows that if $\beta(0)$ is in D^*, then there exists $P(t)$ which satisfies (9) and

$$|\beta(t) - P(t)| \leq \frac{|\beta(0) - P(0)|}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2t}} + \sum_{j=0}^{t-1} \frac{\varepsilon}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2j}}$$

for all $t \in \mathbb{N}_0$, where $\beta(t)$ is a solution of (2). In our talk settings, $\beta(0) = P(0)$, so the first term on the right-hand side is 0. The second term can be evaluated:

$$\sup_{t \in \mathbb{N}_0} \sum_{j=0}^{t-1} \frac{\varepsilon}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2j}} = \frac{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2}}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2} - 1}\varepsilon.$$

Thus, we have their HUS constant κ^*.

Remark: Jung & Nam Comparison

Compare our three important constants obtained earlier with Jung & Nam's $\mathcal{S}^*$, D^*, and κ^*, but note that the negative region of D^* is omitted since it is not of interest in our talk. First, we compare our result with theirs for the upper bound of ε. Using $A = 1 + r$ and $C = \frac{r}{K}$, we have

$$\frac{A\sqrt{A} - 2A + \sqrt{A}}{(A - \sqrt{A} + 1)C} = \frac{K\left(\sqrt{1+r} - 1\right)^2}{r} \times \frac{\sqrt{1+r}}{2 + r - \sqrt{1+r}} < \frac{K\left(\sqrt{1+r} - 1\right)^2}{r}.$$

Our constant (RHS) is larger, allowing ε to be larger and still maintain HUS. Next, we compare the condition on initial values. Since

$$\frac{A - \sqrt{A}}{C} = \frac{K\left(\sqrt{1+r} - 1\right)}{r}\sqrt{1+r} > \frac{K\left(\sqrt{1+r} - 1\right)}{r},$$

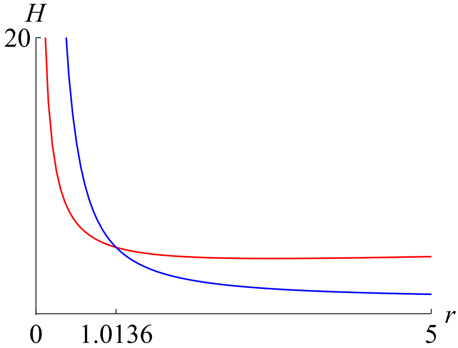
holds, we can conclude that our result guarantees HUS for larger ε and smaller initial values.

Remark: Stability Constant Comparison

Finally, for $r > 0$ compare the HUS constants

$$H(r) = \kappa = \frac{1+r}{\sqrt{1+r}-1} \quad \text{and} \quad H^*(r) = \kappa^* = \frac{\left(\sqrt{1+r}+\frac{1}{\sqrt{1+r}}-1\right)^2}{\left(\sqrt{1+r}+\frac{1}{\sqrt{1+r}}-1\right)^2-1}$$

shown here. The red curve is H and the blue curve is H^*.



Remark: HUS Constants

Note that $r \approx 1.013624$ solves $H(r) = H^*(r)$. Thus, if $0 < r < 1.013624$, then our Hyers–Ulam stability constant $H(r) = \kappa$ is better than theirs. However, this statement may be reversed if $r > 1.013624$.

There is a reason why the HUS constants diverge as r approaches 0. If $h = 1$ and $r = 0$, then (9) (resp., (1)) and (2) become $\Delta P(t) = 0$ and $\Delta\beta(t) = q(t)$ with $|q(t)| \le \varepsilon$ for all $t \in \mathbb{N}_0$. We put $q(t) \equiv \varepsilon$. Then we have a soluion $\beta(t) = \varepsilon t$. Since $P(t) \equiv C$ is any solution of the equation $\Delta P(t) = 0$, where C is an arbitrary constant, we see that

$$\lim_{t\to\infty} |\beta(t) - P(t)| = \lim_{t\to\infty} |\varepsilon t - C| = \infty.$$

This means that (9) is not Hyers–Ulam stable on $\mathbb{N}_0$. Therefore, it is a natural consequence that $\lim_{r\to 0^+} H(r) = \lim_{r\to 0^+} H^*(r) = \infty$. In addition, we have $\lim_{r\to 0^+}(H^*(r) - H(r)) = \infty$. That is, $H^*(r)$ is much larger near $r = 0$ than $H(r)$.

