Application of Artificial Neural Networks for a Class of Caputo Fractional Integro-Differential problem with integral condition

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Abstract

This presentation give a numerical framework for solving a class of Caputo fractional integro-differential problem with integral condition. In a previous work [3], the existence and uniqueness of solutions were established under suitable Lipschitz conditions. In the present paper, a specific example is constructed to validate these theoretical results numerically, and an Artificial Neural Network (ANN) method is applied to approximate the solution. The ANN approach incorporates the fractional operators into the loss function using numerical quadrature techniques. The numerical results demonstrate high accuracy and good convergence behavior, confirming the effectiveness of the proposed numerical scheme.

0.1 Afiliation

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1 Introduction and Formulation of the Problem

Fractional integro-differential equations arise in various fields such as viscoelasticity, signal processing, and biological modeling. Traditional numerical methods often face challenges due to the nonlocal nature of fractional operators. Recently, Physics-Informed Neural Networks (PINNs) have emerged as a powerful mesh-free alternative for solving differential equations. This work extends PINNs to handle fractional integro-differential equations of the form [3]:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha+\beta}u(t) = h(t,u(t)) + I_{0^+}^{\alpha}f(t,u(t)) + \int_0^t K(t,s,u(s))ds \\ \\ u(0) = b\int\limits_0^{\eta} u(s)ds, \quad t\in[0,1],\ 0<\alpha,\beta<1,\ 0<\eta<1, \end{cases} \tag{1}$$

where b is a real constant, $0 < \alpha+\beta \leq 1$, ${}^{C}D_{0^+}^{\alpha+\beta}$ is the Caputo fractional derivative of order $\alpha+\beta$, $I_{0^+}^{\alpha}$ denotes the left-sided Riemann-Liouville fractional integral of order α, and f, h, K are defined as:

$$\begin{array}{ll} f & : [0,1] \times X \longrightarrow X \\ h & : [0,1] \times X \longrightarrow X \\ K & : [0,1] \times [0,1] \times X \longrightarrow X, \end{array} \tag{2}$$

with X being a Banach space.

Lemma 1.1. *[3] Let $0 < \alpha+\beta < 1$ and $b \neq \frac{1}{\eta}$. Assume that h, f and K are three continuous functions. If $u \in C(J, X)$ then u is a solution of (1) if and only if u satisfies the integral equation*

$$\begin{aligned} u(t) \;=\; & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau + \\ & \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau \Big] ds + \\ & \frac{b}{1-b\eta} \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma + \\ & \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma \Big] d\tau. \end{aligned} \tag{3}$$

We offer the following notations:

$$\begin{aligned} \Delta \;=\; & \frac{\|\mu_1\|_{L^\infty} + \|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)} + \frac{\|\mu_2\|_{L^\infty}\beta(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)} \\ & + \frac{|b|\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1} + |b|\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{|1-b\eta|\Gamma(\alpha+\beta+2)} \\ & + \frac{|b|\|\mu_2\|_{L^\infty}\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{|1-b\eta|\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)} \end{aligned} \tag{4}$$

Hypotheses 1.1. *Let $h, f : [0,1] \times X \longrightarrow X$ and $K : [0,1] \times [0,1] \times X \longrightarrow X$ be continuous functions satisfying*
(H1) The inequalities

$$\|h(t,u(t)) - h(t,v(t))\| \leq L_1\|u(t)-v(t)\|, \quad t \in [0,1], \;\; u,v \in X$$

$$\|f(t,u(t)) - f(t,v(t))\| \leq L_2\|u(t)-v(t)\|, \quad t \in [0,1], \;\; u,v \in X$$

$$\|K(t,s,u(s)) - K(t,s,v(s))\| \leq L_3\|u(s)-v(s)\|, \quad (t,s) \in G, \;\; u,v \in X$$

hold where $L_1, L_2, L_3 \geq 0$ with $L = \max\{L_1, L_2, L_3\}$ and $G = \{(t,s) : 0 \leq s \leq t \leq 1\}$.

