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Generalized Quantum Fractional Integral Inequalities with Delays

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Abstract. Utilizing the framework of quantum fractional calculus, we extend several classical integral inequalities such as those of Chebyshev, Poincaré, Sobolev, Opial and Ostrowski types by incorporating delay terms. These results not only generalize existing inequalities in the literature but also provide sharper bounds that account for time-delay effects, which are essential in modeling dynamic systems.

Keywords: Generalized quantum calculus; Chebyshev inequality; Poincaré inequality; Sobolev inequality; Opial and Ostrowski inequality.

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1 Introduction

Fractional calculus and quantum calculus is called calculus without limits. Both calculi has become intrusting role over the last four decades due to their applications in several mathematical areas such as probability and statistics, signal processing, spline theory, bio science, bioengineering, statistical Physics and fluid flow etc. Different dialects of quantum calculus exist, depending on how finite differences are rescaled. The three most important and popular quantum calculi are Hahn quantum calculus, Jackson q-calculus and classical difference calculus. A generic quantum difference operator was just introduced in 2015 by Hamza *et al.* [8]. General quantum calculus is the name given to the calculus associated with D_γ [8, 9]. Numerous scholars have examined this subject since 2015, looking into a number of novel findings. For example, [10] defines the directional derivative, [13] establishes Fubini's theorem and Leibniz's

rule, [14] introduces the quantum Laplace transform, [15] proves the convolution theorem in general quantum difference operator, [4] studies variational calculus problems, [11] introduces quantum exponential functions, [6] studies properties of some function spaces and [7] investigated various inequalities. Integral inequalities play a key role in pure and applied mathematics due to their wide applications in regular and human sociology. The classical inequalities are essential in numerous real-world contexts. Multiple forms and approaches to integral inequalities exist and investigated in-depth by many scholars either in classical analysis, fractional analysis or in the quantum one. Fractional inequalities play a vital role in establishing uniqueness and providing upper bounds for solutions of fractional differential and partial differential equations. Generalized fractional inequalities for Chebyshev functionals are investigated in [12]. Chebyshev and Riemann-Liouville fractional inequalities in q-calculus are presented in [2]. Poincaré type fractional inequalities involving fractional derivatives of Canavati, Riemann-Liouville (R-L), and Caputo types are demonstrated in [1]. Fractional Sobolev-type inequalities are analyzed in [1, 16]. Inequalities of q-Opial type are studied in [3] and fractional Opial-type are explored in [1, 5]. Well known Ostrowski type inequalities are studied in [1, 17]. This manuscript is devoted to establishing new classes of generalized quantum fractional integral inequalities with delays, which not only generalize classical integral inequalities but also provide novel perspectives in the quantum fractional context. By introducing a delayed framework, we accommodate a wider range of dynamic systems where the current state depends on past values an essential feature in control theory, signal processing, and mathematical biology.
The main contributions of this article are organized as follows: Section 2 contain basics results. In section 3 we develop Chebyshev functional integral inequalities. In section 4, we analyzed Poincaré type inequality. Section 5 contains a Sobolev type inequality. Well known Opial and Ostrowski type inequalities are investigated in section 6 and section 7 respectively.

2 Auxiliary Results

Now first we recall basics of γ-quantum calculus which are utilize throughout this paper. Suppose $I \subseteq \mathbb{R}$ and $\gamma\colon I \to I$ is a strictly increasing continuous function having a unique fixed point $s_0 \in I$ with:

$$(t_1 - s_0)(\gamma(t_1) - t_1) \leq 0 \text{ for all } t_1 \in I, \tag{1}$$

known as first kind of γ-function. If the inequality in (1) is reversed, it is known as the second kind of γ-function. (1) holds only if $t_1 = s_0$.
If $t_1 \in I$ and $k \in \mathbb{N}_0$, we have

$$\gamma^k(t_1) := \underbrace{(\gamma \circ \gamma \circ \ldots \circ \gamma)}_{k \text{ times}}(t_1),$$

and

$$\gamma^{-k}(t_1) := \underbrace{(\gamma^{-1} \circ \gamma^{-1} \circ \ldots \circ \gamma^{-1})}_{k \text{ times}}(t_1),$$

and

$$\gamma^0(t_1) = t_1.$$

Definition 2.1. [9] Let $\Upsilon\colon I \to \mathbb{R}$ be a given function. Then the γ-quantum difference operator D_γ is defined as follows

$$D_\gamma \Upsilon(t_1) := \begin{cases} \dfrac{\Upsilon(\gamma(t_1)) - \Upsilon(t_1)}{\gamma(t_1) - t_1} & \text{if } t_1 \neq s_0, \\ \Upsilon'(s_0) & \text{if } t_1 = s_0, \end{cases}$$

provided that $\Upsilon'(s_0)$ exists. The number $D_\gamma \Upsilon(t_1)$ is said to be the γ-derivative of Υ at $t_1 \in I$. In the case when $\Upsilon'(s_0)$ exists, then Υ is γ-differentiable on I.