Example of HUS

Let $h = 1$, $r = \frac{1}{3}$, $K = 9$, $\varepsilon = \frac{3}{5}$, and $P_0 = 9\left(-3 + 2\sqrt{3}\right)$. According to the main theorem, since

$$0 < \varepsilon \leq \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} = 63 - 36\sqrt{3} \approx 0.646171$$

and

$$P_0 \geq \frac{K\left(\sqrt{1+hr}-1\right)}{hr} = 9\left(-3+2\sqrt{3}\right),$$

exact solution P and approximate solutions ℓ and u with initial condition $P_0 = \ell(0) = u(0)$ satisfy $T_P = T_\ell = T_u = \infty$ and

$$|\ell(t) - P(t)|, |u(t) - P(t)| \leq \frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon = \frac{4}{5}\left(3+2\sqrt{3}\right) \approx 5.17128$$

for all $t \in [0, \infty)_h$.

Example continued

Note that in this specific instance we have

$$P(t) = \frac{1}{\frac{1}{9} + 2^{1-2t}3^{-\frac{5}{2}+t}}, \quad \ell(t) = 3 + \frac{12}{5 + 3^{\frac{3}{2}-3t}25^t},$$

and $u(t)$ is

$$\frac{9\left(53-\sqrt{109}\right)^t\left(21-16\sqrt{3}+\rho\right)+9\left(53+\sqrt{109}\right)^t\left(16\sqrt{3}-21+\rho\right)}{\left(53-\sqrt{109}\right)^t\left(53-30\sqrt{3}+\sqrt{109}\right)+\left(53+\sqrt{109}\right)^t\left(-53+30\sqrt{3}+\sqrt{109}\right)}$$

where $\rho = \sqrt{327\left(7-4\sqrt{3}\right)}$, so that we have the numerical comparison for $t = 0, \ldots, 10$ given in Table 1.

Example 1: Table of Stability

Table: Solutions and errors with $h = 1$, $r = \frac{1}{3}$, $K = 9$, $\varepsilon = \frac{3}{5}$, and $P(0) = \ell(0) = u(0) = P_0 = 9\left(-3 + 2\sqrt{3}\right)$.

t	$P(t)$	$\ell(t)$	$u(t)$	$P(t) - \ell(t)$	$u(t) - P(t)$
0	4.17691	4.17691	4.17691	0.0	0.0
1	4.82309	4.22309	5.42309	0.6	0.6
2	5.45614	4.26919	6.62136	1.18695	1.16522
3	6.05189	4.31509	7.68981	1.7368	1.63792
4	6.59169	4.36065	8.58024	2.23105	1.98855
5	7.06428	4.40574	9.28147	2.65853	2.21719
6	7.46571	4.45025	9.80946	3.01546	2.34375
7	7.79806	4.49404	10.1937	3.30401	2.39569
8	8.0674	4.53702	10.4666	3.53039	2.39917
9	8.28195	4.57908	10.6569	3.70287	2.37492
10	8.4505	4.62013	10.788	3.83038	2.33748

Example

If we keep all the parameter values the same but take $\varepsilon = \frac{4}{5}$ instead of $\varepsilon = \frac{3}{5}$, then $\varepsilon > \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} = 63 - 36\sqrt{3} \approx 0.646171$ and the right-hand side of the difference between exact and approximate solutions becomes

$$\frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon = \frac{16}{15}\left(3+2\sqrt{3}\right) \approx 6.89504, \tag{10}$$

so one of the hypotheses of the main theorem is not met. This is illustrated in the next table.