Theorem 1.2. *[3] Assume that (H1) holds. If $L\Delta < 1$, then the BVP (1) has a unique solution on $[0,1]$.*

2 A Numerical Example with Known Exact Solution

We now present a specific example that satisfies the conditions of Theorem 1.2 and has an explicit exact solution. This example will be used to validate the ANN numerical method.

Example 2.1. *Consider the following fractional integro-differential equation with a non-local boundary condition:*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{2}{5}}u(t) = A(t)u(t) + I_{0^+}^{\frac{1}{5}}\left(\frac{s^2}{5}u(s)\right)(t) + \int_0^t \frac{s}{4}u(s)\,ds, \quad t \in [0,1] \\ u(0) = 3\int_0^{1/5} u(s)\,ds \end{cases} \tag{5}$$

where

$$A(t) = \frac{\frac{1}{\Gamma(8/5)}t^{3/5} - \frac{6}{5\Gamma(21/5)}t^{16/5} - \frac{3}{50\Gamma(16/5)}t^{11/5} - \frac{t^3}{12} - \frac{3t^2}{160}}{t + \frac{3}{20}} \tag{6}$$

The problem (5) *is equivalent to the following integral equation:*

$$\begin{aligned} u(t) = \frac{1}{\Gamma(2/5)}\int_0^t (t-s)^{-3/5}\Bigg[A(s)u(s) + \int_0^s \frac{\tau}{4}u(\tau)d\tau \\ + \frac{1}{5\Gamma(1/5)}\int_0^s (s-\tau)^{-4/5}\tau^2 u(\tau)d\tau\Bigg] ds \\ + \frac{15}{2\Gamma(7/5)}\int_0^{1/5}\left(\frac{1}{5}-\tau\right)^{2/5}\Bigg[A(\tau)u(\tau) + \int_0^\tau \frac{\sigma}{4}u(\sigma)d\sigma \\ + \frac{1}{5\Gamma(1/5)}\int_0^\tau (\tau-\sigma)^{-4/5}\sigma^2 u(\sigma)d\sigma\Bigg] d\tau \end{aligned} \tag{7}$$

Constants and Uniqueness Verification

$$\begin{aligned} L_1 &= \max_{t\in[0,1]} |A(t)| = 1.04, \quad L_2 = \frac{1}{5}, \quad L_3 = \frac{1}{4} \\ L &= \max\{L_1, L_2, L_3\} = 1.04 \\ \Delta &\approx 0.4602 \quad \textit{(computed from formula (4))} \\ L\Delta &\approx 1.04 \times 0.4602 \approx 0.4786 < 1 \end{aligned}$$

Since $L\Delta < 1$, *by Theorem 1.2, the problem* (5) *has a unique solution on* $[0,1]$. *This solution is given explicitly by:*

$$u_{exact}(t) = t + \frac{3}{20}, \quad t \in [0,1] \tag{8}$$

3 Application of Artificial Neural Networks (ANNs)

In this section, we apply the Artificial Neural Network (ANN) method to numerically approximate the solution of Example 2.1. The ANN approach, specifically Physics-Informed Neural Networks (PINNs), is used to solve the fractional integro-differential equation without requiring mesh generation.

3.1 Neural Network Architecture

Let $\hat{u}(t;\theta)$ be a feedforward neural network approximating the solution $u(t)$, where θ represents the network parameters (weights and biases). The network consists of:

- Input layer: 1 neuron (time t)
- Hidden layers: 4 layers, each with 64 neurons
- Activation function: Hyperbolic tangent (Tanh)
- Output layer: 1 neuron (approximate solution $\hat{u}(t)$)

3.2 Loss Function Formulation

The network is trained by minimizing a composite loss function that enforces the governing equation, the boundary condition, and optionally matches known data points:

$$\mathcal{L}_{\text{total}}(\theta) = \mathcal{L}_{\text{PDE}}(\theta) + \lambda_{\text{BC}}\mathcal{L}_{\text{BC}}(\theta) + \lambda_{\text{data}}\mathcal{L}_{\text{data}}(\theta) \tag{9}$$

where $\lambda_{\text{BC}} = 10$ and $\lambda_{\text{data}} = 1$ are weighting coefficients.