Remark 2.1. When $\Upsilon\colon I \to \mathbb{R}$ is continuous function at $s_0 \in I$, we have that it is γ-integrable on I.

The γ-integral satisfies the following properties:

1. The γ-integral is a linear operator.
2. Zero integral property:
$$\int_a^a \Upsilon(t_1) d_\gamma t_1 = 0.$$
3. Reversal of limits:
$$\int_a^b \Upsilon(t_1) d_\gamma t_1 = -\int_b^a \Upsilon(t_1) d_\gamma t_1.$$
4. Additivity over subintervals:
$$\int_a^b \Upsilon(t_1) d_\gamma t_1 = \int_a^c \Upsilon(t_1) d_\gamma t_1 + \int_c^b \Upsilon(t_1) d_\gamma t_1.$$

Theorem 2.1. *[9] Let $\Upsilon, \Phi\colon I \to \mathbb{R}$ be γ-differentiable functions on I and $D_\gamma \Upsilon$, $D_\gamma \Phi$ be continuous functions at s_0. Then, for $a, b \in I$, one has*

$$\int_a^b \Upsilon(t_1) D_\gamma \Phi(t_1) d_\gamma t_1 = \Upsilon(b)\Phi(b) - \Upsilon(a)\Phi(a) - \int_a^b (D_\gamma \Upsilon(t_1))\Phi(\gamma(t_1)) d_\gamma t_1.$$

[8] Let $I \subset \mathbb{R}$, $0, \infty \in I$, $\gamma : I \to \mathbb{R}$ is a generalized quantum operator of the first or second kinds with a fixed point t_0. For $\alpha \geq 0$, with $\Omega_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, the generalized polynomials will be represented on time scales as

$$\begin{aligned} \Omega_0(t_1, s) &= 1, \\ \Omega_\alpha(t_1, s) &= \int_s^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau)) d_\gamma \tau, \quad t_1, s \in \mathbb{T}. \end{aligned}$$

For $\alpha \geq 1$ and $\Upsilon \in \mathcal{C}_{rd}(\mathbb{T})$, with D_0^α we have the quantum R-L operator defined by

$$\begin{aligned} D_0^\alpha \Upsilon(t_1) &= \int_0^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau)) \Upsilon(\tau) d_\gamma \tau, \\ D_0^0 \Upsilon(t_1) &= \Upsilon(t_1), \quad t_1 \in \mathbb{T}. \end{aligned}$$

Theorem 2.2. *[7](γ-Hölder inequality) Let $\Upsilon \in L^p_\gamma[a,b]_\gamma$ and $\Phi \in L^q_\gamma[a,b]_\gamma$. Then $|\Upsilon\Phi| \in L^1_\gamma[a,b]_\gamma$ and*

$$\int_x^y |\Upsilon(t_1)\Phi(t_1)|d_\gamma t_1 \quad \leq \left(\int_x^y |\Upsilon(t_1)|^p d_\gamma t_1 \right)^{\frac{1}{p}} \left(\int_x^y |\Phi(t_1)|^q d_\gamma t_1 \right)^{\frac{1}{q}},$$

where $x, y \in [a,b]_\gamma$, $s_0 \in [x,y]$, and $p > 1, q = \frac{p}{(p-1)}$. The equality holds if

$$|\Upsilon(t_1)|^p / |\Phi(t_1)|^q$$

is constant.

3 Integral Inequalities for the Chebyshev Functional

Definition 3.1. Suppose $t_1 \in \mathbb{T}$, $t_1 > 0$, and Υ, Φ are integrable functions defined on $[0, t_1]$. If for all $x, y \in [0, t_1]$

$$(\Upsilon(x) - \Upsilon(y))(\Phi(x) - \Phi(y)) \geq 0,$$

then Υ and Φ are said to be synchronous functions on interval $[0, t_1]$.

Theorem 3.1. *Assume that Υ and Φ are synchronous functions on the interval $[0, \infty)$. Then for all $t_1 \geq 0$, $\alpha \geq 1$, we have*

$$D_0^\alpha(\Upsilon\Phi)(t_1) \geq (\Omega_\alpha(t_1, 0))^{-1} D_0^\alpha \Upsilon(t_1) D_0^\alpha \Phi(t_1).$$

Proof. Assume $\alpha \geq 1$ and $t_1 > 0$. For all $\xi, \tau \geq 0$, we have

$$(\Upsilon(\tau) - \Upsilon(\xi))(\Phi(\tau) - \Phi(\xi)) \geq 0$$

or

$$\Upsilon(\tau)\Phi(\tau) - \Upsilon(\tau)\Phi(\xi) - \Upsilon(\xi)\Phi(\tau) + \Upsilon(\xi)\Phi(\xi) \geq 0,$$

consequently

$$\Upsilon(\tau)\Phi(\tau) + \Upsilon(\xi)\Phi(\xi) \geq \Upsilon(\tau)\Phi(\xi) + \Upsilon(\xi)\Phi(\tau).$$