Example 2: Table of Instability

Table: Solutions and errors with $h = 1$, $r = \frac{1}{3}$, $K = 9$, $\varepsilon = \frac{4}{5}$, and $P(0) = \ell(0) = P_0 = 9\left(-3 + 2\sqrt{3}\right)$ for equations (1) and (3), respectively.

t	$P(t)$	$\ell(t)$	$P(t) - \ell(t)$
0	4.17691	4.17691	0.0
1	4.82309	4.02309	0.8
2	5.45614	3.86849	1.58764
3	6.05189	3.71158	2.3403
4	6.59169	3.5507	3.04099
15	8.86323	0.546773	8.31646
16	8.89703	−0.08544	8.98247
17	8.92255	−0.914282	9.83684
18	8.94179	−2.06177	11.0036
19	8.95627	−3.7763	12.7326
20	8.96716	−6.6538	15.621

Example 3: HUS but Jung & Nam Doesn't Apply

Table: Solutions and errors with $h = K = 1$, $r = 3$, $\varepsilon = \frac{1}{3}$, and $P(0) = \ell(0) = u(0) = P_0 = \frac{1}{3}$.

t	$P(t)$	$\ell(t)$	$u(t)$	$P(t) - \ell(t)$	$u(t) - P(t)$
0	0.333333	0.333333	0.33333	0.0	0.0
1	0.666667	0.333333	1.0	0.333333	0.333333
2	0.888889	0.333333	1.33333	0.555556	0.444444
3	0.969697	0.333333	1.4	0.636364	0.430303
4	0.992248	0.333333	1.41026	0.658915	0.418008
5	0.998051	0.333333	1.41176	0.664717	0.413714
6	0.999512	0.333333	1.41199	0.666179	0.412473
7	0.999878	0.333333	1.41202	0.666545	0.412139
8	0.999969	0.333333	1.41202	0.666636	0.412052
9	0.999992	0.333333	1.41202	0.666659	0.41203
10	0.999998	0.333333	1.41202	0.666665	0.412025

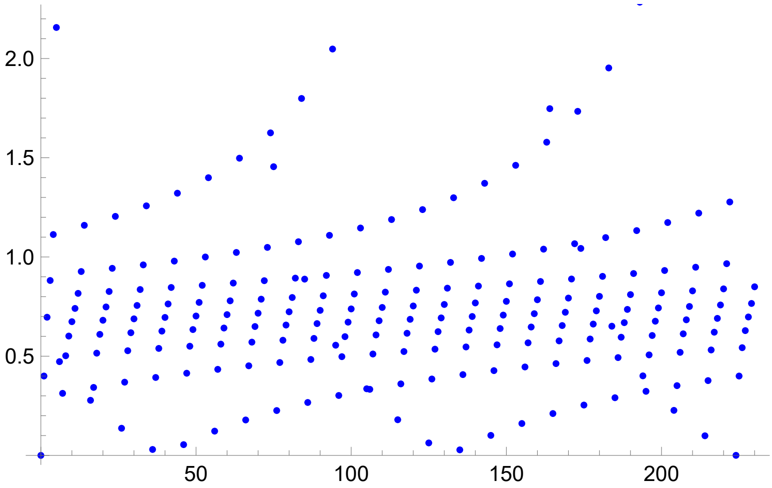
Example 4: Instability due to change of ε from $\frac{1}{3}$ to $\frac{2}{5}$

Table: Solutions and errors with $h = K = 1$, $r = 3$, $\varepsilon = \frac{2}{5}$, and $P(0) = \ell(0) = P_0 = \frac{1}{3}$.

t	$P(t)$	$\ell(t)$	$\lvert \ell(t) - P(t) \rvert$
1	0.666667	0.266667	0.4
2	0.888889	0.192593	0.696296
3	0.969697	0.0882629	0.881434
4	0.992248	−0.120861	1.11311
5	0.998051	−1.15844	2.15649
6	0.999512	1.47198	0.47247
7	0.999878	0.687147	0.312731
13	1.0	0.0733356	0.926664
14	1.0	−0.159557	1.15956
15	1.0	−1.62423	2.62423
16	1.0	1.27762	0.277625