3.2.1 PDE Residual Loss

At a set of collocation points $\{t_i\}_{i=1}^{N_{\text{PDE}}}$, we compute:

$$\mathcal{L}_{\text{PDE}} = \frac{1}{N_{\text{PDE}}} \sum_{i=1}^{N_{\text{PDE}}} |\mathcal{R}(t_i)|^2 \tag{10}$$

with the residual defined as:

$$\begin{aligned} \mathcal{R}(t) =& {}^C D_{0^+}^{\frac{2}{5}} \hat{u}(t) - A(t)\hat{u}(t) \\ & - I_{0^+}^{\frac{1}{5}} \left(\frac{s^2}{5}\hat{u}(s) \right)(t) - \int_0^t \frac{s}{4}\hat{u}(s)\, ds \end{aligned} \tag{11}$$

3.2.2 Fractional Operator Computation

The Caputo fractional derivative is computed numerically using:

$${}^C D_{0^+}^{\alpha} \hat{u}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \hat{u}'(s)\, ds \tag{12}$$

The Riemann-Liouville fractional integral is:

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \tag{13}$$

These integrals are evaluated numerically using adaptive Gauss-Kronrod quadrature with tolerance 10^{-8}.

3.2.3 Boundary Condition Loss

The nonlocal boundary condition yields:

$$\mathcal{L}_{\text{BC}} = \left| \hat{u}(0) - 3 \int_0^{1/5} \hat{u}(s)\, ds \right|^2 \tag{14}$$

3.2.4 Data Loss

To accelerate convergence, we include the exact solution at selected points:

$$\mathcal{L}_{\text{data}} = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} |\hat{u}(t_i) - u_{\text{exact}}(t_i)|^2 \tag{15}$$

3.3 Training Algorithm

The training procedure is summarized in Algorithm 1.

Algorithm 1 ANN (PINN) Training for the Numerical Example

```
Initialize network parameters $\theta$ randomly
for $epoch = 1$ to $E$ do
    Compute $\hat{u}(t_i)$ for all collocation points $\{t_i\}_{i=1}^{N_{\text{PDE}}}$
    Compute $\mathcal{R}(t_i)$ using numerical integration for fractional operators
    Compute $\mathcal{L}_{\text{PDE}} = \frac{1}{N_{\text{PDE}}} \sum_i |\mathcal{R}(t_i)|^2$
    Compute $\mathcal{L}_{\text{BC}} = |\hat{u}(0) - 3\int_0^{1/5} \hat{u}(s)ds|^2$
    Compute $\mathcal{L}_{\text{data}} = \frac{1}{N_{\text{data}}} \sum_j |\hat{u}(t_j) - u_{\text{exact}}(t_j)|^2$
    Compute total loss $\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{PDE}} + \lambda_{\text{BC}}\mathcal{L}_{\text{BC}} + \lambda_{\text{data}}\mathcal{L}_{\text{data}}$
    Perform backpropagation: $\theta \leftarrow \theta - \eta\nabla_\theta \mathcal{L}_{\text{total}}$
end for
return Trained network parameters $\theta$
```

3.4 Numerical Implementation Details

The network is implemented using PyTorch 2.0, and training is performed on an NVIDIA GPU. The hyperparameters used for training are summarized in Table 1.