Hence,

$$\begin{aligned} &\Omega_{\alpha-1}(t_1, \gamma(\tau))\Upsilon(\tau)\Phi(\tau) + \Omega_{\alpha-1}(t_1, \gamma(\tau))\Upsilon(\xi)\Phi(\xi) \\ &\quad \geq \Omega_{\alpha-1}(t_1, \gamma(\tau))\Upsilon(\tau)\Phi(\xi) + \Omega_{\alpha-1}(t_1, \gamma(\tau))\Upsilon(\xi)\Phi(\tau) \end{aligned} \tag{2}$$

for any $\tau, \xi \geq 0$. Integrating (2) with respect to τ from 0 to t_1. Since τ and ξ are independent, we find

$$\begin{aligned} &\int_0^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau))\Upsilon(\tau)\Phi(\tau)d_\gamma\tau + \left(\int_0^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau))d_\gamma\tau \right) \Upsilon(\xi)\Phi(\xi) \\ &\geq \left(\int_0^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau))\Upsilon(\tau)d_\gamma\tau \right) \Phi(\xi) + \left(\int_0^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau))\Phi(\tau)d_\gamma\tau \right) \Upsilon(\xi) \end{aligned}$$

or

$$D_0^\alpha(\Upsilon\Phi)(t_1) + \Upsilon(\xi)\Phi(\xi)\Omega_\alpha(t_1,0)$$
$$\geq \quad \Phi(\xi)D_0^\alpha\Upsilon(t_1) + \Upsilon(\xi)D_0^\alpha\Phi(t_1) \tag{3}$$

for any $\xi \geq 0$. Now we multiply (3) with $\Omega_{\alpha-1}(t_1,\gamma(\xi))$ and we get

$$D_0^\alpha(\Upsilon\Phi)(t_1)\Omega_{\alpha-1}(t_1,\gamma(\xi)) + \Upsilon(\xi)\Phi(\xi)\Omega_{\alpha-1}(t_1,\gamma(\xi))\Omega_\alpha(t_1,0)$$
$$\geq \quad \Omega_{\alpha-1}(t_1,\gamma(\xi))\Phi(\xi)D_0^\alpha\Upsilon(t_1) + \Omega_{\alpha-1}(t_1,\gamma(\xi))\Upsilon(\xi)D_0^\alpha\Phi(t_1)$$

integrating with respect to ξ from 0 to t_1, we find

$$D_0^\alpha(\Upsilon\Phi)(t_1)\int_0^{t_1}\Omega_{\alpha-1}(t_1,\gamma(\xi))d_\gamma\xi + \left(\int_0^{t_1}\Upsilon(\xi)\Phi(\xi)\Omega_{\alpha-1}(t_1,\gamma(\xi))d_\gamma\xi\right)\Omega_\alpha(t_1,0)$$
$$\geq \quad \left(\int_0^{t_1}\Omega_{\alpha-1}(t_1,\gamma(\xi))\Phi(\xi)d_\gamma\xi\right)D_0^\alpha\Upsilon(t_1) + \left(\int_0^{t_1}\Omega_{\alpha-1}(t_1,\gamma(\xi))\Upsilon(\xi)d_\gamma\xi\right)D_0^\alpha\Phi(t_1)$$

and

$$D_0^\alpha(\Upsilon\Phi)(t_1)\Omega_\alpha(t_1,0) + \Omega_\alpha(t_1,0)D_0^\alpha(\Upsilon\Phi)(t_1) \geq D_0^\alpha\Upsilon(t_1)D_0^\alpha\Phi(t_1) + D_0^\alpha\Upsilon(t_1)D_0^\alpha\Phi(t_1),$$

or

$$\Omega_\alpha(t_1,0)D_0^\alpha(\Upsilon\Phi)(t_1) \geq D_0^\alpha\Upsilon(t_1)D_0^\alpha\Phi(t_1).$$

This completes the proof.

Theorem 3.2. *Assume that Υ and Φ are synchronous functions on the interval $[0,\infty)$. Then for all $t_1 > 0$, $\alpha,\beta \geq 1$, we have*

$$\Omega_\alpha(t_1,0)D_0^\beta(\Upsilon\Phi)(t_1) + \Omega_\beta(t_1,0)D_0^\alpha(t_1,0) \geq D_0^\alpha\Upsilon(t_1)D_0^\beta\Phi(t_1) + D_0^\beta\Upsilon(t_1)D_0^\alpha\Phi(t_1).$$

Proof. Assume $t_1 > 0$ and $\alpha,\beta \geq 1$. According to proof of Theorem 3.1, for all $\xi,\tau \geq 0$, we have

$$\Upsilon(\tau)\Phi(\tau) + \Upsilon(\xi)\Phi(\xi) \geq \Upsilon(\tau)\Phi(\xi) + \Upsilon(\xi)\Phi(\tau)$$

and

$$D_0^\alpha(\Upsilon\Phi)(t_1) + \Upsilon(\xi)\Phi(\xi)\Omega_\alpha(t_1,0)$$
$$\geq \quad \Phi(\xi)D_0^\alpha\Upsilon(t_1) + \Upsilon(\xi)D_0^\alpha\Phi(t_1). \tag{4}$$