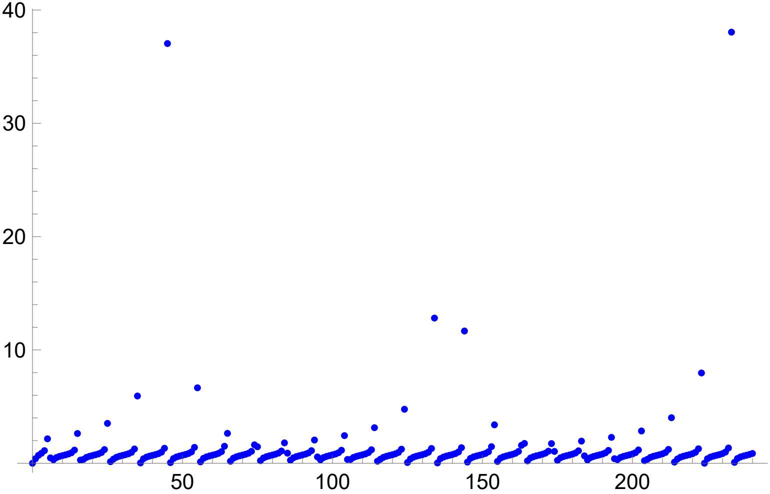
Example 4: Instability in the Difference

This is the difference between the exact solution and the approximate one (zoom in).



Example 4: Instability in the Difference

This is the difference between the exact solution and the approximate one (zoom out).



Example 5: Blow up in finite time

Table: Solutions and errors with $h = K = 1$, $r = 3$, $\varepsilon = \frac{1}{3}$, and $P(0) = \ell(0) = P_0 = \frac{1}{4}$, which is too small.

t	$P(t)$	$\ell(t)$	$\lvert\ell(t) - P(t)\rvert$
0	0.25	0.25	0.0
1	0.571429	0.238095	0.333333
2	0.842105	0.222222	0.619883
3	0.955224	0.2	0.755224
4	0.988417	0.166667	0.82175
5	0.997079	0.111111	0.885968
6	0.999268	0.0	0.999268
7	0.999817	-0.33333	1.33315
8	0.999954	∞	∞

Sensitivity Analysis

Since equation (1) can be rewritten as

$$P(t+h) = \frac{K(1+hr)P(t)}{K+hrP(t)},$$

if we further define $n := \frac{t}{h}$ and $x(n) := P(hn)$, we obtain the following difference equation:

$$x(n+1) = \frac{K(1+hr)x(n)}{K+hrx(n)}. \tag{11}$$

Sensitivity Analysis: K

First, we perform a sensitivity analysis of the parameter K, which represents the carrying capacity. Differentiating (11) with respect to K, we obtain

$$\frac{\partial x(n+1)}{\partial K} = \frac{hr(1+hr)}{\left(\frac{K}{x(n)} + hr\right)^2}.$$

Therefore, the sensitivity coefficient for the parameter K is dependent on the population size, $x(n)$. Given that $0 < x(n) = P(ht) < K$, we observe that the sensitivity is low when the population is small (when $x(n)$ approaches 0), and the sensitivity is high when the population is large (when $x(n)$ approaches K).

Sensitivity Analysis: r

Next, we perform a sensitivity analysis of the parameter r, which represents the growth rate. Differentiating (11) with respect to r, we obtain

$$\frac{\partial x(n+1)}{\partial r} = \frac{hKx(n)(K - x(n))}{(K + hrx(n))^2}.$$

Define the function $S(x) := \frac{hKx(K-x)}{(K+hrx)^2}$ for $0 < x < K$. Then

$$S'(x) = \frac{hK^2(K - 2x)}{(K + hrx)^2}.$$

This demonstrates that the sensitivity is low when the population is small or large (when $x(n)$ approaches 0 or K), and the sensitivity is high when the population is at an intermediate level (when $x(n)$ approaches $\frac{K}{2}$).