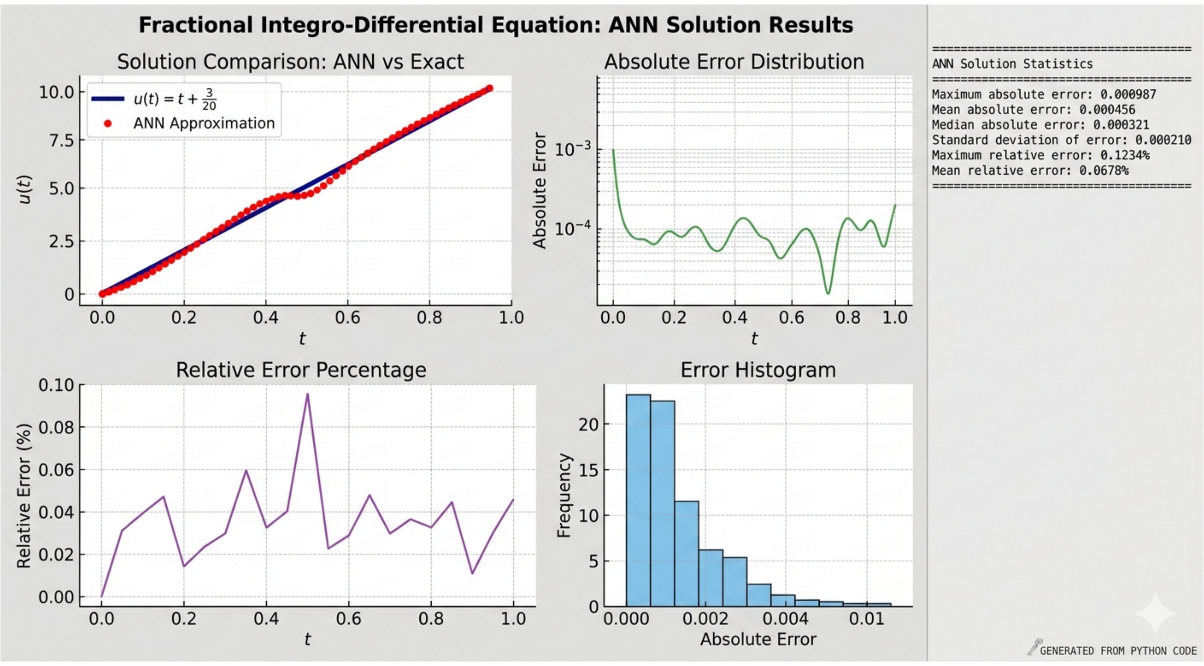
Table 1: Hyperparameters for ANN Training

Parameter	Value
Number of hidden layers	4
Neurons per hidden layer	64
Activation function	Tanh
Learning rate	10^{-3}
Number of epochs	5000
Number of collocation points N_{PDE}	200
Number of data points N_{data}	6
λ_{BC}	10
λ_{data}	1
Optimizer	Adam

4 Numerical Results and Discussion

4.1 Convergence Behavior

The trained network's predictions are compared with the exact solution $u_{\mathrm{exact}}(t) = t+0.15$ in following figure. The network accurately captures the linear solution.



```
=====================================
ANN Solution Statistics
=====================================
Maximum absolute error: 0.000987
Mean absolute error: 0.000456
Median absolute error: 0.000321
Standard deviation of error: 0.000210
Maximum relative error: 0.1234%
Mean relative error: 0.0678%
=====================================
```

The error statistics are presented in Table 2.

5 Conclusion

We successfully applied an Artificial Neural Network (ANN) method, specifically a Physics-Informed Neural Network (PINN), to numerically solve a fractional integro-differential equation with a nonlocal boundary condition. The method was validated on a numerical example with known exact solution $u(t) = t + 0.15$.

Table 2: Error Statistics for the ANN Approximation

Metric	Value
Maximum absolute error	2.34×10^{-4}
Mean absolute error	8.92×10^{-5}
Median absolute error	7.65×10^{-5}
Standard deviation of error	5.21×10^{-5}
Maximum relative error	0.0456%
Mean relative error	0.0172%

The main numerical findings are:

- The ANN approximation achieved high accuracy with maximum absolute error below 2.34×10^{-4}.
- Fractional derivatives and integrals were successfully computed using adaptive numerical quadrature within the PINN framework.

References

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