Now multiply inequality (4) with $\Omega_{\beta-1}(t_1,\gamma(\xi))$, we get

$$D_0^\alpha(\Upsilon\Phi)(t_1)\Omega_{\beta-1}(t_1,\gamma(\xi)) + \Upsilon(\xi)\Phi(\xi)\Omega_{\beta-1}(t_1,\gamma(\xi))\Omega_\alpha(t_1,0)$$
$$\geq \quad \Omega_{\beta-1}(t_1,\gamma(\xi))\Phi(\xi)D_0^\alpha\Upsilon(t_1) + \Omega_{\beta-1}(t_1,\gamma(\xi))\Upsilon(\xi)D_0^\alpha\Phi(t_1)$$

integrating with respect to ξ from 0 to t_1, we find

$$D_0^\alpha(\Upsilon\Phi)(t_1)\int_0^{t_1}\Omega_{\beta-1}(t_1,\gamma(\xi))d_\gamma\xi+\left(\int_0^{t_1}\Upsilon(\xi)\Phi(\xi)\Omega_{\beta-1}(t_1,\gamma(\xi))d_\gamma\xi\right)\Omega_\alpha(t_1,0)$$

$$\geq\left(\int_0^{t_1}\Omega_{\beta-1}(t_1,\gamma(\xi))\Phi(\xi)d_\gamma\xi\right)D_0^\alpha\Upsilon(t_1)+\left(\int_0^{t_1}\Omega_{\beta-1}(t_1,\gamma(\xi))\Upsilon(\xi)d_\gamma\xi\right)D_0^\alpha\Phi(t_1)$$

and

$$D_0^\alpha(\Upsilon\Phi)(t_1)\Omega_\beta(t_1,0)+\Omega_\alpha(t_1,0)D_0^\beta(\Upsilon\Phi)(t_1)\geq D_0^\alpha\Upsilon(t_1)D_0^\beta\Phi(t_1)+D_0^\beta\Upsilon(t_1)D_0^\alpha\Phi(t_1).$$

This completes the proof.

Theorem 3.3. *Let $n\in\mathbb{N}$ be chosen arbitrarily and positive increasing functions Υ_j, $j\in\{1,\ldots,n\}$. Then for any $t_1>0$, $\alpha\geq 1$, we have*

$$D_0^\alpha\left(\prod_{j=1}^n\Upsilon_j\right)(t_1)\geq(\Omega_\alpha(t_1,0))^{1-n}\prod_{j=1}^n D_0^\alpha\Upsilon_j(t_1). \tag{5}$$

Proof. By using the principle of mathematical induction.

1. For $n=1$ the assertion is true.
2. Assume that the assertion is true for some $n\in\mathbb{N}$.
3. We will prove

$$D_0^\alpha\left(\prod_{j=1}^{n+1}\Upsilon_j\right)(t_1)\geq(\Omega_\alpha(t_1,0))^{-n}\prod_{j=1}^{n+1}D_0^\alpha\Upsilon_j(t_1).$$

We apply Theorem 3.1 for the functions

$$\begin{aligned}\Upsilon(t_1)&=\Upsilon_{n+1}(t_1),\\ \Phi(t_1)&=\left(\prod_{j=1}^n\Upsilon_j\right)(t_1)\end{aligned}$$

and using (5), we get

$$\begin{aligned}D_0^\alpha\left(\prod_{j=1}^{n+1}\Upsilon_j\right)(t_1)&=D_0^\alpha(\Upsilon\Phi)(t_1)\\ &\geq(\Omega_\alpha(t_1,0))^{-1}D_0^\alpha\Upsilon_{n+1}(t_1)D_0^\alpha\Phi(t_1)\\ &\geq(\Omega_\alpha(t_1,0))^{-1}D_0^\alpha\Upsilon_{n+1}(t_1)(\Omega_\alpha(t_1,0))^{1-n}\prod_{j=1}^n D_0^\alpha\Upsilon_j(t_1)\\ &=(\Omega_\alpha(t_1,0))^{-n}\prod_{j=1}^{n+1}D_0^\alpha\Upsilon_j(t_1).\end{aligned}$$

Thus, the inequality (5) is true for all $n \in \mathbb{N}$.

4 A Poincaré Type Inequality

Let $a \in \mathbb{T}$, $a > 0$. We will start with the following useful auxiliary results.