Sensitivity: Conclusion

Therefore, we can conclude that the carrying capacity K is sensitive when the population is large, but even if some perturbation is added to the equation, it does not affect the error between the approximate solution and the true solution, so it is a parameter that does not need to be treated very delicately. On the other hand, r and h exhibit sensitivity when the population is at an intermediate level, and they also influence the error between the approximate solution and the true solution. In many cases, h is fixed in advance, and from a biological perspective, it is important to investigate how the population changes from the intermediate stage. Therefore, the parameter to which we should truly pay attention is r, which represents the growth rate.

Example

In (1) and (2) take $h = K = 1$. According to Theorem 5, if

$$0 < \varepsilon \leq \frac{\left(\sqrt{1+r}-1\right)^2}{r} \quad \text{and} \quad P_0 \geq \frac{\sqrt{1+r}-1}{r}$$

hold, then solutions $P : [0, T_P)_1 \to \mathbb{R}$ and $\beta : [0, T_\beta)_1 \to \mathbb{R}$ of (1) and (2), respectively, with initial condition (5) satisfy $T_P = T_\beta = \infty$ and

$$|\beta(t) - P(t)| \leq \frac{1+r}{\sqrt{1+r}-1}\varepsilon$$

for all $t \in [0, \infty)_1$. Table 6 shows the upper bounds of ε, the lower bounds of P_0, and the Hyers–Ulam stability constants, all of which depend on r.

Table

Table: Upper bounds of ε, lower bounds of P_0, and HUS constants, all dependent on r.

r	$\frac{\left(\sqrt{1+r}-1\right)^2}{r}$	$\frac{\sqrt{1+r}-1}{r}$	$\frac{1+r}{\sqrt{1+r}-1}$
0.1	0.023823	0.488088	22.5369
0.2	0.0455488	0.477226	12.5727
0.3	0.0654972	0.467251	9.27409
0.4	0.0839202	0.45804	7.64126
0.5	0.101021	0.44949	6.67423
0.6	0.116963	0.441518	6.03976
0.7	0.131884	0.434058	5.59504
0.8	0.145898	0.427051	5.26869

Figure

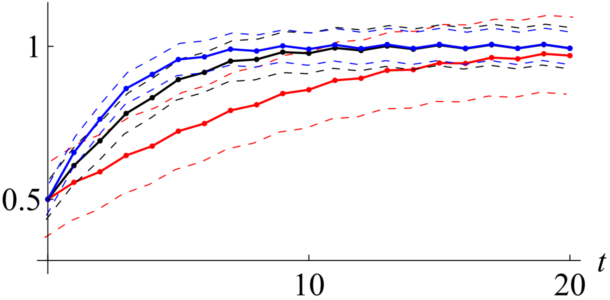


Figure: The solution orbits with $r = 0.2$ (red), 0.5 (black), and 0.8 (blue).

Conclusion

We establish robust conditional Hyers–Ulam stability results for the logistic h-difference equation, also known as the Beverton-Holt equation if $h = 1$, for any constant step-size $h > 0$. As h tends to zero, our results recover known results for the conditional stability of the continuous logistic-growth model. Additionally, departing from the methodology employed by Jung and Nam in the case $h = 1$, we introduce a novel approach to derive sharper results. Specifically, we explicitly determine the optimal lower bound for the initial value region and the upper bound for the perturbation amplitude, demonstrating an improvement over their findings. Furthermore, our analysis yields a sharper Hyers–Ulam constant, which quantifies the error between the true and approximate solutions. Given that a smaller Hyers–Ulam constant indicates greater stability and is desirable for practical applications, our results offer a substantial advancement in precision. The sharpness of our derived bounds and constants is substantiated through illustrative examples.

Future Directions

Extend to periodic parameters for h-difference equations, and to general time scales.

Thanks to my co-author, Masakazu Onitsuka, Okayama University of Science

Thanks for Listening!