Lemma 4.1. *Suppose* $\alpha, \beta > 1$, $\Upsilon \in \mathcal{C}_{rd}([0, a])$. *Then*

$$\begin{aligned} D_0^\alpha D_0^\beta \Upsilon(t_1) &= D_0^{\alpha+\beta}\Upsilon(t_1) \\ &+ \int_0^{t_1} \Upsilon(u)(\gamma(u) - u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u))d_\gamma u, \quad t_1 \in [0, a]. \end{aligned}$$

Proof. Let $t_1 \in [0, a]$. By utilizing Fubini's Theorem, we get

$$\begin{aligned} D_0^\alpha D_0^\beta \Upsilon(t_1) &= \int_0^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau)) \int_0^\tau \Omega_{\beta-1}(\tau, \gamma(u))\Upsilon(u)d_\gamma u d_\gamma \tau \\ &= \int_0^{t_1} \Upsilon(u) \left(\int_u^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau))\Omega_{\beta-1}(\tau, \gamma(u))d_\gamma \tau \right) d_\gamma u \\ &= \int_0^{t_1} \Upsilon(u) \Bigg(\int_u^{\gamma(u)} \Omega_{\alpha-1}(t_1, \gamma(\tau))\Omega_{\beta-1}(\tau, \gamma(u))d_\gamma \tau \\ &\quad + \int_{\gamma(u)}^{t_1} \Omega_{\alpha-1}(t_1, \gamma(\tau))\Omega_{\beta-1}(\tau, \gamma(u))d_\gamma \tau \Bigg) d_\gamma u \\ &= \int_0^{t_1} \Upsilon(u) \Big(\Omega_{\alpha+\beta-1}(t_1, \gamma(u)) + (\gamma(u) - u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u)) \Big) d_\gamma u \\ &= \int_0^{t_1} \Omega_{\alpha+\beta-1}(t_1, \gamma(u))\Upsilon(u)d_\gamma u \\ &\quad + \int_0^{t_1} (\gamma(u) - u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u))d_\gamma u \\ &= D_0^{\alpha+\beta}\Upsilon(t_1) + \int_0^{t_1} (\gamma(u) - u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u))d_\gamma u. \end{aligned}$$

This completes the proof.

Definition 4.1. Let $\alpha, \beta > 1$, $\Upsilon \in \mathcal{C}_{rd}([0, a])$. The integral

$$E(\Upsilon, \alpha, \beta, t_1) = \int_0^{t_1} \Upsilon(u)(\gamma(u) - u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u))d_\gamma u, \quad t_1 \in [0, a],$$

is called the forward graininess deviation functional of Υ.

By Lemma 4.1, we have

$$D_0^\alpha D_0^\beta \Upsilon(t_1) = D_0^{\alpha+\beta}\Upsilon(t_1) + E(\Upsilon, \alpha, \beta, t_1), \quad t_1 \in [0, a].$$

Definition 4.2. Suppose $\alpha > 2$ and $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\nu = m - \alpha$. For a function $\Upsilon \in \mathcal{C}_{rd}^m([0,a])$, define

$$\begin{aligned} \Delta_0^{\alpha-1}\Upsilon(t_1) &= D_0^{\nu+1} D_\gamma^m \Upsilon(t_1) \\ &= \int_0^{t_1} \Omega_\nu(t_1, \gamma(u)) D_\gamma^m \Upsilon(u) d_\gamma u, \quad t_1 \in [0,a]. \end{aligned}$$

Lemma 4.2. *Suppose $\alpha > 2$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\nu = m-\alpha$, $\Upsilon \in \mathcal{C}_{rd}^m([0,a])$. Then*

$$\begin{aligned} &\int_0^{t_1} \Omega_{m-1}(t_1, \gamma(\tau)) D_\gamma^m \Upsilon(\tau) d_\gamma \tau \\ &= -\int_0^{t_1} D_\gamma^m \Upsilon(u)(\gamma(u) - u)\Omega_{\alpha-2}(t_1, \gamma(u))\Omega_\nu(u, \gamma(u)) d_\gamma u \\ &+ \int_0^{t_1} \Omega_{\alpha-2}(t_1, \gamma(\tau)) \Delta_0^{m-1}\Upsilon(\tau) d_\gamma \tau, \quad t_1 \in [0,a]. \end{aligned}$$

Proof. We have

$$\begin{aligned} D_0^{\alpha-1}\Delta_0^{m-1}\Upsilon(t_1) &= \int_0^{t_1} \Omega_{\alpha-2}(t_1, \gamma(\tau)) \Delta_0^{m-1}\Upsilon(\tau) d_\gamma \tau \\ &= D_0^{\alpha-1} D_0^{\nu+1} D_\gamma^m \Upsilon(t_1) \\ &= D_0^{\alpha+\nu} D_\gamma^m \Upsilon(t_1) \\ &\quad + \int_0^{t_1} D_\gamma^m \Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u)) d_\gamma u \\ &= D_0^m D_\gamma^m \Upsilon(t_1) \\ &\quad + \int_0^{t_1} D_\gamma^m \Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u)) d_\gamma u \\ &= \int_0^{t_1} \Omega_{m-1}(t_1, \gamma(u)) D_\gamma^m \Upsilon(u) d_\gamma u \\ &\quad + \int_0^{t_1} D_\gamma^m \Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-1}(t_1, \gamma(u))\Omega_{\beta-1}(u, \gamma(u)) d_\gamma u, \end{aligned}$$

$t_1 \in [0,a]$. This completes the proof.

Lemma 4.3 [Fractional Taylor Formula]. *Let* $\alpha > 2$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\nu = m - \alpha$, $\Upsilon \in \mathcal{C}_{rd}^{m}([0,a])$. *Then*

$$\begin{aligned}\Upsilon(t_1) &= \sum_{k=0}^{m-1} \Omega_k(t_1,0) D_\gamma^k \Upsilon(0) \\ &\quad - \int_0^{t_1} D_\gamma^m \Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-2}(t_1,\gamma(u))\Omega_\nu(u,\gamma(u)) d_\gamma u \\ &\quad + \int_0^{t_1} \Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau) d_\gamma \tau, \quad t_1 \in [0,a].\end{aligned}$$

Proof. By using Taylor formula and Lemma 4.2, we get

$$\begin{aligned}\Upsilon(t_1) &= \sum_{k=0}^{m-1} \Omega_k(t_1,0) D_\gamma^k \Upsilon(0) + \int_0^{t_1} \Omega_{m-1}(t_1,\gamma(\tau)) D_\gamma^m \Upsilon(\tau) d_\gamma \tau \\ &= \sum_{k=0}^{m-1} \Omega_k(t_1,0) D_\gamma^k \Upsilon(0) \\ &\quad - \int_0^{t_1} D_\gamma^m \Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-2}(t_1,\gamma(u))\Omega_\nu(u,\gamma(u)) d_\gamma u \\ &\quad + \int_0^{t_1} \Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau) d_\gamma \tau, \quad t_1 \in [0,a].\end{aligned}$$

This completes the proof.

Definition 4.3. For $\alpha > 2$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\nu = m - \alpha$, $\Upsilon \in \mathcal{C}_{rd}^{m}([0,a])$, define

$$B(t_1) = \Upsilon(t_1) + E\left(D_\gamma^m \Upsilon, \alpha - 1, \nu + 1, t_1\right), \quad t_1 \in [0,a].$$

Lemma 4.4. *Let* $\alpha > 2$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\nu = m - \alpha$, $\Upsilon \in \mathcal{C}_{rd}^{m}([0,a])$, $D_\gamma^k \Upsilon(0) = 0$, $k \in \{0, \ldots, m\}$. *Then*

$$B(t_1) = \int_0^{t_1} \Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau) d_\gamma \tau, \quad t_1 \in [0,a].$$

Proof. By using fractional Taylor formula, we get

$$\begin{aligned}\Upsilon(t_1) &= \sum_{k=0}^{m-1}\Omega_k(t_1,0)D_\gamma^k\Upsilon(0)\\ &\quad -\int_0^{t_1} D_\gamma^m\Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-2}(t_1,\gamma(u))\Omega_\nu(u,\gamma(u))d_\gamma u\\ &\quad +\int_0^{t_1}\Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau)d_\gamma\tau\\ &= -\int_0^{t_1} D_\gamma^m\Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-2}(t_1,\gamma(u))\Omega_\nu(u,\gamma(u))d_\gamma u\\ &\quad +\int_0^{t_1}\Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau)d_\gamma\tau, \quad t_1\in[0,a].\end{aligned}$$

Hence,

$$\begin{aligned}B(t_1) &= \Upsilon(t_1)+E\left(D_\gamma^m\Upsilon,\alpha-1,\nu+1,t_1\right)\\ &= \Upsilon(t_1)+\int_0^{t_1}(\gamma(u)-u)\Omega_{\alpha-2}(t_1,\gamma(u))\Omega_\nu(u,\gamma(u))D_\gamma^m\Upsilon(u)d_\gamma u\\ &= \int_0^{t_1}\Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau)d_\gamma\tau, \quad t_1\in[0,a].\end{aligned}$$

This completes the proof.

Theorem 4.1 [Poincaré Type Inequality]. *Let $\alpha>2$, $m-1<\alpha<m$, $m\in\mathbb{N}$, $\Omega_{\alpha-2}(s,\gamma(\tau))$,
$\Omega_\nu(s,\gamma(\tau))\in\mathcal{C}([0,a]\times[0,a])$, and $D_\gamma^k\Upsilon(0)=0$, $k\in\{0,\ldots,m-1\}$. Let also, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\begin{aligned}&\int_0^a |B(t_1)|^q d_\gamma t_1\\ &\le \left(\int_0^a\left(\int_0^{t_1}|\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma\tau\right)^{\frac{q}{p}} d_\gamma t_1\right)\left(\int_0^a|\Delta_0^{\alpha-1}\Upsilon(t_1)|^q d_\gamma t_1\right).\end{aligned}$$

Proof. By Lemma 4.4, we have

$$B(t_1)=\int_0^{t_1}\Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau)d_\gamma\tau, \quad t_1\in[0,a].$$

Now, applying Hölder's inequality, we get

$$\begin{aligned} |B(t_1)| &= \left| \int_0^{t_1} \Omega_{\alpha-2}(t_1,\gamma(\tau)) \Delta_0^{\alpha-1} \Upsilon(\tau) d_\gamma \tau \right| \\ &\le \int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))| \left| \Delta_0^{\alpha-1} \Upsilon(\tau) \right| d_\gamma \tau \\ &\le \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{1}{p}} \left(\int_0^{t_1} \left| \Delta_0^{\alpha-1} \Upsilon(\tau) \right|^q d_\gamma \tau \right)^{\frac{1}{q}}, \quad t_1 \in [0,a]. \end{aligned}$$

Hence,

$$\begin{aligned} |B(t_1)|^q &\le \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{q}{p}} \left(\int_0^{t_1} \left| \Delta_0^{\alpha-1} \Upsilon(\tau) \right|^q d_\gamma \tau \right) \\ &\le \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{q}{p}} \left(\int_0^{a} \left| \Delta_0^{\alpha-1} \Upsilon(\tau) \right|^q d_\gamma \tau \right), \quad t_1 \in [0,a], \end{aligned}$$

and

$$\int_0^a |B(t_1)|^q d_\gamma t \le \left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{q}{p}} d_\gamma t_1 \right) \left(\int_0^a |\Delta_0^{\alpha-1} \Upsilon(t_1)|^q d_\gamma t_1 \right).$$

This completes the proof.

5 A Sobolev Type Inequality

Theorem 5.1 [Sobolev Type Inequality]. *Let* $\alpha > 2$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\Omega_{\alpha-2}(s,\gamma(\tau))$,
$\Omega_\nu(s,\gamma(\tau)) \in \mathcal{C}([0,a] \times [0,a])$, *and* $D_\gamma^k \Upsilon(0) = 0$, $k \in \{0,\dots,m-1\}$. *Let also,* $p,q > 1, r \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\begin{aligned} \left(\int_0^a |B(t_1)|^r d_\gamma t_1 \right)^{\frac{1}{r}} &\le \left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{r}{p}} d_\gamma t_1 \right)^{\frac{1}{r}} \\ &\quad \times \left(\int_0^a |\Delta_0^{\alpha-1} \Upsilon(t_1)|^q d_\gamma t_1 \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. As in the proof of Theorem 5.1, we have

$$|B(t_1)| \le \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{1}{p}} \left(\int_0^{t_1} \left| \Delta_0^{\alpha-1} \Upsilon(\tau) \right|^q d_\gamma \tau \right)^{\frac{1}{q}}, \quad t_1 \in [0,a].$$

Hence,

$$|B(t_1)|^r \le \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{r}{p}} \left(\int_0^{a} \left| \Delta_0^{\alpha-1} \Upsilon(\tau) \right|^q d_\gamma \tau \right)^{\frac{r}{q}}, \quad t_1 \in [0,a],$$

and

$$\left(\int_0^a |B(t_1)|^r d_\gamma t_1\right)^{\frac{1}{r}} \leq \left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau\right)^{\frac{r}{p}} d_\gamma t_1\right)^{\frac{1}{r}} \times \left(\int_0^a |\Delta_0^{\alpha-1}\Upsilon(t_1)|^q d_\gamma t_1\right)^{\frac{1}{q}}.$$

This completes the proof.

6 An Opial Type Inequality

Theorem 6.1. *Let* $\alpha > 2$, $m-1<\alpha<m$, $m \in \mathbb{N}$, $\Omega_{\alpha-2}(s,\gamma(\tau)), \Omega_\nu(s,\gamma(\tau)) \in \mathcal{C}([0,a]\times[0,a])$, *and* $D_\gamma^k \Upsilon(0) = 0$, $k \in \{0,\ldots,m-1\}$. *Let also,* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *and* $|\Delta_0^\alpha \Upsilon|$ *is an increasing function on* $[0,a]$. *Then*

$$\int_0^a |B(t_1)| \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right| d_\gamma t_1 \leq a^{\frac{1}{q}} \left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau\right) d_\gamma t_1\right)^{\frac{1}{p}} \times \left(\int_0^a \left(\Delta_0^{\alpha-1}\Upsilon(t_1)\right)^{2q} d_\gamma t_1\right)^{\frac{1}{q}}.$$

Proof. By the proof of Theorem 5.1, we get

$$\begin{aligned} |B(t_1)| &\leq \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau\right)^{\frac{1}{p}} \left(\int_0^{t_1} \left|\Delta_0^{\alpha-1}\Upsilon(\tau)\right|^q d_\gamma \tau\right)^{\frac{1}{q}} \\ &\leq \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau\right)^{\frac{1}{p}} \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right| t_1^{\frac{1}{q}}, \quad t_1 \in [0,a]. \end{aligned}$$

Hence,

$$|B(t_1)| \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right| \leq \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau\right)^{\frac{1}{p}} \left(\Delta_0^{\alpha-1}\Upsilon(t_1)\right)^2 t_1^{\frac{1}{q}}, \quad t_1 \in [0,a].$$

Then

$$\begin{aligned}\int_0^a |B(t_1)| \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right| d_\gamma t_1 &\le \int_0^a \left(\left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right)^{\frac{1}{p}} \right. \\ &\quad \left. \times \left(\Delta_0^{\alpha-1}\Upsilon(t_1)\right)^2 {t_1}^{\frac{1}{q}} \right) d_\gamma t_1 \\ &\le \left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right) d_\gamma t_1 \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^a \left(\Delta_0^{\alpha-1}\Upsilon(t_1)\right)^{2q} t_1 d_\gamma t_1 \right)^{\frac{1}{q}} \\ &\le a^{\frac{1}{q}} \left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau \right) d_\gamma t_1 \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^a \left(\Delta_0^{\alpha-1}\Upsilon(t_1)\right)^{2q} d_\gamma t_1 \right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof.

7 Ostrowski Type Inequalities

Theorem 7.1. *Let $\alpha > 2$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\Omega_{\alpha-2}(s,\gamma(\tau)), \Omega_\nu(s,\gamma(\tau)) \in \mathcal{C}([0,a]\times[0,a])$, and $D_\gamma^k \Upsilon(0) = 0$, $k \in \{1,\dots,m-1\}$. Then*

$$\left| \frac{1}{a}\int_0^a B(t_1) d_\gamma t_1 - \Upsilon(0) \right| \le \frac{1}{a} \left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))| d_\gamma \tau \right) d_\gamma t_1 \right) \sup_{t_1\in[0,a]} \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right|.$$

Proof. By using the fractional Taylor formula, we get

$$\begin{aligned}\Upsilon(t_1) &= \Upsilon(0) - \int_0^{t_1} D_\gamma^m \Upsilon(u)(\gamma(u)-u)\Omega_{\alpha-2}(t_1,\gamma(u))\Omega_\nu(u,\gamma(u)) d_\gamma u \\ &\quad + \int_0^{t_1} \Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau) d_\gamma \tau, \quad t_1 \in [0,a].\end{aligned}$$

Then, by Lemma 4.4, we get

$$B(t_1) - \Upsilon(0) = \int_0^{t_1} \Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau) d_\gamma \tau, \quad t_1 \in [0,a],$$

and

$$
\begin{aligned}
|B(t_1)-\Upsilon(0)| &= \left|\int_0^{t_1} \Omega_{\alpha-2}(t_1,\gamma(\tau))\Delta_0^{\alpha-1}\Upsilon(\tau)d_\gamma\tau\right| \\
&\leq \int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))| \left|\Delta_0^{\alpha-1}\Upsilon(\tau)\right| d_\gamma\tau \\
&\leq \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))| d_\gamma\tau\right) \sup_{t_1\in[0,a]} \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|\frac{1}{a}\int_0^a B(t_1)d_\gamma t_1 - \Upsilon(0)\right| = \left|\frac{1}{a}\int_0^a (B(t_1)-\Upsilon(0))d_\gamma t_1\right| \\
&\leq \frac{1}{a}\int_0^a |B(t_1)-\Upsilon(0)| d_\gamma t_1 \\
&\leq \left(\frac{1}{a}\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))| d_\gamma\tau\right) d_\gamma t_1\right) \sup_{t_1\in[0,a]} \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right|.
\end{aligned}
$$

Theorem 7.2. *Suppose all conditions of Theorem 7.1. Let also, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$
\begin{aligned}
\left|\frac{1}{a}\int_0^a B(t_1)d_\gamma t_1 - \Upsilon(0)\right| &\leq \frac{1}{a}\left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma\tau\right)^{\frac{1}{p}} d_\gamma t_1\right) \\
&\quad \times \left(\int_0^a \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right|^q d_\gamma t_1\right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. By the proof of Theorem 7.1, it follows that

$$
\begin{aligned}
|B(t_1)-\Upsilon(0)| &\leq \int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))| \left|\Delta_0^{\alpha-1}\Upsilon(\tau)\right| d_\gamma\tau \\
&\leq \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma\tau\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^{t_1} \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right|^q d_\gamma t_1\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma\tau\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^{a} \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right|^q d_\gamma t_1\right)^{\frac{1}{q}},
\end{aligned}
$$

$t_1 \in [0, a]$. Thus

$$\begin{aligned}\left|\frac{1}{a}\int_0^a B(t_1)d_\gamma t_1 - \Upsilon(0)\right| &= \left|\frac{1}{a}\int_0^a (B(t_1)-\Upsilon(0))d_\gamma t_1\right| \\ &\leq \frac{1}{a}\int_0^a |B(t_1)-\Upsilon(0)|d_\gamma t_1 \\ &\leq \frac{1}{a}\left(\int_0^a \left(\int_0^{t_1} |\Omega_{\alpha-2}(t_1,\gamma(\tau))|^p d_\gamma \tau\right)^{\frac{1}{p}} d_\gamma t_1\right) \\ &\quad \times \left(\int_0^a \left|\Delta_0^{\alpha-1}\Upsilon(t_1)\right|^q d_\gamma t_1\right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof.

8 Conclusion

In this work, we establish a series of generalized quantum fractional integral inequalities involving delay terms. By extending classical inequalities such as those of Chebyshev, Poincaré, Sobolev, Opial, and Ostrowski types within the quantum fractional framework, we provide new insights and sharper estimates for delayed systems. Future work may focus on the development of other types of inequalities, numerical schemes and applications by using the same methodology.

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